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Asymptotic formulas for a scalar linear delay differential equation

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

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Abstract. The linear delay differential equation

$$x'(t) = p(t)x(t-r)$$

is considered, where $r > 0$ and the coefficient $p : [t_0, \infty) \to \mathbb{R}$ is a continuous function such that $p(t) \to 0$ as $t \to \infty$. In a recent paper [M. Pituk, G. Röst, *Bound. Value Probl.* 2014:114] an asymptotic description of the solutions has been given in terms of a special solution of the associated formal adjoint equation and the initial data. In this paper, we give a representation of the special solution of the formal adjoint equation. Under some additional conditions, the representation theorem yields explicit asymptotic formulas for the solutions as $t \to \infty$.

Keywords: delay differential equation, formal adjoint equation, asymptotic formulas.

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1 Introduction

Consider the delay differential equation

$$x'(t) = p(t)x(t-r), \tag{1.1}$$

where $r > 0$ and $p : [t_0, \infty) \to \mathbb{R}$ is a continuous function. The initial value problem associated with (1.1) has the form

$$x(t) = \phi(t), \qquad t_1 - r \le t \le t_1, \tag{1.2}$$

where $t_1 \ge t_0$ and $\phi : [t_1 - r, t_1] \to \mathbb{R}$ is a continuous function. Recently, under the smallness condion

$$\int_t^{t+r} |p(s)|\, ds \to 0 \qquad \text{as } t \to \infty, \tag{1.3}$$

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we have given an asymptotic description of the solution of the initial value problem (1.1) and (1.2) in terms of a special solution of the formal adjoint equation

$$y'(t) = -p(t+r)y(t+r). \tag{1.4}$$

We have shown the following theorem (see Theorems 3.1–3.3 in [10]).

Theorem 1.1. *Suppose* (1.3) *holds. Then up to a constant multiple the adjoint equation* (1.4) *has a unique solution y on $[t_0, \infty)$ which is positive for all large t and satisfies*

$$\limsup_{t\to\infty} \frac{y(t+r)}{y(t)} < \infty. \tag{1.5}$$

Furthermore, if x is the solution of the initial value problem (1.1) *and* (1.2)*, then*

$$x(t) = \frac{1}{y(t)}\big(\, c + o(1)\,\big), \qquad t \to \infty, \tag{1.6}$$

where c is a constant given by

$$c = \phi(t_1)y(t_1) + \int_{t_1-r}^{t_1} p(s+r)\phi(s)y(s+r)\,ds. \tag{1.7}$$

In the sequel, the solution y of the adjoint equation described in Theorem 1.1 will be called a *special solution* of Eq. (1.4).

A close look at the proof of Theorem 3.1 in [10] shows that the special solution of the adjoint equation y has the following additional properties: if $t_1 \ge t_0$ is chosen such that

$$\int_t^{t+r} p_-(s)\,ds < \frac{1}{e}, \qquad t \ge t_1, \tag{1.8}$$

where p_- is the negative part of p defined by $p_-(t) = \max\{\,0, -p(t)\,\}$ for $t \ge t_0$, then

$$y(t) > 0, \qquad t \ge t_1, \tag{1.9}$$

and

$$\frac{y(t+r)}{y(t)} \le e, \qquad t \ge t_1. \tag{1.10}$$

Note that in view of the inequality $0 \le p_- \le |p|$ assumption (1.3) implies that condition (1.8) is satisfied for all sufficiently large t_1.

We emphasize that (1.6) gives a genuine asymptotic representation of the solutions of Eq. (1.1) in the sense that there exists a solution x of (1.1) for which the constant c in (1.6) is nonzero. Indeed, if t_1 is chosen such that (1.8) is satisfied, then for the solution x of (1.1) with initial data (1.2) defined by

$$\phi(t) = \frac{1}{y(t+r)}, \qquad t_1 - r \le t \le t_1,$$

we have (by (1.7)),

$$\begin{aligned} c &= \frac{y(t_1)}{y(t_1+r)} + \int_{t_1-r}^{t_1} p(s+r)\,ds \ge \frac{y(t_1)}{y(t_1+r)} - \int_{t_1-r}^{t_1} p_-(s+r)\,ds \\ &\ge \frac{1}{e} - \int_{t_1-r}^{t_1} p_-(s+r)\,ds = \frac{1}{e} - \int_{t_1}^{t_1+r} p_-(u)\,du > 0, \end{aligned}$$

the second and the last inequality being a consequence of (1.10) and (1.8), respectively.

Our previous study [10] was motivated by the Dickman–de Bruijn equation (see [1,2,5])

$$x'(t) = -\frac{x(t-1)}{t} \tag{1.11}$$

for which the special solution of the associated adjoint equation

$$y'(t) = \frac{y(t+1)}{t+1} \tag{1.12}$$

can be given explicitly by $y(t) = t$ for $t \geq 1$. Thus, in this case (1.6) leads to the explicit asymptotic representation

$$x(t) = \frac{1}{t}\big(c + o(1)\big), \qquad t \to \infty. \tag{1.13}$$

For similar qualitative results, see [3,4,6–8] and the references therein.

In contrast with the Dickman–de Bruijn equation (1.11), in most cases we do not know an explicit formula for the special solution of the adjoint equation (1.4). Therefore the purpose of the present paper is to describe the special solution of the adjoint equation (1.4) in terms of the coefficient p and the delay r. In Section 2, we prove a new representation theorem for the special solution of the adjoint equation (1.4) (see Theorem 2.1 below). In Section 3, in Theorem 3.1, we show that under some additional conditions the representation theorem yields explicit asymptotic formulas for the solutions of the linear delay differential equation (1.1).

2 Representation of the special solution of the adjoint equation

To simplify the calculations instead of (1.3) we will assume the slightly stronger condition

$$p(t) \to 0 \qquad \text{as } t \to \infty. \tag{2.1}$$

This implies that if $t_1 \geq t_0$ is sufficiently large, then

$$q = \sup_{t \geq t_1} |p(t)| < \frac{1}{re}. \tag{2.2}$$

Clearly, condition (2.2) implies (1.8). Therefore, under condition (2.2), the special solution y of the adjoint equation has properties (1.9) and (1.10).

In order to formulate our main representation theorem, we need to introduce some auxiliary functions. Define

$$\alpha_1(t,s) = -p(s+r) \qquad \text{for } s \geq t \geq t_0, \tag{2.3}$$

and

$$\alpha_{k+1}(t,s) = -p(s+r)\int_t^{s+r} \alpha_k(t,u)\,du \qquad \text{for } s \geq t \geq t_0 \tag{2.4}$$

for $k = 1,2,3,\ldots$

Theorem 2.1. *Suppose that (2.1) holds. If $t_1 \geq t_0$ is chosen such that (2.2) is satisfied, then the unique special solution y of the adjoint equation (1.4) with property $y(t_1) = 1$ is given by*

$$y(t) = \exp\left(\int_{t_1}^{t} \sigma(s)\,ds\right), \qquad t \geq t_1, \tag{2.5}$$

where $\sigma : [t_1, \infty) \to \mathbb{R}$ *is defined by*

$$\sigma(t) = \sum_{k=1}^{\infty} \alpha_k(t,t), \qquad t \geq t_1, \tag{2.6}$$

the function series on the righ-hand side being uniformly convergent on $[t_1, \infty)$.

Before we give a proof of Theorem 2.1, we establish some auxiliary results. Suppose (2.1) and (2.2) hold. As noted above, if y is a special solution of Eq. (1.4), then conditions (1.9) and (1.10) are satisfied. Define

$$\beta_1(t,s) = -p(s+r)\frac{y(s+r)}{y(t)} \qquad \text{for } s \geq t \geq t_1, \tag{2.7}$$

and

$$\beta_{k+1}(t,s) = -p(s+r)\int_t^{s+r} \beta_k(t,u)\,du \qquad \text{for } s \geq t \geq t_1 \tag{2.8}$$

and $k = 1, 2, 3, \ldots$

In the following lemmas, we prove some useful identities involving the functions $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=1}^{\infty}$ defined by (2.3), (2.4), (2.7) and (2.8), respectively.

Lemma 2.2. *Supppose* (2.1) *and* (2.2) *hold. If* y *is a special solution of Eq.* (1.4)*, then for every positive integer* k*,*

$$\alpha_k(t,s) + \beta_{k+1}(t,s) = \beta_k(t,s) \qquad \textit{whenever } s \geq t \geq t_1. \tag{2.9}$$

Proof. We will prove (2.9) by induction on k. We have for $s \geq t \geq t_1$,

$$\begin{aligned}
\alpha_1(t,s) + \beta_2(t,s) &= -p(s+r) - p(s+r)\int_t^{s+r} \beta_1(t,u)\,du \\
&= -p(s+r) + p(s+r)\int_t^{s+r} p(u+r)\frac{y(u+r)}{y(t)}\,du \\
&= -p(s+r) + \frac{p(s+r)}{y(t)}\int_t^{s+r} p(u+r)y(u+r)\,du \\
&= -p(s+r) - \frac{p(s+r)}{y(t)}\int_t^{s+r} y'(u)\,du \\
&= -p(s+r) - \frac{p(s+r)}{y(t)}(y(s+r) - y(t)) = \beta_1(t,s).
\end{aligned}$$

Thus, (2.9) holds for $k = 1$. Now assume that (2.9) holds for some positive integer k. Then

$$\begin{aligned}
\alpha_{k+1}(t,s) + \beta_{k+2}(t,s) &= -p(s+r)\int_t^{s+r} \alpha_k(t,u)\,du - p(s+r)\int_t^{s+r} \beta_{k+1}(t,u)\,du \\
&= -p(s+r)\int_t^{s+r} [\alpha_k(t,u) + \beta_{k+1}(t,u)]\,du \\
&= -p(s+r)\int_t^{s+r} \beta_k(t,u)\,du = \beta_{k+1}(t,s)
\end{aligned}$$

for $s \geq t \geq t_1$. This proves that (2.9) holds for all $k = 1, 2, 3, \ldots$ □

Lemma 2.3. *Supppose* (2.1) *and* (2.2) *hold. If* y *is a special solution of Eq.* (1.4)*, then for every positive integer* n*, we have*

$$y'(t) = \left(\sum_{k=1}^{n} \alpha_k(t,t) + \beta_{n+1}(t,t)\right) y(t), \qquad t \geq t_1. \tag{2.10}$$

Proof. We will prove (2.10) by induction on n. We have for $t \geq t_1$,

$$\begin{aligned}
y'(t) &= -p(t+r)y(t+r) = -p(t+r)y(t) - p(t+r)\big(y(t+r) - y(t)\big) \\
&= -p(t+r)y(t) - p(t+r)\int_t^{t+r} y'(u)\,du \\
&= -p(t+r)y(t) + p(t+r)\int_t^{t+r} p(u+r)y(u+r)\,du \\
&= \left(-p(t+r) + p(t+r)\int_t^{t+r} p(u+r)\frac{y(u+r)}{y(t)}\,du\right)y(t) \\
&= \left(-p(t+r) - p(t+r)\int_t^{t+r} \beta_1(t,u)\,du\right)y(t) \\
&= \big(\alpha_1(t,t) + \beta_2(t,t)\big)y(t).
\end{aligned}$$

Thus, (2.10) holds for $n = 1$. Now suppose that (2.10) holds for some positive integer n. Then for $t \geq t_1$,

$$\begin{aligned}
y'(t) &= \left(\sum_{k=1}^{n} \alpha_k(t,t) + \beta_{n+1}(t,t)\right)y(t) \\
&= \left(\sum_{k=1}^{n+1} \alpha_k(t,t) + \beta_{n+2}(t,t) - \big(\alpha_{n+1}(t,t) + \beta_{n+2}(t,t) - \beta_{n+1}(t,t)\big)\right)y(t) \\
&= \left(\sum_{k=1}^{n+1} \alpha_k(t,t) + \beta_{n+2}(t,t)\right)y(t),
\end{aligned}$$

the last equality being a consequence of conclusion (2.9) of Lemma 2.2. This proves that (2.10) holds for all n. □

Now we are in a position to give a proof of Theorem 2.1.

Proof. We will show that the series (2.6) converges uniformly on $[t_1, \infty)$. First we prove by induction that for every positive integer k,

$$|\alpha_k(t,s)| \leq \frac{q^k}{(k-1)!}(s - t + (k-1)r)^{k-1} \qquad \text{whenever } s \geq t \geq t_1, \tag{2.11}$$

where q is defined by (2.2). By virtue of (2.2) and (2.3), we have

$$|\alpha_1(t,s)| = |p(s+r)| \leq q \qquad \text{whenever } s \geq t \geq t_1.$$

Thus, (2.11) holds for $k = 1$. Now suppose that (2.11) holds for some positive integer k. Then for $s \geq t \geq t_1$,

$$\begin{aligned}
|\alpha_{k+1}(t,s)| &= |p(s+r)|\left|\int_t^{s+r} \alpha_k(t,u)\,du\right| \leq q\int_t^{s+r} |\alpha_k(t,u)|\,du \\
&\leq q\int_t^{s+r} \frac{q^k}{(k-1)!}(u - t + (k-1)r)^{k-1}\,du \\
&= q^{k+1}\left[\frac{(u - t + (k-1)r)^k}{k!}\right]_t^{s+r} \leq \frac{q^{k+1}}{k!}(s - t + kr)^k.
\end{aligned}$$

This proves that (2.11) holds for all k. From (2.11), we find that for every positive integer k,

$$|\alpha_k(t,t)| \le q(qr)^{k-1}\frac{(k-1)^{k-1}}{(k-1)!}, \qquad t \ge t_1.$$

From this and the inequality

$$\frac{(k-1)^{k-1}}{(k-1)!} \le \sum_{j=0}^{\infty}\frac{(k-1)^j}{j!} = e^{k-1}, \tag{2.12}$$

we obtain for every positive integer k,

$$|\alpha_k(t,t)| \le q(qre)^{k-1}, \qquad t \ge t_1.$$

Since $qre < 1$, this implies the uniform convergence of the function series (2.6) on $[t_1,\infty)$.

Next we show that $\beta_n(t,t) \to 0$ uniformly on $[t_1,\infty)$ as $n \to \infty$. It is easy to show that if $qre < 1$, then the equation

$$\lambda = qe^{\lambda r}$$

has a unique root λ_0 in $(0, qe)$. Moreover, for every $\lambda \in (\lambda_0, 1/r)$, we have

$$\frac{qe^{\lambda r}}{\lambda} < 1. \tag{2.13}$$

Choose

$$\lambda \in (qe, 1/r) \tag{2.14}$$

so that (2.13) holds. We will show by induction that for every positive integer k,

$$|\beta_k(t,s)| \le \frac{q^k}{\lambda^{k-1}}\exp(\lambda(s-t+kr)) \qquad \text{whenever } s \ge t \ge t_1. \tag{2.15}$$

First observe that by virtue of (1.9), (1.10) and (2.2), we have for $t \ge t_1$,

$$y'(t) = -p(t+r)\frac{y(t+r)}{y(t)}y(t) \le qey(t).$$

Hence

$$\frac{y'(t)}{y(t)} \le qe, \qquad t \ge t_1.$$

Integrating the last inequality from t to $s+r$, we find for $s \ge t \ge t_1$,

$$\ln\frac{y(s+r)}{y(t)} \le qe(s+r-t).$$

Hence

$$\frac{y(s+r)}{y(t)} \le \exp(qe(s+r-t)) \le \exp(\lambda(s+r-t)) \qquad \text{whenever } s \ge t \ge t_1,$$

where the last inequality is a consequence of (2.14). From this, (2.2) and (2.7), we find for $s \ge t \ge t_1$,

$$|\beta_1(t,s)| \le q\exp(\lambda(s+r-t)).$$

Thus, (2.15) holds for $k = 1$. Now suppose that (2.15) holds for some positive integer k. Then for $s \geq t \geq t_1$,

$$\begin{aligned}|\beta_{k+1}(t,s)| &= |p(s+r)| \left| \int_t^{s+r} \beta_k(t,u)\,du \right| \leq q \int_t^{s+r} |\beta_k(t,u)|\,du \\ &\leq q \int_t^{s+r} \frac{q^k}{\lambda^{k-1}} \exp(\lambda(u-t+kr))\,du = q^{k+1} \left[\frac{\exp(\lambda(u-t+kr))}{\lambda^k} \right]_t^{s+r} \\ &\leq \frac{q^{k+1}}{\lambda^k} \exp(\lambda(s-t+(k+1)r)).\end{aligned}$$

This proves that (2.15) holds for all k. From (2.15), we obtain for every positive integer n,

$$|\beta_n(t,t)| \leq \lambda \left(\frac{qe^{\lambda r}}{\lambda} \right)^n, \qquad t \geq t_1.$$

In view of (2.13), the last inequality implies that $\beta_n(t,t) \to 0$ uniformly on $[t_1, \infty)$ as $n \to \infty$. Finally letting $n \to \infty$ in conclusion (2.10) of Lemma 2.3, we obtain that the special solution y of Eq. (1.4) satisfies the ordinary differential equation

$$y'(t) = \sigma(t) y(t), \qquad t \geq t_1,$$

where σ is defined by (2.6). Since $y(t_1) = 1$, this implies that y has the form (2.5) and the proof of Theorem 2.1 is complete. □

3 Explicit asymptotic formulas

From Theorems 1.1 and 2.1, we can deduce explicit asymptotic formulas for the solutions of Eq. (1.1).

Theorem 3.1. *Suppose that there exists a positive monotone decreasing function $a : [t_0, \infty) \to (0, \infty)$ such that*

$$|p(t)| \leq a(t), \qquad t \geq t_0, \tag{3.1}$$

and

$$\int_{t_0}^{\infty} a^{n+1}(t)\,dt < \infty \qquad \text{for some positive integer } n. \tag{3.2}$$

Then for every solution x of Eq. (1.1) there exists a constant γ such that

$$x(t) = \exp\left(- \int_{t_0}^{t} \sigma_n(s)\,ds \right) (\gamma + o(1)), \qquad t \to \infty, \tag{3.3}$$

where σ_n is the n^{th} partial sum of the function series (2.6),

$$\sigma_n(t) = \sum_{k=1}^{n} \alpha_k(t,t), \qquad t \geq t_0. \tag{3.4}$$

Moreover, the asymptotic formula (3.3) is genuine in the sense that there exists a solution x of (1.1) for which the constant γ in (3.3) is nonzero.

Proof. First we prove by induction that under the hypotheses of the theorem for all positive integer k,

$$|\alpha_k(t,s)| \le a^k(t+r)\frac{(s-t+(k-1)r)^{k-1}}{(k-1)!} \qquad \text{whenever } s \ge t \ge t_0. \tag{3.5}$$

By virtue of (2.3) and (3.1), we have for $s \ge t \ge t_0$,

$$|\alpha_1(t,s)| = |p(s+r)| \le a(s+r) \le a(t+r),$$

where the last inequality is a consequence of the monotonicity of a. Thus, (3.5) holds for $k = 1$. Now suppose that (3.5) holds for some positive integer k. Then we have for $s \ge t \ge t_1$,

$$\begin{aligned}
|\alpha_{k+1}(t,s)| = |p(s+r)|\left|\int_t^{s+r}\alpha_k(t,u)\,du\right| &\le a(s+r)\int_t^{s+r}|\alpha_k(t,u)|\,du \\
&\le a(s+r)\int_t^{s+r}a^k(t+r)\frac{(u-t+(k-1)r)^{k-1}}{(k-1)!}\,du \\
&= a(s+r)a^k(t+r)\left[\frac{(u-t+(k-1)r)^k}{k!}\right]_t^{s+r} \\
&\le a(s+r)a^k(t+r)\frac{(s-t+kr)^k}{k!} \le a^{k+1}(t+r)\frac{(s-t+kr)^k}{k!},
\end{aligned}$$

the last inequality being a consequence of the monotonicity of a. This proves that (3.5) holds for all k.

From (3.5), we find that for all positive k,

$$|\alpha_k(t,t)| \le a^k(t+r)\frac{((k-1)r)^{k-1}}{(k-1)!}, \qquad t \ge t_0.$$

From this, using inequality (2.12) and taking into account that a is monotone decreasing, we obtain for all k,

$$|\alpha_k(t,t)| \le a^k(t)(re)^{k-1}, \qquad t \ge t_0. \tag{3.6}$$

Choose $q > 0$ such that $qre < 1$. Since a is monotone decreasing and (3.2) holds, it follows that $a(t) \to 0$ as $t \to \infty$. Therefore there exists $t_1 \ge t_0$ such that

$$\sup_{t\ge t_1}|p(t)| \le \sup_{t\ge t_1}a(t) \le q < \frac{1}{re}. \tag{3.7}$$

By the application of Theorem 2.1, we conclude that the special solution y of the adjoint equation (1.4) with property $y(t_1) = 1$ has the form (2.5). This, combined with Theorem 1.1, implies that every solution x of Eq. (1.1) satisfies the asymptotic relation

$$x(t) = \exp\left(-\int_{t_1}^t \sigma(s)\,ds\right)(c+o(1)), \qquad t \to \infty, \tag{3.8}$$

where c is a constant depending on x. Moreover, as shown in Section 1, there exists a solution x of Eq. (1.1) for which $c > 0$. For $t \ge t_1$, define

$$\rho_n(t) = \sigma(t) - \sigma_n(t) = \sum_{k=n+1}^{\infty}\alpha_k(t,t). \tag{3.9}$$

We will show that

$$\rho_n(t) = O(a^{n+1}(t)) \qquad \text{as } t \to \infty. \tag{3.10}$$

From (3.6) and (3.7), we obtain for $t \geq t_1$,

$$\begin{aligned} |\rho_n(t)| &\leq \sum_{k=n+1}^{\infty} |\alpha_k(t,t)| \leq \sum_{k=n+1}^{\infty} a^k(t)(re)^{k-1} \\ &= a^{n+1}(t) \sum_{k=n+1}^{\infty} a^{k-n-1}(t)(re)^{k-1} \leq a^{n+1}(t) \sum_{k=n+1}^{\infty} q^{k-n-1}(re)^{k-1} \\ &= a^{n+1}(t)\, q^{-n} \sum_{k=n+1}^{\infty} (qre)^{k-1} = a^{n+1}(t) \frac{(re)^n}{1-qre} \end{aligned}$$

which proves (3.10). From conditions (3.2) and (3.10), it follows that the improper Riemann integral $\int_{t_1}^{\infty} \rho_n(t)\, dt$ converges. Since $\sigma_n = \sigma - \rho_n$, we have for $t \geq t_1$,

$$x(t) \exp\left(\int_{t_1}^{t} \sigma_n(s)\, ds\right) = x(t) \exp\left(\int_{t_1}^{t} \sigma(s)\, ds\right) \exp\left(-\int_{t_1}^{t} \rho_n(s)\, ds\right). \tag{3.11}$$

From this and the asymptotic representation (3.8), we obtain

$$x(t) \exp\left(\int_{t_1}^{t} \sigma_n(s)\, ds\right) \longrightarrow d = c \exp\left(-\int_{t_1}^{\infty} \rho_n(s)\, ds\right)$$

as $t \to \infty$. Thus, (3.3) holds with

$$\gamma = d \exp\left(\int_{t_0}^{t_1} \sigma_n(s)\, ds\right).$$

Clearly, if c is nonzero, then so is d and hence γ. This completes the proof of the theorem. $\square$

Remark 3.2. To illustrate the importance of hypothesis (3.2) in Theorem 3.1 condsider Eq. (1.1), where $p : [t_0, \infty) \to (-\infty, 0)$ is a negative monotone increasing function which tends to zero as $t \to \infty$. Clearly, in this case condition (3.1) holds with $a = |p|$. Suppose that condition (3.2) does not hold, that is

$$\int_{t_0}^{\infty} |p(t)|^{n+1}\, dt = \infty \qquad \text{for every positive integer } n. \tag{3.12}$$

(An example of such a p is the function $p(t) = -\ln^{-1} t$ defined for $t \geq 2$.) We will show that if σ_n has the meaning from Theorem 3.1, then for every positive integer n,

$$x(t) \exp\left(\int_{t_0}^{t} \sigma_n(s)\, ds\right) \longrightarrow 0 \qquad \text{as } t \to \infty. \tag{3.13}$$

Thus, in this case the constant γ in the asymptotic relation (3.3) is always zero. Therefore if hypothesis (3.2) is not satisfied, then (3.3) in general does not give a genuine asymptotic description of the solutions as $t \to \infty$.

Now we prove (3.13). Using the facts that p is negative and $|p|$ is monotone decreasing, it follows by easy induction that for all positive k,

$$\alpha_k(t,s) \geq r^{k-1} |p(s+kr)|^k > 0 \qquad \text{whenever } s \geq t \geq t_0. \tag{3.14}$$

As noted in the proof of Theorem 3.1, if $t_1 \geq t_0$ is chosen such that (3.7) is satisfied, then for every solution x of (1.1) the asymptotic formula (3.8) holds. If ρ_n is defined by (3.9), then by virtue of (3.14), we have for $t \geq t_1$,

$$\rho_n(t) \geq \alpha_{n+1}(t,t) \geq r^n |p(t+(n+1)r)|^{n+1}.$$

From this and (3.12), we find that

$$\int_{t_1}^{\infty} \rho_n(t)\,dt = \infty \qquad \text{for all } n.$$

This, together with (3.8) and (3.11), implies (3.13).

Example 3.3. As an application of Theorem 3.1, we will describe the asymptotic behavior of the solutions of the equation

$$x'(t) = -\frac{x(t-r)}{\sqrt{t}}, \tag{3.15}$$

which is a special case of Eq. (1.1) when

$$p(t) = -\frac{1}{\sqrt{t}}, \qquad t \geq 1.$$

In contrast with the Dickman–de Bruijn equation (1.11) in this case we do not know an explicit formula for the special solution of the associated formal adjoint equation

$$y'(t) = \frac{y(t+r)}{\sqrt{t+r}}. \tag{3.16}$$

Therefore conclusion (1.6) of Theorem 1.1 does not give an explicit asymptotic description of the solutions of Eq. (3.15). We will determine the asymptotic behavior of the solutions of Eq. (3.15) by applying Theorem 3.1 with

$$a(t) = \frac{1}{\sqrt{t}}, \qquad t \geq 1$$

and $n = 2$. By simple calculations, we obtain for $t \geq 1$,

$$\alpha_1(t,t) = \frac{1}{\sqrt{t+r}} \tag{3.17}$$

and

$$\alpha_2(t,t) = 2\left(\sqrt{\frac{t+2r}{t+r}} - 1\right) = 2\left(\sqrt{1 + \frac{r}{t+r}} - 1\right). \tag{3.18}$$

From (3.17), we find for $t \geq 1$,

$$\exp\left(\int_1^t \alpha_1(s,s)\,ds\right) = \exp(2\sqrt{t+r})\exp(-2\sqrt{1+r}). \tag{3.19}$$

By Taylor's theorem, we have

$$2(\sqrt{1+x} - 1) = x + O(x^2) \qquad \text{as } x \to 0.$$

This, combined with (3.18), yields

$$\alpha_2(t,t) = \frac{r}{t+r} + \psi(t), \tag{3.20}$$

where

$$\psi(t) = O\left(\frac{1}{t^2}\right) \qquad \text{as } t \to \infty.$$

In particular, the improper Riemann integral $\int_1^\infty \psi(t)\,dt$ converges. From this and (3.20), we find that

$$(t+r)^{-r}\exp\left(\int_1^t \alpha_2(s,s)\,ds\right) = (1+r)^{-r}\exp\left(\int_1^t \psi(s)\,ds\right) \longrightarrow \varkappa \qquad \text{as } t\to\infty,$$

where

$$\varkappa = (1+r)^{-r}\exp\left(\int_1^\infty \psi(s)\,ds\right).$$

From the last limit relation and (3.19), by the application of Theorem 3.1, we conclude that every solution x of Eq. (3.15) has the form

$$x(t) = \frac{1}{t^r\exp(2\sqrt{t})}\big(\delta + o(1)\big) \qquad \text{as } t\to\infty, \tag{3.21}$$

where δ is a constant depending on x. Moreover, there exists a solution x of Eq. (3.15) for which $\delta \neq 0$. Thus, every solution of Eq. (3.15) converges to zero as $t\to\infty$ and formula (3.21) describes how the rate of convergence depends on the size of the delay r.

Example 3.4. In [8] we have considered the equation

$$x'(t) = \frac{\sin t}{\sqrt{t}}x(t-r), \qquad t\geq 1, \tag{3.22}$$

where $r>0$. We have shown that the solutions of Eq. (3.22) can be asymptotically stable, stable or unstable depending on r (see Corollary 3.2 in [8]). As a refinement of the results presented in [8], we will show that Theorem 3.1 enables us to determine the precise asymptotics of the solutions of Eq. (3.22). Note that Eq. (3.22) is a special case of (1.1) when

$$p(t) = \frac{\sin t}{\sqrt{t}}, \qquad t\geq 1.$$

The hypotheses of Theorem 3.1 are satisfied with

$$a(t) = \frac{1}{\sqrt{t}}, \qquad t\geq 1$$

and $n=2$. By simple calculations, we obtain for $t\geq 1$,

$$\alpha_1(t,t) = -\frac{\sin(t+r)}{\sqrt{t+r}}$$

and

$$\alpha_2(t,t) = \frac{\sin(t+r)}{\sqrt{t+r}}\int_t^{t+r}\frac{\sin(u+r)}{\sqrt{u+r}}\,du.$$

By the Dirichlet convergence test for improper integrals, the improper integral

$$\int_1^\infty \frac{\sin(s+r)}{\sqrt{s+r}}\,ds$$

converges. Hence

$$\exp\left(\int_1^t \alpha_1(s,s)\,ds\right) \longrightarrow \varkappa_1 = \exp\left(-\int_1^\infty \frac{\sin(s+r)}{\sqrt{s+r}}\,ds\right), \qquad t\to\infty. \tag{3.23}$$

Further, we have for $t \geq 1$,

$$\alpha_2(t,t) = f(t) + g(t), \tag{3.24}$$

where

$$f(t) = \frac{\sin(t+r)}{\sqrt{t+r}} \int_t^{t+r} \sin(u+r)\left(\frac{1}{\sqrt{u+r}} - \frac{1}{\sqrt{t+r}}\right) du$$

and

$$g(t) = \frac{\sin(t+r)}{t+r} \int_t^{t+r} \sin(u+r)\, du = \frac{\sin(t+r)}{t+r}\big(\cos(t+r) - \cos(t+2r)\big).$$

Clearly,

$$\begin{aligned}|f(t)| &\leq \frac{1}{\sqrt{t+r}} \int_t^{t+r} \left(\frac{1}{\sqrt{t+r}} - \frac{1}{\sqrt{u+r}}\right) du \leq \frac{r}{\sqrt{t+r}}\left(\frac{1}{\sqrt{t+r}} - \frac{1}{\sqrt{t+2r}}\right) \\ &= \frac{r}{\sqrt{t+r}} \frac{\sqrt{t+2r} - \sqrt{t+r}}{\sqrt{t+r}\sqrt{t+2r}} = \frac{r}{(t+r)\sqrt{t+2r}} \frac{r}{\sqrt{t+2r} + \sqrt{t+r}} \leq \frac{r^2}{2t^2}\end{aligned}$$

for $t \geq 1$. Therefore the improper integral $\int_1^\infty f(s)\, ds$ converges and

$$\exp\left(\int_1^t f(s)\, ds\right) \longrightarrow \varkappa_2 = \exp\left(\int_1^\infty f(s)\, ds\right), \qquad t \to \infty. \tag{3.25}$$

Using the trigonometric rules

$$\begin{aligned}\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}, \\ \sin\alpha\sin\beta &= \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)), \\ 2\sin\alpha\cos\alpha &= \sin 2\alpha,\end{aligned}$$

we obtain for $t \geq 1$,

$$\begin{aligned}g(t) &= 2\sin\frac{r}{2}\frac{1}{t+r}\sin(t+r)\sin(t+\frac{3r}{2}) = \sin\frac{r}{2}\frac{1}{t+r}(\cos\frac{r}{2} - \cos(2t+\frac{5r}{2})) \\ &= \frac{\sin r}{2}\frac{1}{t+r} - \sin\frac{r}{2}\frac{\cos\big(2t+\frac{5r}{2}\big)}{t+r}.\end{aligned}$$

From this, we find for $t \geq 1$,

$$\int_1^t g(s)\, ds = \frac{\sin r}{2}\ln\frac{t+r}{1+r} - \sin\frac{r}{2}\int_1^t \frac{\cos\big(2s+\frac{5r}{2}\big)}{s+r}\, ds$$

and hence

$$\exp\left(\int_1^t g(s)\, ds\right) = \left(\frac{t+r}{1+r}\right)^{\frac{\sin r}{2}} \exp\left(-\sin\frac{r}{2}\int_1^t \frac{\cos\big(2s+\frac{5r}{2}\big)}{s+r}\, ds\right).$$

Taking into account that according to the Dirichlet convergence test the improper integral

$$\int_1^\infty \frac{\cos\big(2s+\frac{5r}{2}\big)}{s+r}\, ds$$

converges, it follows that

$$(t+r)^{-\frac{\sin r}{2}} \exp\left(\int_1^t g(s)\,ds\right) \longrightarrow \varkappa_3, \qquad t \to \infty, \tag{3.26}$$

where

$$\varkappa_3 = (1+r)^{\frac{-\sin r}{2}} \exp\left(-\sin\frac{r}{2}\int_1^\infty \frac{\cos\left(2s+\frac{5r}{2}\right)}{s+r}\,ds\right).$$

From (3.24), (3.25) and (3.26), we find that

$$(t+r)^{-\frac{\sin r}{2}} \exp\left(\int_1^t \alpha_2(s,s)\,ds\right) \longrightarrow \varkappa, \qquad t \to \infty,$$

where $\varkappa = \varkappa_2\varkappa_3 > 0$. From this and (3.23), by the application of Theorem 3.1, we conclude that every solution x of Eq. (3.22) has the form

$$x(t) = t^{-\frac{\sin r}{2}}\big(\eta + o(1)\big) \qquad \text{as } t \to \infty, \tag{3.27}$$

where η is a constant depending on x. Moreover, there exists a solution x of Eq. (3.22) for which $\eta \neq 0$.

Note that the asymptotic representation (3.27) implies the following interesting stability criteria for Eq. (3.22) (see Corollary 3.2 in [8]).

(i) The zero solution of Eq. (3.22) is asymptotically stable if and only if

$$r \in \bigcup_{k\in\mathbb{Z}^+} (2k\pi, (2k+1)\pi),$$

where $\mathbb{Z}^+$ denotes the set of nonnegative integers.

(ii) The zero solution of Eq. (3.22) is stable if and only if

$$r \in \bigcup_{k\in\mathbb{Z}^+} [2k\pi, (2k+1)\pi].$$

(iii) The zero solution of Eq. (3.22) is stable, but it is not asymptotically stable if and only if

$$r = k\pi \qquad \text{for some } k \in \mathbb{Z}^+.$$

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