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Asymptotic behavior of solutions of forced fractional differential equations

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

Said R. Grace[1], **John R. Graef**[✉,2] and **Ercan Tunç**[3]

[1]Department of Engineering Mathematics, Faculty of Engineering, Cairo University
Orman, Giza 12613, Egypt
[2]Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
[3]Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University
60240, Tokat, Turkey

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Abstract. The authors study the boundedness of nonoscillatory solutions of forced fractional differential equations of the form

$$^{C}D_c^{\alpha} y(t) = e(t) + f(t, x(t)), \qquad c > 1, \quad \alpha \in (0,1),$$

where $y(t) = (a(t)x'(t))'$, $c_0 = \frac{y(c)}{\Gamma(1)} = y(c)$, and c_0 is a real constant. The technique used in obtaining their results will apply to related fractional differential equations with Caputo derivatives of any order. Examples illustrate the results obtained in this paper.

Keywords: integro-differential equations, fractional differential equations, asymptotic behavior, nonoscillatory solutions.

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1 Introduction

We consider the forced fractional differential equation

$$^{C}D_c^{\alpha} y(t) = e(t) + f(t, x(t)), \qquad c > 1, \quad \alpha \in (0,1), \tag{1.1}$$

where $y(t) = (a(t)x'(t))'$, $c_0 = \frac{y(c)}{\Gamma(1)} = y(c)$, c_0 is a real constant, and $^{C}D_c^{\alpha} u(t)$ is the Caputo derivative of order α, which is defined as

$$^{C}D_a^{\alpha} u(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds \tag{1.2}$$

with $n = \lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

In the remainder of the paper we assume that:

✉Corresponding author. Email: John-Graef@utc.edu

(i) $a : [c, \infty) \to \mathbb{R}^+ = (0, \infty)$ is a continuous function;

(ii) $e : [c, \infty) \to \mathbb{R}$ is a continuous function;

(iii) $f : [c, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a continuous function $h : [c, \infty) \to (0, \infty)$ and a real number λ with $0 < \lambda < 1$ such that

$$0 \le x f(t, x) \le h(t) \, |x|^{\lambda+1} \quad \text{for } x \ne 0 \text{ and } t \ge c.$$

We only consider those solutions of equation (1.1) that are continuable and nontrivial in any neighborhood of ∞. Such a solution is said to be *oscillatory* if there exists an increasing sequence $\{t_n\} \subseteq [c, \infty)$ with $t_n \to \infty$ as $n \to \infty$ such that $x(t_n) = 0$, and it is *nonoscillatory* otherwise.

Fractional differential and integro-differential equations are receiving considerably more attention in the last twenty years due to their importance in applications in many areas of science and engineering such as in modeling systems and processes in physics, mechanics, chemistry, aerodynamics, and the electrodynamics of complex media. In this regard we refer the reader to the monographs [1,13,14,18–21].

Results on the oscillatory and asymptotic behavior of solutions of fractional and integro-differential equations are relatively scarce in the literature; some results can be found, for example, in [2,7,8,10,11,16] and the references contained therein. Currently there does not appear to be any such results for forced fractional differential equations of the type (1.1) other than those in [9]. We are particularly interested in obtaining results that guarantee the boundedness of all nonoscillatory solutions of equation (1.1).

Equation (1.1) is equivalent to the nonlinear Volterra type integral equation

$$y(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} \left[e(s) + f(s, x(s)) \right] ds, \qquad c > 1, \quad \alpha > 0, \tag{1.3}$$

provided the right hand side of equation (1.1), namely $e(t) + f(t, x)$, belongs to the class of absolutely continuous functions AC, and $y(t)$ in (1.3) belongs to the class $AC^2 = \{y \in C^1 : y' \in AC\}$ (see Theorems 2.4 (iii) and 2.7 in [15]). In obtaining our results, we introduce a technique that can be applied to some related fractional differential equations involving Caputo fractional derivatives of any order. Recall that

$${}^{C}D_a^{\alpha} x(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds$$

is the Caputo derivative of the order $\alpha \in (n-1, n)$ of a C^n-scalar valued function $x(t)$ defined on the interval $[c, \infty)$, where $x^{(n)}(t) = \frac{d^n x(t)}{dt^n}$. For $\alpha \in (0, 1)$, this definition was given by Caputo [4]; for the definition of the Caputo derivative of order $\alpha \in (n-1, n)$, $n \ge 1$, see [1,5,6].

2 Main results

In what follows $\Gamma(x)$ is the usual Gamma function given by

$$\Gamma(x) = \int_0^{\infty} s^{x-1} e^{-s} ds, \qquad x > 0.$$

The next two lemmas will be used to prove our main results.

Lemma 2.1. ([3,17]) *Let α and p be positive constants such that*

$$p(\alpha-1)+1>0.$$

Then

$$\int_0^t (t-s)^{p(\alpha-1)}e^{ps}ds \le Qe^{pt}, \qquad t\ge 0.$$

where

$$Q=\frac{\Gamma\left(1+p(\alpha-1)\right)}{p^{1+p(\alpha-1)}}.$$

Lemma 2.2 ([12]). *If X and Y are nonnegative and $0<\lambda<1$, then*

$$X^{\lambda}-(1-\lambda)Y^{\lambda}-\lambda XY^{\lambda-1}\le 0, \tag{2.1}$$

where equality holds if and only if $X=Y$.

We begin with a result that gives sufficient conditions for every nonoscillatory solution x of equation (1.1) to be bounded.

Theorem 2.3. *Let conditions (i)–(iii) hold and assume that there exist real numbers $p>1$ and $0<\alpha<1$ such that $p(\alpha-1)+1>0$, there are numbers $S>0$ and $\sigma>1$ such that*

$$\left(\frac{t}{a(t)}\right)\le Se^{-\sigma t}, \tag{2.2}$$

and there exists a continuous function $m:[c,\infty)\to(0,\infty)$ such that

$$\int_c^{\infty} e^{-qs}m^{q}(s)ds<\infty, \quad \text{where} \;\; q=\frac{p}{p-1}. \tag{2.3}$$

If

$$\limsup_{t\to\infty}\frac{1}{t}\int_c^t (t-s)^{\alpha}e(s)ds<\infty, \qquad \liminf_{t\to\infty}\frac{1}{t}\int_c^t (t-s)^{\alpha}e(s)ds>-\infty, \tag{2.4}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_c^t\int_{t_1}^u (u-s)^{\alpha-1}\left(m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s)\right)dsdu<\infty, \tag{2.5}$$

then any nonoscillatory solution $x(t)$ of equation (1.1) is bounded.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t)>0$ for $t\ge t_1$ for some $t_1\ge c$. If we let $F(t)=f(t,x(t))$, and use (i)–(iii), we see that equation (1.1) can be written as

$$\begin{aligned}\left(a(t)x'(t)\right)'\le c_0&+\frac{1}{\Gamma(\alpha)}\int_c^{t_1}(t-s)^{\alpha-1}|F(s)|\,ds+\frac{1}{\Gamma(\alpha)}\int_c^{t_1}(t-s)^{\alpha-1}|e(s)|\,ds\\&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}e(s)ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}m(s)x(s)ds\\&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\left[h(s)x^{\lambda}(s)-m(s)x(s)\right]ds.\end{aligned} \tag{2.6}$$

Using the fact that $(t-s)^{\alpha-1} \le (t_1-s)^{\alpha-1}$ in the first and second integrals in (2.6), we obtain

$$\begin{aligned}
\big(a(t)x'(t)\big)' &\le c_0 + \frac{1}{\Gamma(\alpha)}\int_c^{t_1}(t_1-s)^{\alpha-1}|F(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_c^{t_1}(t_1-s)^{\alpha-1}|e(s)|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}e(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}m(s)x(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\Big[h(s)x^{\lambda}(s) - m(s)x(s)\Big]\,ds \\
&\le c_1 + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}e(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}m(s)x(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\Big[h(s)x^{\lambda}(s) - m(s)x(s)\Big]\,ds, \qquad (2.7)
\end{aligned}$$

where

$$c_1 = c_0 + \frac{1}{\Gamma(\alpha)}\int_c^{t_1}(t_1-s)^{\alpha-1}|F(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_c^{t_1}(t_1-s)^{\alpha-1}|e(s)|\,ds.$$

Applying Lemma 2.2 with

$$X = h^{1/\lambda}(s)x(s) \quad\text{and}\quad Y = \left(\frac{1}{\lambda}m(s)h^{-1/\lambda}(s)\right)^{1/(\lambda-1)},$$

we obtain

$$h(s)x^{\lambda}(s) - m(s)x(s) \le (1-\lambda)\lambda^{\lambda/(1-\lambda)}m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s),$$

and substituting this into (2.7), we have

$$\begin{aligned}
\big(a(t)x'(t)\big)' &\le c_1 + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}e(s)ds \\
&\quad + \frac{\Big((1-\lambda)\lambda^{\lambda/(1-\lambda)}\Big)}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\Big[m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s)\Big]\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}m(s)x(s)ds. \qquad (2.8)
\end{aligned}$$

An integration of (2.8) from t_1 to t yields

$$\begin{aligned}
a(t)x'(t) &\le a(t_1)x'(t_1) + c_1(t-t_1) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\int_{t_1}^{u}(u-s)^{\alpha-1}e(s)dsdu \\
&\quad + \int_{t_1}^{t}\int_{t_1}^{u}(u-s)^{\alpha-1}\Big(m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s)\Big)\,dsdu \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\int_{t_1}^{u}(u-s)^{\alpha-1}m(s)x(s)dsdu \\
&= a(t_1)x'(t_1) + c_1(t-t_1) + \int_{t_1}^{t}\int_{t_1}^{u}(u-s)^{\alpha-1}\Big(m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s)\Big)\,dsdu \\
&\quad + \frac{1}{\Gamma(\alpha+1)}\int_{t_1}^{t}(t-s)^{\alpha}e(s)ds + \frac{1}{\Gamma(\alpha+1)}\int_{t_1}^{t}(t-s)^{\alpha}m(s)x(s)ds \\
&\le a(t_1)x'(t_1) + c_1(t-t_1) + \int_{t_1}^{t}\int_{t_1}^{u}(u-s)^{\alpha-1}\Big(m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s)\Big)\,dsdu \\
&\quad + \frac{1}{\Gamma(\alpha+1)}\int_{t_1}^{t}(t-s)^{\alpha}e(s)ds + \frac{t}{\Gamma(\alpha+1)}\int_{t_1}^{t}(t-s)^{\alpha-1}m(s)x(s)ds.
\end{aligned}$$

In view of (2.4) and (2.5), the last inequality implies

$$a(t)x'(t) \le c_2 + c_3 t + \frac{t}{\Gamma(\alpha+1)} \int_{t_1}^{t} (t-s)^{\alpha-1} m(s)x(s)ds, \tag{2.9}$$

for some positive constants c_2 and c_3.

Integrating (2.9) from t_1 to t and noting condition (2.2), we see that

$$\begin{aligned} x(t) &\le x(t_1) + c_2 \int_{t_1}^{t} \frac{1}{a(s)} ds + c_3 \int_{t_1}^{t} \frac{s}{a(s)} ds + \frac{1}{\Gamma(\alpha+1)} \int_{t_1}^{t} \frac{u}{a(u)} \int_{t_1}^{u} (u-s)^{\alpha-1} m(s)x(s)dsdu \\ &\le c_4 + \frac{1}{\Gamma(\alpha+1)} \int_{t_1}^{t} \frac{u}{a(u)} \int_{t_1}^{u} (u-s)^{\alpha-1} m(s)x(s)dsdu \end{aligned} \tag{2.10}$$

for some constant $c_4 > 0$.

Applying Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned} &\int_{t_1}^{u} \left((u-s)^{\alpha-1} e^{s}\right)\left(e^{-s} m(s)x(s)\right) ds \\ &\quad \le \left(\int_{t_1}^{u} (u-s)^{p(\alpha-1)} e^{ps} ds\right)^{1/p} \left(\int_{t_1}^{u} e^{-qs} m^q(s) x^q(s) ds\right)^{1/q} \\ &\quad \le \left(\int_{0}^{u} (u-s)^{p(\alpha-1)} e^{ps} ds\right)^{1/p} \left(\int_{t_1}^{u} e^{-qs} m^q(s) x^q(s) ds\right)^{1/q} \\ &\quad \le (Qe^{pu})^{1/p} \left(\int_{t_1}^{u} e^{-qs} m^q(s) x^q(s) ds\right)^{1/q} \\ &\quad = Q^{1/p} e^{u} \left(\int_{t_1}^{u} e^{-qs} m^q(s) x^q(s) ds\right)^{1/q}. \end{aligned} \tag{2.11}$$

From (2.2), (2.10), and (2.11),

$$\begin{aligned} x(t) &\le c_4 + \frac{Q^{1/p}}{\Gamma(\alpha+1)} \int_{t_1}^{t} \frac{ue^{u}}{a(u)} \left(\int_{t_1}^{u} e^{-qs} m^q(s) x^q(s) ds\right)^{1/q} du \\ &\le c_4 + \frac{Q^{1/p} S}{\Gamma(\alpha+1)} \int_{t_1}^{t} e^{-(\sigma-1)u} \left(\int_{t_1}^{u} e^{-qs} m^q(s) x^q(s) ds\right)^{1/q} du. \end{aligned} \tag{2.12}$$

Since $\sigma > 1$ and the integral on the far right in (2.12) is increasing, we obtain the estimate

$$x(t) \le 1 + c_4 + K \left(\int_{t_1}^{t} e^{-qs} m^q(s) x^q(s) ds\right)^{1/q}, \tag{2.13}$$

where $K = \frac{Q^{1/p} S}{(\sigma-1)\Gamma(\alpha+1)}$.

Applying the inequality

$$(x+y)^q \le 2^{q-1}(x^q + y^q) \quad \text{for } x, y \ge 0 \ \text{ and } \ q > 1,$$

to (2.13) gives

$$x^q(t) \le 2^{q-1}(1+c_4)^q + 2^{q-1} K^q \left(\int_{t_1}^{t} e^{-qs} m^q(s) x^q(s) ds\right). \tag{2.14}$$

Setting $P_1 = 2^{q-1}(1+c_4)^q$, $Q_1 = 2^{q-1}K^q$, and $w(t) = x^q(t)$ so that $x(t) = w^{1/q}(t)$, (2.14) becomes

$$w(t) \le P_1 + Q_1 \left(\int_{t_1}^{t} e^{-qs} m^q(s) w(s) ds\right)$$

for $t \geq t_1$. By Gronwall's inequality and condition (2.3), we see that $w(t)$ is bounded, and so $x(t)$ is bounded. Clearly, a similar argument holds if $x(t)$ is an eventually negative solution of (1.1). This completes the proof of the theorem. □

Next, we consider the fractional differential equation

$$^{C}D_{c}^{\alpha}y(t) = e(t) + f(t, x(t)), \qquad c > 1, \quad \alpha \in (0,1), \tag{2.15}$$

where $y(t) = a(t)x'(t)$ and $c_0 = \frac{y(c)}{\Gamma(1)} = y(c)$ is a real constant. We now give sufficient conditions under which any nonoscillatory solution x of equation (2.15) is bounded.

Theorem 2.4. *Let conditions (i)–(iii) hold and suppose that there exist real numbers $p > 1$ and $0 < \alpha < 1$ such that $p(\alpha - 1) + 1 > 0$. In addition, assume that there exists a continuous function $m : [c, \infty) \to (0, \infty)$ such that (2.3) holds and*

$$\left(\frac{1}{a(t)}\right) \leq Se^{-\sigma t} \tag{2.16}$$

for some $S > 0$ and $\sigma > 1$. If

$$\limsup_{t\to\infty} \int_c^t (t-s)^{\alpha-1} e(s)ds < \infty, \qquad \liminf_{t\to\infty} \int_c^t (t-s)^{\alpha-1} e(s)ds > -\infty, \tag{2.17}$$

and

$$\limsup_{t\to\infty} \int_c^t (t-s)^{\alpha-1} \left(m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)\right) ds < \infty, \tag{2.18}$$

then any nonoscillatory solution $x(t)$ of equation (2.15) is bounded.

Proof. Let $x(t)$ be an eventually positive solution of equation (2.15). We may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. Again let $F(t) = f(t, x(t))$. In view of (i)–(iii), equation (2.15) can be written as

$$\begin{aligned} a(t)x'(t) \leq c_0 &+ \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |e(s)|\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} e(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} m(s)x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \left[h(s)x^{\lambda}(s) - m(s)x(s)\right] ds. \end{aligned} \tag{2.19}$$

Proceeding as in the proof of Theorem 2.3, from (2.19) we obtain (see (2.7))

$$\begin{aligned} a(t)x'(t) \leq c_1 &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} e(s)ds \\ &+ \frac{\left((1-\lambda)\lambda^{\lambda/(1-\lambda)}\right)}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \left[m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)\right] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} m(s)x(s)ds, \\ \leq M &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} m(s)x(s)ds, \end{aligned} \tag{2.20}$$

for some positive constant M.

An integration of (2.20) from t_1 to t yields

$$x(t) \leq x(t_1) + M \int_{t_1}^{t} \frac{1}{a(s)} ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} (u-s)^{\alpha-1} m(s) x(s) ds du.$$

The remainder of the proof is similar to that of Theorem 2.3 and hence is omitted. □

Similar reasoning to that used in the sublinear case guarantees the following theorems for the integro-differential equations (1.1) and (2.15) in case $\lambda = 1$.

Theorem 2.5. *Let $\lambda = 1$ and the hypotheses of Theorems 2.3–2.4 hold with $m(t) = h(t)$. Then the conclusion of Theorems 2.3–2.4 holds.*

Example 2.6. Consider the equation

$$y(t) = a(t)x'(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_2^t (t-s)^{\alpha-1} \left[e^{-2s} + h(s) |x(s)|^{\lambda-1} x(s) \right] ds, \tag{2.21}$$

with $0 < \lambda < 1$. Here we have $c = 2$, $e(t) = e^{-2t}$, $f(t, x(t)) = h(t) |x(t)|^{\lambda-1} x(t)$, and we take $a(t) = e^{2t}/S$ with $S > 0$, $h(t) = e^{-t}$, $\alpha = 1/2$, and $p = 3/2 > 1$. Then $q = \frac{p}{p-1} = 3$ and $p(\alpha-1)+1 = 1/4 > 0$. With $\sigma = 2$ and $h(t) = m(t)$, conditions (2.16) and (2.3) become

$$\frac{1}{a(t)} = \frac{1}{\frac{e^{2t}}{S}} = \frac{S}{e^{2t}} \leq Se^{-2t}$$

and

$$\int_c^\infty e^{-qs} m^q(s) ds = \int_2^\infty e^{-3s} e^{-3s} ds \leq \frac{1}{6} < \infty,$$

and so conditions (2.16) and (2.3) hold, respectively. With $h(t) = m(t)$, we have

$$\int_c^t (t-s)^{\alpha-1} \left(m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s) \right) ds = \int_2^t (t-s)^{\alpha-1} m(s) ds = \int_2^t (t-s)^{-1/2} e^{-s} ds. \tag{2.22}$$

Letting $u = t - s + 2$ in (2.22), we obtain

$$\begin{aligned}
\int_2^t (t-s)^{-1/2} e^{-s} ds &= -\int_t^2 (u-2)^{-1/2} e^{u-t-2} du \\
&= \frac{1}{e^{t+2}} \int_2^t (u-2)^{-1/2} e^u du \\
&= \frac{1}{e^{t+2}} \left[\int_2^4 (u-2)^{-1/2} e^u du + \int_4^t (u-2)^{-1/2} e^u du \right] \\
&= \frac{1}{e^{t+2}} \left[\lim_{b \to 2^+} \int_b^4 (u-2)^{-1/2} e^u du \right] + \frac{1}{e^{t+2}} \left[\int_4^t (u-2)^{-1/2} e^u du \right] \\
&= \frac{1}{e^{t+2}} \lim_{b \to 2^+} e^4 \int_b^4 (u-2)^{-1/2} du + \frac{(4-2)^{-1/2}}{e^{t+2}} \int_4^t e^u du \\
&= \frac{2^{3/2} e^4}{e^{t+2}} + \frac{1}{\sqrt{2} e^{t+2}} \left(e^t - e^4 \right),
\end{aligned}$$

so (2.18) holds. Finally,

$$\begin{aligned}\int_c^t (t-s)^{\alpha-1} e(s)ds &= \int_2^t (t-s)^{-1/2} e^{-2s} ds \\ &= -\int_t^2 (u-2)^{-1/2} e^{-2(t-u+2)} du \\ &= \frac{1}{e^{2t+4}} \int_2^t (u-2)^{-1/2} e^{2u} ds \\ &< \infty,\end{aligned}$$

so condition (2.17) is satisfied. Hence, by Theorem 2.4, every nonoscillatory solution x of equation (2.21) is bounded.

Example 2.7. Consider the equation

$$y(t) = \left(a(t)x'(t)\right)' = c_0 + \frac{1}{\Gamma(\alpha)} \int_2^t (t-s)^{\alpha-1} \left[\frac{1}{s^2} + h(s) \left|x(s)\right|^{\lambda-1} x(s)\right] ds, \tag{2.23}$$

with $0 < \lambda < 1$. Here we have $c = 2$, $e(t) = 1/t^2$, $f(t, x(t)) = h(t) \left|x(t)\right|^{\lambda-1} x(t)$, and we take $a(t) = te^{2t}/S$ with $S > 0$, $h(t) = e^{-t}$, $\alpha = 1/2$, and $p = 3/2 > 1$. Then $q = \frac{p}{p-1} = 3$ and $p(\alpha - 1) + 1 = 1/4 > 0$. With $\sigma = 2$ and $h(t) = m(t)$, conditions (2.2) and (2.3) become

$$\frac{t}{a(t)} = \frac{t}{\frac{te^{2t}}{S}} = \frac{S}{e^{2t}} \le Se^{-2t}$$

and

$$\int_c^\infty e^{-qs} m^q(s)ds = \int_2^\infty e^{-3s} e^{-3s} ds \le \frac{1}{6} < \infty,$$

and so conditions (2.2) and (2.3) hold. From Example 2.1, we see that

$$\int_c^t (t-s)^{\alpha-1} \left(m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)\right) ds < 2^{3/2} e^2 + \frac{1}{\sqrt{2}} < \infty,$$

so clearly condition (2.5) holds.

Finally,

$$\begin{aligned}\frac{1}{t} \int_c^t (t-s)^{\alpha} e(s)ds &= \frac{1}{t} \int_2^t (t-s)^{1/2} \frac{1}{s^2} ds \\ &\le \frac{(t-2)^{1/2}}{t} \int_2^t \frac{1}{s^2} ds \\ &\le -\frac{1}{t^{3/2}} + \frac{1}{2t^{1/2}} < \infty,\end{aligned}$$

so condition (2.4) is satisfied. Hence, by Theorem 2.3, every nonoscillatory solution x of equation (2.23) is bounded.

In conclusion, we wish to point out that the results in this paper are presented in a form that can be extended to fractional differential equations of the type (1.1) of order $\alpha \in (n-1, n)$, $n \ge 1$. It would also be of interest to study equation (1.1) in case f satisfies condition (iii) with $\lambda > 1$.

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