



A computer assisted proof of multiple periodic orbits in some first order non-linear delay differential equation

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

Robert Szczelina 

Malopolska Center of Biotechnology, Jagiellonian University,
Gronostajowa street 7A, Krakow, 30-387, Poland

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Abstract. We present an application of a recently developed algorithm for rigorous integration forward in time of delay differential equations (DDEs) to a computer assisted proof of the existence of several periodic orbits in a DDE obtained by a singular perturbation limit method from the classical logistic map. The proofs are done near the parameter value for which multistability was numerically observed.

Keywords: delay differential equations, Poincaré map, topological methods, fixed point theorem, rigorous numerics.

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1 Introduction

In this work we are dealing with the delay differential equation (DDE) of the form:


$$\dot{x}(t) = -\mu \cdot x(t) + \lambda \cdot x(t-1) \cdot (1 + x(t-1)), \quad x \in \mathbb{R}. \quad (1.1)$$

Equation (1.1) is a special form of the delayed feedback equation:

$$\dot{x}(t) = -\mu \cdot x(t) + f(x(t-\tau)), \quad x \in \mathbb{R}, \quad (1.2)$$

which is extensively studied in the literature under various assumptions on the feedback function f , both analytically and numerically.

Numerical studies show that complex dynamics is common in nonlinear systems of the form (1.2). A famous example was proposed by Mackey and Glass, for which there is an evidence of a series of period doubling bifurcations that apparently lead to a chaotic attractor [17]. Losson, Mackey and Longtin numerically studied multistability for Equation (1.1),

 Email: robert.szczelina@uj.edu.pl

which was named there as so-called “logistic” DDE, as it is obtained by a singular perturbation limit procedure on the famous logistic map [16]. The authors of [16] found that the basins of attraction for different stable periodic orbits may have a fractal structure.

Analytic investigation of dynamics in Equation (1.2) usually proves extremely difficult for arbitrary feedback function f , so many analytical results, despite being very impressive, are proved only for f close to a piecewise constant function. For such systems we can find several important results. Mallet-Paret and Sell proposed to use discrete Lyapunov functionals to prove a Poincaré–Bendixson type theorem in the case of monotone systems close to a step function [18]. The conditions for existence and the shape of a global attractor were studied by Krisztin, Walther and Wu for systems with positive monotone feedback i.e. for monotone functions such that $x \cdot f(x) > 0$ for $x \neq 0$, see [8] and references therein. Under the same assumptions, Krisztin and Vas constructed equations for which large-amplitude, slowly oscillatory periodic solutions (LSOPs) exist, such that they revolve around more than one equilibrium [9]. A global picture of the attractor was also obtained. Vas showed that a positive monotone feedback function f may be chosen such that the global attractor contains any number of unstable LSOPs. [27]. Earlier, Vas also showed that systems with negative monotone feedback (i.e. $x \cdot f(x) < 0$ for $x \neq 0$) can exhibit multistability and that the number of stable periodic orbits can be infinite [26]. Lani-Wayda and Walther were able to construct systems of the form $\dot{x} = f(x(t-1))$ for which they proved the existence of dynamics conjugated to a symbol shift (Smale’s horseshoe) [10]. Srzednicki and Lani-Wayda proved the existence of multiple periodic orbits and the existence of chaos for some tooth-shaped (piecewise affine) periodic function f by the use of the generalized Lefschetz fixed point theorem [11]. There are some results for equations with unimodal feedback function (i.e. smooth f that has a unique extremum), such as in Equation (1.1) see [14, 15] and references therein.

Most of the mentioned works concentrate on the construction of special kinds of feedback functions for which it is possible to analytically compute the action of a semiflow associated with (1.2). Then, the system may be studied by an application of arguments from the geometric theory of dynamical systems, namely, by construction of suitable Poincaré maps. A possible approach to treat arbitrary equations such as (1.1) is to use rigorous (validated) numerical methods. In recent years, there were many computer assisted proofs of various properties of dynamical systems ranging from ordinary differential equations to (dissipative) partial differential equations, see for example [2, 19, 21, 25, 31] and references therein. A computer assisted proof is a computer program which rigorously checks assumptions of abstract theorems. Such results, in the context of continuous dynamical systems, usually require some kind of an algorithm for rigorous integration of the flow forward in time.

In this paper we present an application of the recently developed [22, 23] forward in time integration algorithm for rigorous solving of Initial Value Problems (IVPs) of the form:

$$\dot{x} = f(x(t-1), x(t)), \quad t \geq 0, \quad (1.3)$$

$$x(t) = \psi(t), \quad t \in [-1, 0]. \quad (1.4)$$

By the rigorous integration we understand a computer procedure which, given some *a priori* bounds on ψ , produces bounds for the solution $x(t)$ to (1.3) with $x|_{[-1,0]} \equiv \psi$ (we assume the r.h.s. f to be such that this solution is unique). Using the integrator, we rigorously construct images of sets by Poincaré maps, then we show, under some assumptions, that those maps are compact. Finally, we apply Schauder fixed point theorem to obtain existence of periodic orbits. We do not prove that the orbits are attracting, as this would require some C^1 -computations (see [30]) which are for now beyond the scope of our methods.

There are several papers that deal with computer assisted proofs of periodic solutions to DDEs [12, 13, 28], however their approach is very different from our method. These works transform the question of the existence of periodic orbits into a boundary value problem (BVP). The BVP is then solved by using either Newton–Kantorovich theorem [12, 13] or the local Brouwer degree [28]. It is clear that rigorous integration in the phase space may be used to obtain a more diverse spectrum of results such as connecting orbits and the existence of chaos.

Recently Bánhelyi *et al.* made progress towards proving the long-standing Wright’s conjecture, by combining analytical and computer-assisted arguments [3]. In their paper, the global attractivity of the stationary solution to Wright’s equation was shown to be equivalent to the non-existence of slowly oscillating periodic (SOP) solutions with amplitudes greater than a certain constant $\varepsilon > 0$. Next, a procedure to find a set of bounding piecewise constant functions for any SOP solution to the equation was presented. Finally, the bounding functions were used to show that the SOP solution, if it exists, has an amplitude less than ε . The idea of obtaining bounds for a solution in a form suitable for rigorous computations is similar in spirit to our methods, however we use more general representation of the phase space. Moreover, the authors of [3] use the validated numerical integration to obtain the bounds of the solution only over a fixed interval and it is not clear if their method may be adopted to construct Poincaré maps.

This paper is organized as follows. In Section 2, we summarize the algorithm for rigorous construction of Poincaré maps for DDEs. In Section 3, we apply our method to prove the existence of several attracting periodic orbits in (1.1) for various values of parameter λ near 5.81, for which multistability occurs. In Section 4, we apply our method to find numerical approximations of apparently unstable orbits in the system by direct application of Newton algorithm to a nonrigorous version of our integrator. Finally in Section 5, we discuss possible future developments of the rigorous integration methods.

1.1 Notation

In this work we use the following notation. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, by $f^{(k)}$ we denote the k -th derivative of f . By $f^{[k]}$ we denote the term $\frac{1}{k!} \cdot f^{(k)}$. In the context of piecewise smooth maps by $f^{(k)}(t^-)$ and $f^{(k)}(t^+)$ we denote left and right one sided derivatives of f w.r.t. t , respectively.

For a given set A by $\text{cl}(A)$ and $\text{int } A$ we denote the closure and interior of A , respectively (in a given topology e.g. defined by the norm in the considered Banach space).

We will abuse notation and we will write $C^r = C^r([-\tau, 0], \mathbb{R})$ for the space of all functions of class C^r over $[-\tau, 0]$ equipped with the supremum norm: $\|g\| = \sum_{i=0}^r \sup_{x \in [-\tau, 0]} |g^{(i)}(x)|$.

By \mathbb{R}_0 we denote the set $\mathbb{R}_+ \cup \{0\}$.

For a given function $x : [-1, a) \rightarrow \mathbb{R}$, $a \in \mathbb{R}_0 \cup \{\infty\}$ for any $t \in [0, a)$ we denote by $x_t \in C^0$ the function such that $x_t(s) = x(t + s)$ for all $s \in [-1, 0]$.

For $v \in \mathbb{R}^n$ by $\pi_i v$ for $i \in \{1, 2, \dots, n\}$ we denote the projection of v onto the i -th coordinate. For vectors $u, v \in \mathbb{R}^n$ by $u \cdot v$ we denote the standard scalar product: $u \cdot v = \sum_{i=1}^n \pi_i v \cdot \pi_i u$. We will sometimes use the usual notation $v_i = \pi_i v$, but only when the context is absolutely clear, as not to confuse it with the notation of $x_t \in C^0$.

Let $A = \prod_{i=1}^n [a_i, b_i]$, for $a_i \leq b_i$, $a_i, b_i \in \mathbb{R}$. We call A an *interval set* (a product of closed intervals in \mathbb{R}^n). For any $A \subset \mathbb{R}^n$ we denote by $\text{hull}(A)$ a minimal interval set, such that $A \subset \text{hull}(A)$. If $A \subset \mathbb{R}$ is bounded then $\text{hull}(A) = [\inf(A), \sup(A)]$. For any set X by $\text{m}(X)$

we denote the midpoint of interval set $\text{hull}(X)$ and by $\text{diam}(X)$ the diameter of $\text{hull}(X)$.

1.2 Set arithmetic and rigorous numerics

By its very nature, the numerical computations done in computers are finite, so only a small subset of real numbers can be stored in an exact manner. We call them *representable numbers*. Other real values must be approximated to the nearest representable number. The commonly used standard for those operations is IEEE 754 [7]. The idea of rigorous interval numerics is to replace operators and elementary functions ($+$, $-$, \div , \times , \sin , \cos , \exp , etc.) with their set-valued versions such that they operate on intervals. The interval counterpart \diamond of a given operator \diamond guarantees that the result contains all possible outcomes of applying the operator on any two possible arguments, i.e.

$$\{x \diamond y : x \in [a_1, b_1], y \in [a_2, b_2]\} \subset [a_1, b_1] \diamond [a_2, b_2].$$

Therefore, by using interval arithmetic in our computations, we are sure that they contain all possible results – they are *validated*. To be able to implement such interval operations on computers, we need *closedness*, that is, we require $[a_1, b_1] \diamond [a_2, b_2]$ to be again an interval with ends being representable numbers. Fortunately, this can be simply done under the IEEE 754 standard [6], and many software packages already exist (see [1] and references therein). We use [4] for this purpose.

To stress the fact that we are using set arithmetic we will often put a variable name in square brackets, e.g., we will write $[r]$ to denote a set in \mathbb{R}^m . Usually it will happen in formulas used in algorithms. If both variables r and $[r]$ are used simultaneously then usually r represents a value in $[r]$, however this is not implied by default and it will be always stated explicitly. Please note that the notion $[r]$ has no additional meaning and $[r]$ may be simply regarded as a name of another variable.

Further, we will put explicit numerical values into square brackets to stress the fact that in their place we are using an interval with ends being representable numbers that contains the given value. For example $[0.3333]$ should be understood as a representable interval obtained by the rigorous numerical computation of the division operator: $[3333, 3333] \div [10000, 10000]$. If the explicit value in a numerical formula is put without brackets then it is in fact a closest representable IEEE floating point number in a given precision. This happens in formulation of some computer-assisted proofs in Sections 3 and 4.

2 Rigorous forward in time integration and construction of Poincaré maps for DDEs

For the rest of this paper we assume that $\tau = 1$ in (1.3). All computations can be easily redone for any delay τ .

The *semiflow* φ associated with Equation (1.3) is defined as:

$$\varphi : \mathbb{R}_0 \times C^0([-1, 0], \mathbb{R}) \ni (t, \psi) \mapsto x_t^\psi \in C^0([-1, 0], \mathbb{R}). \quad (2.1)$$

where $x^\psi : [-1, a_\psi) \rightarrow \mathbb{R}$ denotes the solution to the Cauchy problem (1.3)-(1.4), such that the solution exists for all $t < a_\psi$. Under some mild assumptions on the r.h.s. f of (1.3) x^ψ is unique for all initial function ψ and the (local) semiflow φ is continuous on $C^0([-1, 0], \mathbb{R})$ [5]. Also, from the classical method of steps, one can obtain the following lemma, which we state without proof.

Lemma 2.1 (Smoothing property). *Assume f is of class C^m , let $n \in \mathbb{N}$, $t \geq n$. If $x_0 \in C^0$, then $x_t = \varphi(t, x_0)$ is of class at least $C^{\min(m+1, n)}$.*

In our computer assisted proofs, we follow a standard algorithm to construct Poincaré maps for flows:

1. Select a finite representation of the initial condition $x_0 \in C^0$, i.e. a set $X_0 \subset C^0$ such that $x_0 \in X_0$ and X_0 can be described with a finite number of parameters.
2. Define a section S in the phase space such that $x_0 \in S$ and the semiflow φ is transversal to S in the neighbourhood of x_0 .
3. Compute estimates on the transition time to section $t_S \in [t_S]$ for $t_S > 0$ such that $\varphi(t_S, x_0) \in S$.
4. Return a (finite representation of) set X_{t_S} such that $\varphi(t_S, X_0) \subset X_{t_S}$.

Our rigorous integration scheme will be given by a family of operators $\{I_{\Delta t}\}_{\Delta t \in [0, h]}$ such that

$$\varphi(\Delta t, X_0) \subset I_{\Delta t}(X_0).$$

If $[t_S] = q \cdot h + [\varepsilon_1, \varepsilon_2]$ for $0 \leq \varepsilon_1 \leq \varepsilon_2 < h$ then Step 4 can be implemented as:

$$X_{t_S} := I_{[\varepsilon_1, \varepsilon_2]} \circ I_h^q(X_0), \quad (2.2)$$

where

$$I_{[\varepsilon_1, \varepsilon_2]}(\cdot) = \bigcup_{\varepsilon \in [\varepsilon_1, \varepsilon_2]} I_\varepsilon(\cdot). \quad (2.3)$$

Notice that, the value of q can be obtained by counting the number of iterations of the operator I_h before the algorithm detects a crossing of the section. Then $[\varepsilon_1, \varepsilon_2]$ may be found by application of any bisection algorithm.

In [22, 23], a rigorous algorithm for construction of I_h for the constant step size $h = \frac{1}{p}$ for some $p \in \mathbb{N}_+$ was proposed. Also, under some assumptions, it is possible to obtain a rigorous version of I_ε for $0 < \varepsilon < h$. The original algorithm extensively relies on the smoothing property of DDEs (Lemma 2.1) so from now on we assume that r.h.s. f of Equation (1.3) is “sufficiently smooth” for various expressions to make sense. In fact, we can assume that $f \in C^\infty$ as this is usually true in the context of the classical equations defined by a finite composition of elementary C^∞ functions. The “logistic” (Eq. (1.1)) and Mackey–Glass equations are both good examples.

Let us recall basic definitions and results from [23]. We assume that integers $n \geq 0$ and $p > 0$ are given and we set $h = \frac{1}{p}$.

Definition 2.2 (Definition 1 in [23]). By C_p^n we denote the set of all functions $g : [-1, 0] \rightarrow \mathbb{R}$ such that for $i \in \{1, \dots, p\}$ we have:

- g is $(n+1)$ -times differentiable on $(-i \cdot h, -i \cdot h + h)$,
- $g^{(k)}(-i \cdot h^+)$ exists for all $k \in \{0, \dots, n+1\}$,
- $g^{(n+1)}$ is continuous and bounded on $(-i \cdot h, -i \cdot h + h)$.

In other words, $g \in C_p^n$ is given by a *piecewise Taylor expansion* on each of the intervals $[-i \cdot h, -i \cdot h + h)$. For $t = -i \cdot h + \varepsilon$ and $1 \leq i \leq p$, $0 \leq \varepsilon < h$ we can write:

$$g(t) = \sum_{k=0}^n g^{[k]}(-i \cdot h) \cdot \varepsilon^k + g^{[n+1]}(-i \cdot h + \zeta(\varepsilon)) \cdot \varepsilon^{n+1}, \quad (2.4)$$

Definition 2.3 (Definition 2 in [23]). Let $g \in C_p^n$ and let $\mathbb{I} : \{1, \dots, p\} \times \{0, \dots, n\} \rightarrow \{1, \dots, p \cdot (n+1)\}$ be any bijection.

A (p, n) -representation of g is a pair $\bar{g} = (a, B)$ such that

$$\begin{aligned} a &\in \mathbb{R}^{p \cdot (n+1) + 1}, & B &\subset \mathbb{R}^p \text{ is an interval set} \\ \pi_1 a &= g(0), & \pi_i B &= \left[\inf_{\zeta \in (0, h)} g^{[n+1]}(-ih + \zeta), \sup_{\zeta \in (0, h)} g^{[n+1]}(-ih + \zeta) \right] \\ \pi_{1+\mathbb{I}(i, k)} a &= g^{[k]}(-ih^+), & \text{for } 1 \leq i \leq p, 0 \leq k \leq n \end{aligned}$$

The graphical presentation of the idea of a (p, n) -representation of some exemplary function is given in Figure 2.1. The following notation is used for any (p, n) -representation \bar{g} :

$$\bar{g}^{0, [0]} := \pi_1 a, \quad \bar{g}^{i, [k]} := \pi_{1+\mathbb{I}(i, k)} a, \quad \bar{g}^{i, [n+1]} := \pi_i B$$

The term $\bar{g}^{i, [k]}$ is called *the (i, k) -th coefficient of the representation* and $\bar{g}^{i, [n+1]}$ *the i -th remainder of the representation*. The interval set B is called *the remainder of the representation*. We will call the constant $M = p \cdot (n+2) + 1$ *the size of the (p, n) -representation*. The parameters n and p are omitted when known from the context.

Definition 2.4 (Definition 3 in [23]). Let $\bar{G} = (A, C) \subset \mathbb{R}^{p \cdot (n+1) + 1} \times \mathbb{R}^p = \mathbb{R}^m$ be a bounded set. Then, we define (p, n) -f-set (or (p, n) -functions set) $\text{fset}(\bar{G}) \subset C_p^n$ as

$$\text{fset}(\bar{G}) = \left\{ f \in C_p^n : \bar{f} \subset \bar{G} \text{ for the } (p, n)\text{-representation } \bar{f} \text{ of } f \right\}. \quad (2.5)$$

Conversely, given a set $G \subset C_p^n$ we will say that \bar{G} is a (p, n) -representation of G if $G = \text{fset}(\bar{G})$.

The two concepts are strongly linked but, formally, a (p, n) -representation is an object in a *finite* dimensional space, where a (p, n) -f-set contains elements of an *infinite* dimensional functional space.

The next lemma follows simply from the definitions of (p, n) -representations and (p, n) -f-sets.

Lemma 2.5 (Lemma 4 in [23]). *Let G, F be (p, n) -f-sets with representations \bar{G} and \bar{F} , respectively. The following statements hold.*

- *If $\bar{G} \subset \bar{F}$, then $G \subset F$.*
- *If $\bar{G} \subset \mathbb{R}^m$ is convex, then G is convex in C_p^n .*
- *If $\bar{G} \subset \mathbb{R}^m$ is convex, then $G \cap C^k$ is convex for any $k \geq 0$.*

In [23], the authors proposed an algorithm for obtaining a (p, n) -f-set that contains all solutions to (1.3), given initial functions defined in an another (p, n) -f-set. Namely, they developed a procedure to compute:

$$\bar{x}_h = I_h(\bar{x}_0).$$

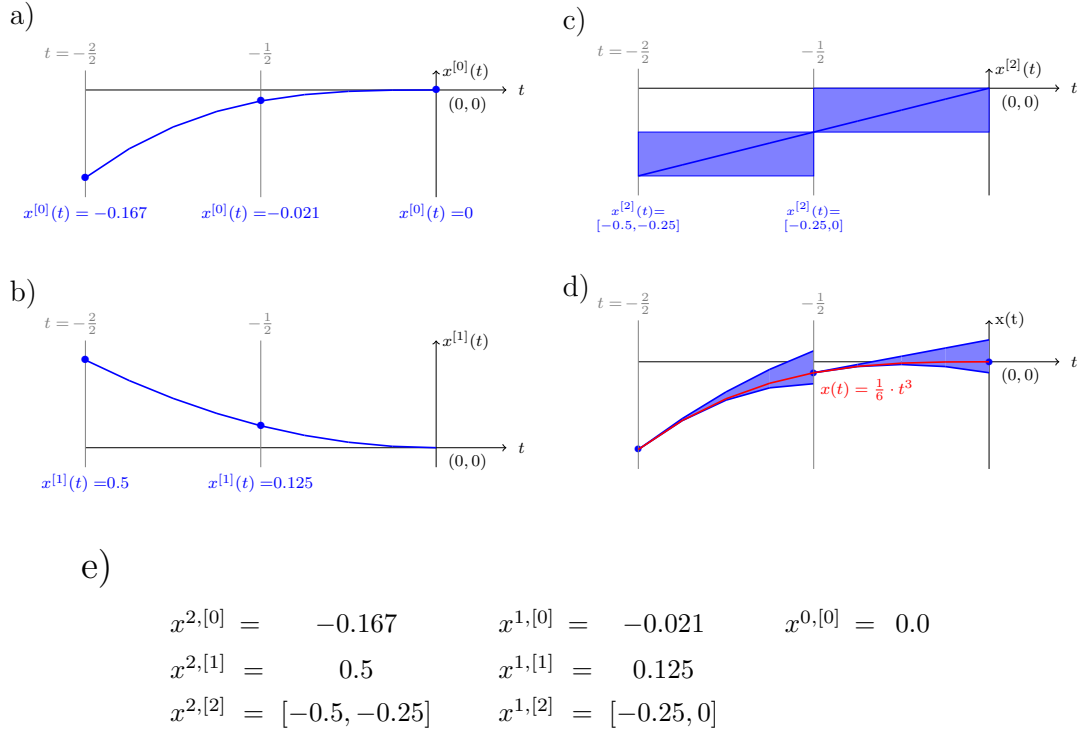


Figure 2.1: An example of constructing the (p, n) -representation $\bar{x} = (a, B)$ for the function $x(t) = \frac{1}{6} \cdot t^3$ (plotted in red in d)). Blue curves in a), b) and c) are $x = x^{[0]}$, $x^{(1)} = x^{[1]}$ and $\frac{1}{2} \cdot x^{(2)} = x^{[2]}$, respectively. The remainder terms B are presented graphically as blue boxes, while the values of a are presented as blue dots. In e) the coefficients a and B are organized into a matrix that resembles their position w.r.t. indices i and k , e.g. element $\bar{x}^{i,[k]}$ lies in i -th column and k -th row. For the sake of the presentation, numbers are rounded up to three decimal places. The reconstruction of the (p, n) -f-set represented by \bar{x} is shown as blue region in d). We see that x is contained in $\text{fset}(\bar{x})$. An animated version of this figure is available in PDF format in [24].

where \bar{x}_0, \bar{x}_h are (p, n) -representations, such that $\varphi(h, \text{fset}(\bar{x}_0)) \subset \text{fset}(\bar{x}_h)$ and $h = \frac{1}{p}$.

Also, under assumption that the iteration time is “long enough”, they proposed a family of algorithms I_ε for $0 < \varepsilon < h$ for which the following is true:

Theorem 2.6 (Theorem 14 in [23]). *Assume that $\varepsilon \in [0, h]$ and \bar{x}_0 is a (p, n) -representation. Let $q \in \mathbb{N}$ be such that $n \cdot p \leq q$. Then*

$$\varphi(q \cdot h + \varepsilon, x) \in \text{fset}(I_\varepsilon(I_h^q(\bar{x}_0))) \quad (2.6)$$

for all $x \in \text{fset}(\bar{x}_0) \cap C^0$.

We skip the details of the construction here. We would like to note that the condition for a “long enough” iteration is $n \cdot p \leq q$. If we denote the transition time $q \cdot h + \varepsilon$ in (2.6) by t , then the condition transforms into $t \geq n$. By Lemma 2.1 it guarantees that $\varphi(t, x)$ is of class at least C^n (we remind that we assumed $\tau = 1$).

Finally, we recall the theorem which guarantee that Poincaré map is well defined and it is a compact operator.

Definition 2.7 (Definition 6 in [23]). Let $n \in \mathbb{N}$, $s : C^n \rightarrow \mathbb{R}$ be a bounded linear operator and $a \in \mathbb{R}$. We define a *global C^n -section* as a hyperplane:

$$\mathcal{S}_a = \{x \in C^n : s(x) = a\}. \quad (2.7)$$

Any convex, bounded subset $S_a \subset \mathcal{S}_a$ is called a *local C^n -section* (or simply a section).

Let $W \subset C^n$ be a convex open set such that

$$\begin{aligned} U &:= W \cap \mathcal{S}, & \text{cl}(U) &= \text{cl}(S) \\ W_- &:= \{x \in W : s(x) < a\} \neq \emptyset, \\ W_+ &:= \{x \in W : s(x) > a\} \neq \emptyset. \end{aligned}$$

A section S_a is said to be *transversal on W* if

$$s(\dot{x}) > 0, \quad \forall x \in W \cap C^{n+1}. \quad (2.8)$$

We will refer to (2.8) as the *transversality condition*.

Definition 2.8 (Definition 7 and Theorem 10 in [23]). Let $\omega \geq n + 1$, S be a section, $V \subset C^0$ and assume there exist t_1, t_2 such that:

- $\omega \leq t_1 < t_2$,
- $\varphi([t_1, t_2], V) \subset W$,
- S is transversal on W ,
- $\varphi(t_1, V) \subset W_-$,
- $\varphi(t_2, V) \subset W_+$.

Then, *the transition map to the section S after (at least) ω* is well defined and given by:

$$P_{\geq \omega} : V \rightarrow S, \quad P_{\geq \omega}(x_0) := \varphi(t_S(x_0), x_0). \quad (2.9)$$

where $t_S(x)$ is a unique time such that $\varphi(t_S(x), x) \in S$ (see Theorem 10 in [23]).

If $V \subset S$, then the map $P_{\geq \omega}$ will be called *the Poincaré return map on the section S after ω* .

Note that the condition on the transition time given by ω is $n + 1$ instead of n (as compared to Theorem 2.6). This extra iteration time guarantees that the derivative of $P(x)$ w.r.t. t is well defined and of class at least C^n .

The main result that allows us to use the rigorous bounds in the actual proofs is as follows.

Theorem 2.9 (Theorem 11 in [23]). *Let S be a C^n -section and let $V \subset S$ be bounded in C^n . Assume $\omega \geq n + 1$ and $t_S \in [t_1, t_2]$ as in Definition 2.8. If $\varphi([t_1, t_2], V)$ is bounded then $P_{\geq \omega} : V \rightarrow S$ is continuous. Moreover, $\text{cl}(P_{\geq \omega}(V))$ is compact in C^n .*

3 Computer assisted proofs of attracting periodic orbits in the logistic equation

We start this section with recall of Schauder's fixed point theorem [20, 29]. We will use it to prove existence of several periodic orbits to (1.1) for various values of parameter λ .

Theorem 3.1 (Schauder's fixed point theorem). *Let X be a Banach space, let $V \subset X$ be a non-empty, convex, bounded set and let $P : V \rightarrow X$ be a continuous mapping such that $P(V) \subset K \subset V$ and K is compact. Then the map P has a fixed point in V .*

We will be working in the space $X = C^n$. We will use routines described in the previous Section to obtain bounds on the set K in Theorem 3.1 for some a priori estimated set $V \in C_p^n \cap C^n$. Theorem 2.9 will guarantee compactness of operator P , thus compactness of the set K .

Since we are using (p, n) -representations with $n = 4$ and $p = 32$ to describe sets in the phase space, it would not be feasible to present data from the proofs directly, as the number of coefficients is equal to the size of the (p, n) -representation: $M = 193$. Also, it would be impossible to provide reasonable initial data by hand. Therefore, we discuss shortly how we automatically obtain bounds on the set of initial functions V on a suitably chosen section S_a .

3.1 (p, n) -sections

We follow notation of [23]. Since we are using (p, n) -representations to describe functions in C^n , it is advisable to define sections in the phase space in such a way that it would be easy to rigorously check if $x \in S$ for all functions represented by a given (p, n) -f-set (or its (p, n) -representation). A straightforward way is to require s in Definition 2.7 to depend only on representation coefficients $\bar{x}^{i,[k]}$ for $k \leq n$. This motivates the following definition.

Definition 3.2. Let $\hat{l}_{i,k} \in \mathbb{R}$ for $(i, k) \in \mathcal{C} = \{1, \dots, p\} \times \{0, \dots, n\} \cup \{(0, 0)\}$. We assume that at least one $\hat{l}_{i,k}$ is not equal to zero. We define a bounded linear operator $s(x)$ by:

$$s(x) = \sum_{(i,k) \in \mathcal{C}} \hat{l}_{i,k} x^{[k]}(-ih), \quad x \in C^n. \quad (3.1)$$

A section $\mathcal{S}_a = \{x \in C^n, s(x) = a\}$ (see Definition 2.7) is called a (p, n) -section. A subset $S_a \subset \mathcal{S}_a$ is called a local (p, n) -section. A vector $l \in \mathbb{R}^{M-p+1}$ (where M is the size of the (p, n) -representation) such that $\pi_{\mathbb{I}(i,k)} l = \hat{l}_{i,k}$ for $(i, k) \in \mathcal{C}$ is called the representation of a (p, n) -section.

Note that $s(x) \equiv l \cdot \bar{x}$ for the (p, n) -representation of a function x .

3.2 Selection of the initial conditions

To find good initial conditions, we will use an approximate flow $\hat{\phi}$ defined by:

$$\hat{\phi}(t, x) = \Phi_\varepsilon \circ \Phi_h^q(x), \quad x \in R^m, t = q \cdot h + \varepsilon, \quad (3.2)$$

where Φ_h, Φ_ε denotes "truncated" versions of the rigorous integration algorithms I_h and I_ε , respectively. Algorithms Φ_h and Φ_ε are called truncated, as they do not take into consideration the effect of remainder terms B of (p, n) -representations (see Section 2.2 of [23] for details). Despite this, we hope that a periodic solution \hat{x} to (3.2) would be close enough to the true solution of (1.3), so that a small neighbourhood $[\hat{x}]$ of \hat{x} can be selected as V in Theorem 3.1.

We start the selection of the initial function by computing $\hat{\phi}(T_0, \hat{x})$ with $\hat{x} = A \cdot \sin(B \cdot t) + C$ for some $A, B, C \in \mathbb{R}$ (discussed later in Section 4) and $T_0 \in \mathbb{R}_+$ large enough so that the solution \hat{x}_{T_0} stabilizes as discussed in [16]. Then, we construct a Poincaré map \hat{P} with a (p, n) -section defined by $s(x) = \pi_{\mathbb{I}(0,0)} x$ and $a = \pi_{\mathbb{I}(0,0)} \hat{x}_{T_0}$. For the map \hat{P} we apply Newton

algorithm to find $\bar{x}_0 \in \mathbb{R}^m$ such that $\|\hat{P}(\bar{x}_0) - \bar{x}_0\| \leq 10^{-13}$. The vector \bar{x}_0 will be the middle point of the (p, n) -representation of the initial set V .

In the next step we choose a (p, n) -section such that it minimizes the difference between transition time t_S for different $x \in V$. A heuristic algorithm to find such section exists (see Section 3.4 in [23]).

In order to demonstrate the idea behind the heuristic algorithm, let us consider for a while an ODE of the form:

$$x' = f(x), \quad f \in C^1, \quad x \in \mathbb{R}^m. \quad (3.3)$$

Let x_0 be a periodic orbit of period T of the flow φ induced by (3.3). Then $f(x_0)$ is the right eigenvector of the monodromy matrix $A = \frac{\partial \varphi}{\partial x}(T, x_0)$ with eigenvalue $\lambda = 1$. Let l be the left eigenvector of A corresponding to $\lambda = 1$, i.e. $l^T \cdot A = l^T$. If x_0 is hyperbolic then the left and right eigenvectors of A are uniquely defined up to a multiplier and we can choose l such that

$$l \cdot f(x_0) = 1 \quad (3.4)$$

Let us consider section $S = \{x \mid l \cdot x = l \cdot x_0\}$ in \mathbb{R}^m . Then, from

$$l \cdot \varphi(t_S(x), x) - l \cdot x_0 = 0, \quad (3.5)$$

we obtain

$$\frac{\partial t_S}{\partial x}(x_0) = -\frac{l \cdot \frac{\partial \varphi}{\partial x}(T, x_0)}{l \cdot f(x_0)}, \quad (3.6)$$

Let l^\perp be the tangent space to l , i.e. $l^\perp = \{x \in \mathbb{R}^m : l \cdot x = 0\}$. Then, by direct computation, we get:

$$\frac{\partial t_S}{\partial x}(x_0) \cdot b = 0, \quad \text{for } b \in l^\perp. \quad (3.7)$$

Therefore, in the ODE setting, choosing left eigenvector of the monodromy matrix A to define (linear) section guarantees that it is transversal to the flow and the return time is constant in the first order approximation.

We use this fact to obtain a good candidate for a section by computing the left eigenvector corresponding to eigenvalue 1 of the matrix $\frac{\partial \hat{\varphi}}{\partial x}(T, x_0)$. Let $l \in \mathbb{R}^{M-p+1}$ be this vector. We define the matrix

$$\hat{C} := \begin{pmatrix} l_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ l_2 & 1 & & & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \ddots & \vdots \\ l_{M-p+1} & & & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & & \\ \vdots & \vdots & \ddots & \vdots & & \ddots & \\ 0 & 0 & \cdots & 0 & & & 1 \end{pmatrix}. \quad (3.8)$$

This is a non-singular matrix. Let C be a matrix obtained from the procedure of orthonormalisation of \hat{C} by the Gram-Schmidt algorithm, so that C is an orthonormal base for the phase space of \mathbb{R}^m . We choose the initial set V to be:

$$V := \hat{x}_0 + C \cdot \zeta \cdot r_0, \quad \zeta \in [-1, 1] \quad (3.9)$$

for some $r_0 \in \mathbb{R}^m$, $\pi_1 r_0 = 0$. Notice, that the set V is contained on the (p, n) -section defined by l as a hyperplane in the phase space. Coefficients of r_0 are chosen experimentally to follow an exponential law in k , i.e. $r_0^{i, [k]} \approx d \cdot q^k$ for some $q > 1$ and $d \in (0, 1)$. They are further refined during the computations to obtain better initial bounds. A method of selecting the middle point \hat{x}_0 for various values of the parameter λ in (1.1) is discussed in details in Section 4.

3.3 Computer assisted proofs

Before we formulate theorems, we once more recall the problem of exact representation of numbers in computer programs. Below we state theorems with floating-point (real) numbers presented up to four digits after the decimal point, so they are not exactly the ones which the software for computer-assisted proof produces. We have rounded them in such a way that they are valid bounds (upper or lower where appropriate) for the values produced in rigorous numerical procedures. Please also remember, that if we write $[5.8]$, then we mean by that a minimal interval with end points being representable numbers such that it contains 5.8.

The computer program and initial data used to obtain estimates in theorems may be found online [24]. There, one can find also a detailed instruction on compiling and running the programs. The compilation requires CAPD library [4], as we use it in underlying rigorous interval numerical procedures. Source codes were tested on a standard laptop PC with Intel® Core™ I7-2860QM CPU (2.50 GHz), 16 GB RAM under 64-bit Linux operating system (Ubuntu 12.04 LTS) and C/C++ compiler gcc version 4.6.3. In the sequel, we will refer to this configuration as a *reference machine*. On this configuration the computation time for each of the proofs is approximately one minute.

Before we state the theorems, we note that it is straightforward to check that $\zeta_0 = 0$ and $\zeta_1 = \frac{\lambda-1}{\lambda}$ are the stationary solutions to (1.1) for $\lambda \neq 0$. Moreover, if x is a bounded solution to (1.1) with initial condition x_0 and $x_0 \equiv x_T$ for some $T > 0$, then x may be extended to the whole \mathbb{R} where it will satisfy (1.1) for any t . By $x(\mathbb{R})$ we denote the image of the extended function over \mathbb{R} .

Now we can proceed to state the theorems, which are proved with computer assistance.

Theorem 3.3. *There exists a periodic solution x to (1.1) with $\lambda = 5.5$ such that $\zeta_1 \in x(\mathbb{R})$ and $\zeta_0 \notin x(\mathbb{R})$.*

Proof. We choose the (32,4)-representation of the phase space and we set $\lambda = [5.5]$ in (1.1).

We choose the (p, n) -section and we construct representation of the (p, n) -f-set of initial conditions V as described in Section 3.2. For those initial conditions we perform the rigorous construction of the image of Poincaré map $P_{\geq \omega}$ as described in Section 2. Let W be the (p, n) -f-set enclosing the image of $P_{\geq \omega}$ obtained from the integration procedure, i.e. $P(V) \subset W = \Phi_{[\varepsilon]} \circ \Phi_h^q(V)$ for some q and $[\varepsilon]$. The program verified that:

- $t_S \in q \cdot h + [\varepsilon_1, \varepsilon_2] = 209 \cdot \frac{1}{32} + [0.0194046, 0.0194166]$,
- $209 = q > (n + 1) \cdot p = 5 \cdot 32 = 160$,
- $\varphi(q \cdot h + \varepsilon_1, V) \in W_-$ and $\varphi(q \cdot h + \varepsilon_2, V) \in W_+$, see the output of the program,
- $s(\dot{x}) > 0.1269$ for $x \in W \cap C^n$, which guarantees transversality,
- $0.2260 < \inf_{t \in \mathbb{R}} x(t) < 0.2284$, $1.0850 < \sup_{t \in \mathbb{R}} x(t) < 1.0862$, $\zeta_1 \in [0.8181, 0.8182]$,

so by the Theorem 2.9 the Poincaré map is well defined and compact on V .

Next, the program checks inclusion $W \subset V$ (see output of the program for details). By Schauder theorem we get the existence of a fixed point in the (p, n) -f-set V which corresponds to a periodic orbit (as $\zeta_0, \zeta_1 \notin V$).

Since x is periodic, it suffices to compute lower and upper estimates on the value of $x(t)$ over an interval containing basic period to obtain estimates of $\inf_{t \in \mathbb{R}} x(t)$ and $\sup_{t \in \mathbb{R}} x(t)$. The

program verifies that

$$\begin{aligned} \zeta_1 \in [0.8181, 0.8182] &> 0.2284 > \inf_{t \in \mathbb{R}} x(t) = \inf_{t \in [0, q \cdot h + \varepsilon_2]} x(t) > 0.2260 > 0 = \zeta_0, \\ \zeta_1 \in [0.8181, 0.8182] &< 1.0850 < \sup_{t \in \mathbb{R}} x(t) = \sup_{t \in [0, q \cdot h + \varepsilon_2]} x(t), \end{aligned}$$

which proves $\zeta_0 \notin x(\mathbb{R})$ and $\zeta_1 \in x(\mathbb{R})$.

The data for this proof can be found in a .zip file at [24] under `logistic/5.5/0_B.initial`. The estimates obtained on the reference machine can be found under `logistic/5.5/0_C.proof`. \square

The approximate orbit which defines the middle point of the initial set V in Theorem 3.3 is presented in Figure 3.1.

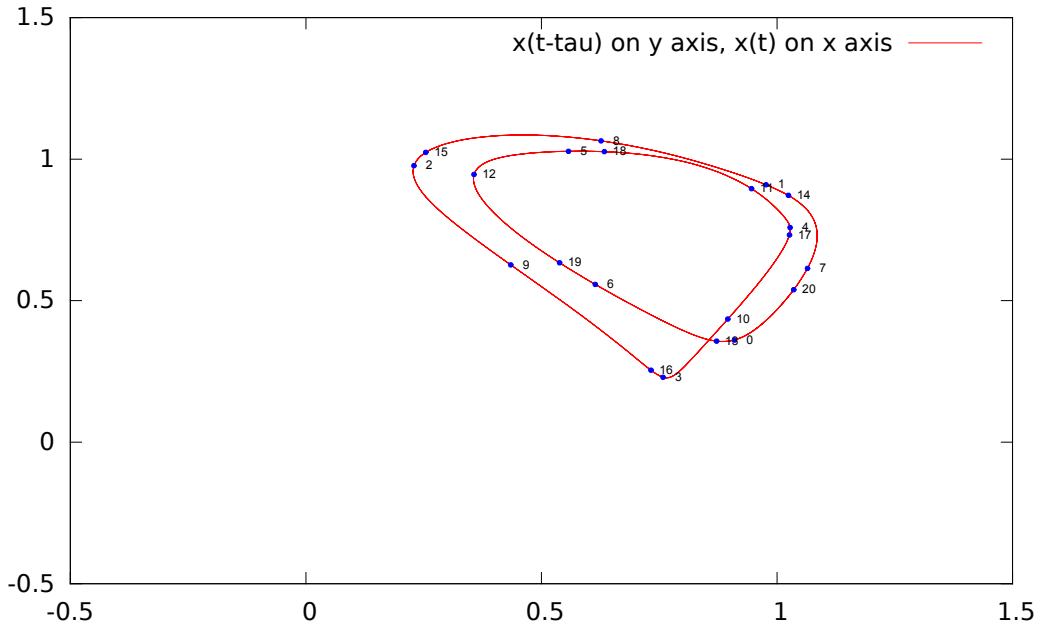


Figure 3.1: Plot of the solution from Theorem 3.3. The dots mark the shift by a next full delay, where the associated number is the corresponding time.

Theorem 3.4. *There exists a periodic solution x to (1.1) with $\lambda = 5.75$ such that $\zeta_1 \in x(\mathbb{R})$ and $\zeta_0 \notin x(\mathbb{R})$.*

Proof. We proceed in the same manner as in the proof of Theorem 3.3. We present those numbers that are needed to verify assumptions:

- $t_S \in q \cdot h + [\varepsilon_1, \varepsilon_2] = 259 \cdot \frac{1}{32} + [0.000719, 0.000741]$,
- $s(\dot{x}) > 0.2167$ for $x \in W \cap C^n$, which guarantees transversality,
- $0.0163 < \inf_{t \in \mathbb{R}} x(t) < 0.0195$, $1.1449 < \sup_{t \in \mathbb{R}} x(t) < 1.1461$, $\zeta_1 \in [0.8260, 0.8261]$

The data for this proof can be found in a .zip file at [24] under `logistic/5.75/1_B.initial`. The estimates obtained on the reference machine can be found under `logistic/5.75/1_C.proof`. \square

The approximate orbit which defines the middle point of the initial set V in Theorem 3.4 is presented in Figure 3.2.

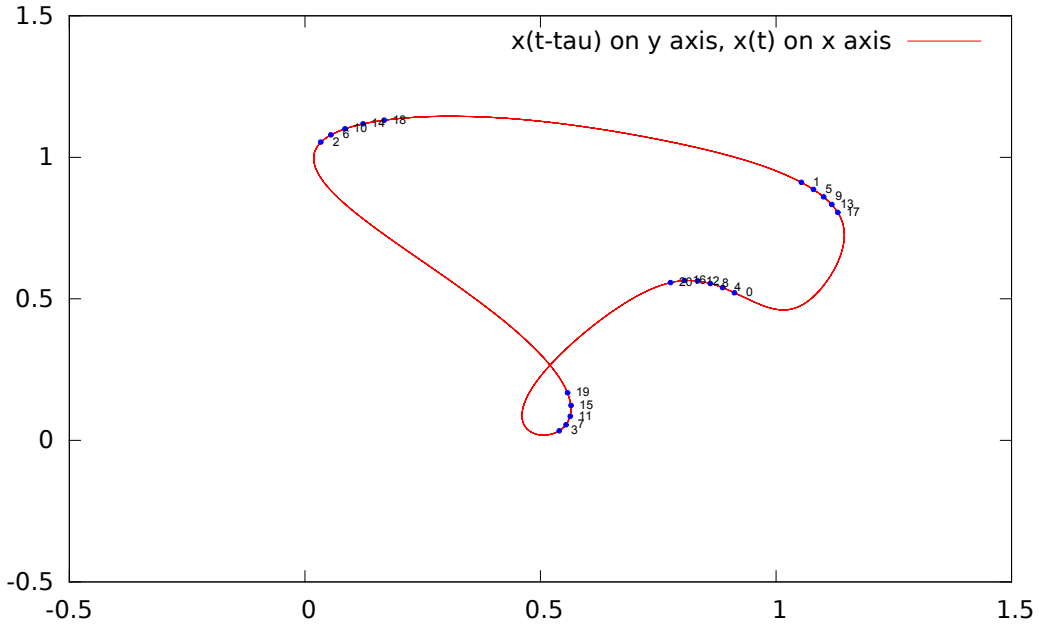


Figure 3.2: Plot of the solution from Theorem 3.4. The dots mark the shift by a next full delay, where the associated number is the corresponding time.

Theorem 3.5. *There exists a periodic solution x to (1.1) with $\lambda = 5.8$ such that $\xi_1 \in x(\mathbb{R})$ and $\xi_0 \in x(\mathbb{R})$.*

Proof. We proceed in the same manner as in the proof of Theorem 3.3. We present only numbers needed to verify assumptions:

- $t_S \in q \cdot h + [\varepsilon_1, \varepsilon_2] = 261 \cdot \frac{1}{32} + [0.0005558, 0.0005740]$,
- $s(\dot{x}) > 0.4290$ for $x \in W \cap C^n$, which guarantees transversality,
- $-0.0128 < \inf_{t \in \mathbb{R}} x(t) < -0.0101$, $1.1523 < \sup_{t \in \mathbb{R}} x(t) < 1.1539$, $\xi_1 \in [0.8275, 0.8276]$.

The data for this proof can be found in a .zip file at [24] under `logistic/5.8/1_B.initial`. The estimates obtained on the reference machine can be found under `logistic/5.8/1_C.proof`. \square

The approximate orbit which defines the middle point of the initial set V in Theorem 3.5 is presented in Figure 3.3.

4 Further numerical investigations of the logistic equation

As we have stated in Section 2, we have generated initial conditions by a very long, non-rigorous integration of (1.1) for initial functions of the form $A \cdot \sin(B \cdot t) + C$.

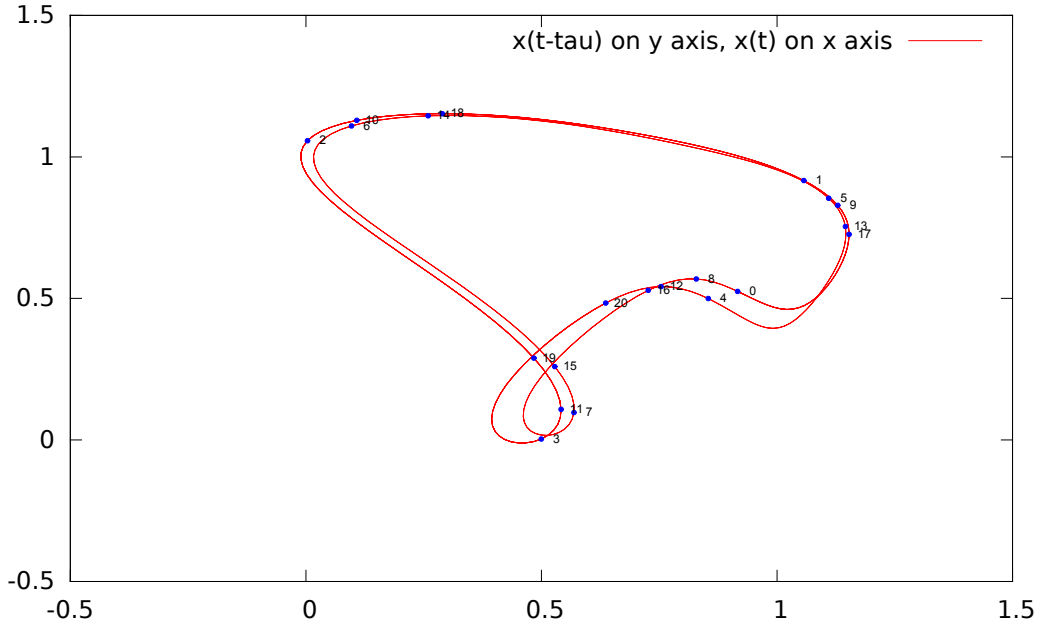


Figure 3.3: Plot of the solution from Theorem 3.5. The dots mark the shift by a next full delay, where the associated number is the corresponding time.

Initially we carried out multiple runs with

$$A \in \{0.1 \cdot i : i \in 0, \dots, 4\},$$

$$B \in \left\{ \frac{0.5 \cdot i}{2 \cdot \pi} : i \in 0, \dots, 4 \right\},$$

$$C \in \{0.4 + 0.1 \cdot i : i \in 1, \dots, 5\},$$

and we have observed to what orbit the solution appeared to converge. Then, we have selected two initial functions which generated distinct behaviours:

$$a_0(t) := 0.4 \cdot \sin\left(\frac{2}{2 \cdot \pi} \cdot t\right) + 0.6 \quad (4.1)$$

$$b_0(t) := 0.2 \cdot \sin\left(\frac{3.25}{2 \cdot \pi} \cdot t\right) + 0.6. \quad (4.2)$$

We call the solutions *a-type* and *b-type*. The *a-type* solutions tend to the periodic orbit with $\xi_0 = 0 \notin x(\mathbb{R})$ while *b-type* solutions has both $\xi_0 = 0 \in x(\mathbb{R})$ and $\xi_1 = \frac{\lambda-1}{\lambda} \in x(\mathbb{R})$ for larger values of parameter λ . For each value of the parameter $\lambda \in \{5.5 + 0.1 \cdot i : i \in 0, \dots, 3\} \cup \{5.75 + 0.01 \cdot i : i \in 0, \dots, 10\}$ we computed 100-th iterates of a_0 and b_0 : $a_{100} := \text{Phi}_h^{100}(a_0)$, $b_{100} := \text{Phi}_h^{100}(b_0)$. Then, Newton algorithm was applied to a_{100} and b_{100} to obtain final candidates a and b such that $\|P(x) - x\| < 10^{-13}$ for $x \in \{a, b\}$.

We ran an automatic procedure to verify (non-rigorously) if the orbit is attracting or repelling by computing (approximate) eigenvalues of the Jacobian of P at $x \in \{a, b\}$. The resulting orbits are shown in Tables 4.1, 4.2 and 4.3. With green box we have marked solutions for which we were able to get a computer assisted proof of existence. An *a-type* periodic solution for $\lambda = 5.5$ for which we have managed to get the computer assisted proof is shown

in Figure 3.1. Solutions presented in Tables 4.1 and 4.2 were classified as attracting, where in Table 4.3 we have gathered initial conditions obtained from Newton algorithm that are apparently unstable (numerical computations show that Jacobian of P at x has one unstable eigenvalue).

We were not able to obtain a computer assisted proof in the case of a -type periodic solution when $\lambda > 5.7$. The reason for this is that the approximate monodromy matrix of φ at x have not contained an eigenvalue 1 in its spectrum. Thus, we were not able to select a good representation of the (p, n) -section. The observed dominant approximate eigenvalue was always complex with the magnitude ≈ 1 . We have tried to repeat computations with larger (p, n) -representations (a (128,4)-representation and a (32,10)-representation), but without success. However, for the (128,4)-representation, the magnitude of the imaginary part of the dominant eigenvalue was much less than in the case of (32,4)-representation. This suggest that the lack of eigenvalue 1 may be only an artefact of numerical computations. We plan to carry out the computations for even larger representations of the phase space, but due to the time constraints, we need first to optimise the procedure of computation of the monodromy matrix as suggested in [22]. With the current implementation we estimate that computations on (256, 4)-representation would take more than a year.

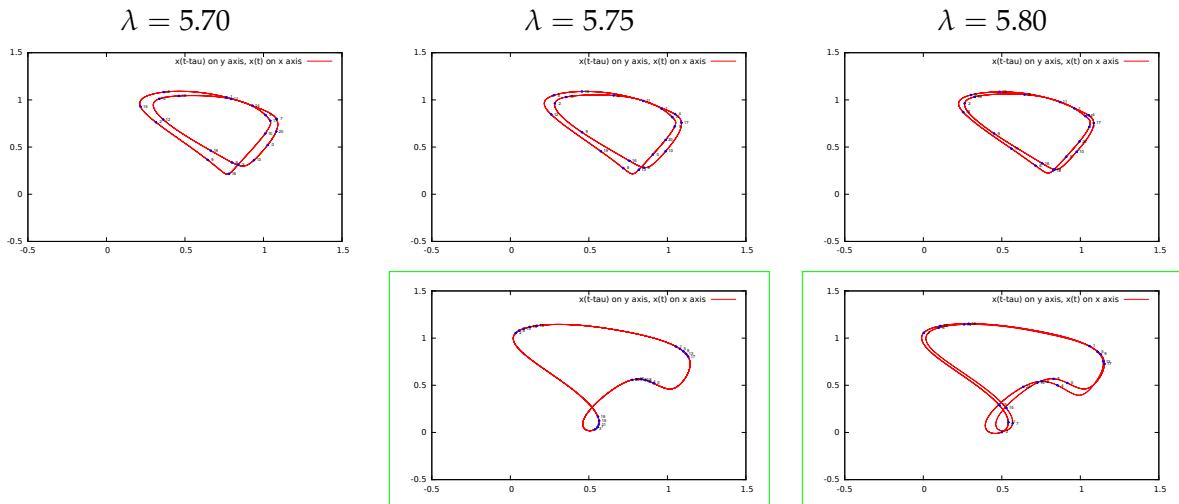


Table 4.1: Numerical investigations of logistic equation (1.1) with various λ . With a green box we have marked periodic solutions we were able to prove rigorously. We see that the double oscillation in the first orbit (upper row) shrinks. Probably, it is the reason why we were not able to prove their existence, as they get closer and closer to the unstable periodic orbit, as seen in Table 4.3.

5 Summary

In this work we have applied a recently developed rigorous procedure for forward in time integration of DDEs to prove existence of several periodic orbits to the “logistic” equation (1.1), for which it is difficult to obtain such results by analytical means.

There is much work left to be done. First of all, we were not able to prove the existence of two coexisting attracting periodic orbits, observed numerically in [16]. A possible approach

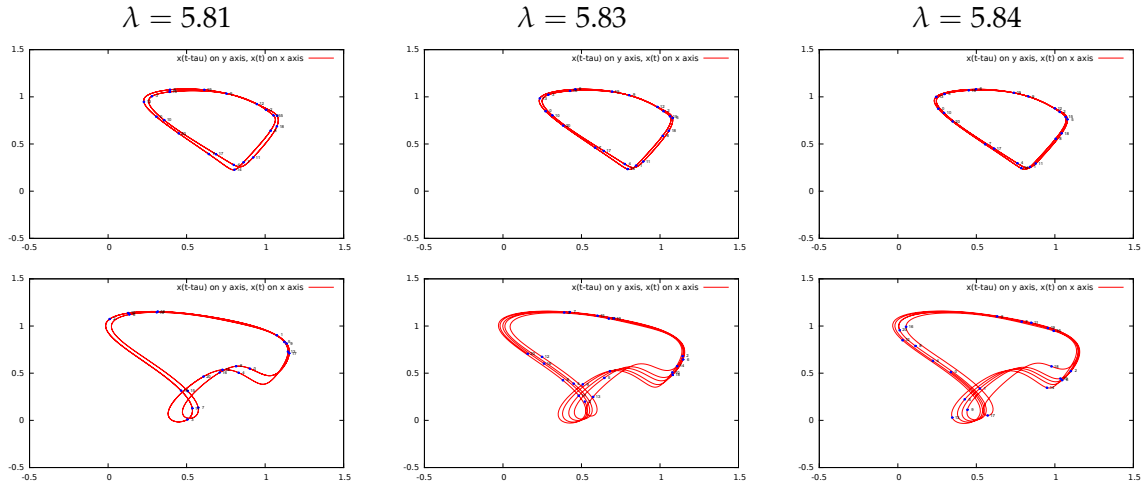


Table 4.2: Numerically attracting periodic orbits in (1.1) for higher values of the λ parameter. Orbits were obtained by integration of the initial conditions (4.1) and (4.2) (upper and lower row respectively) forward in time up to $t = 100$ (i.e. $100 \cdot \tau$). Then, the pictures were obtained by plotting the numerical solution over time $t \in [0, 20]$. It seems that for $\lambda > 5.83$ a chaotic attractor appeared. However, using Newton algorithm, we were able to find apparently unstable periodic orbit as seen in Table 4.3.

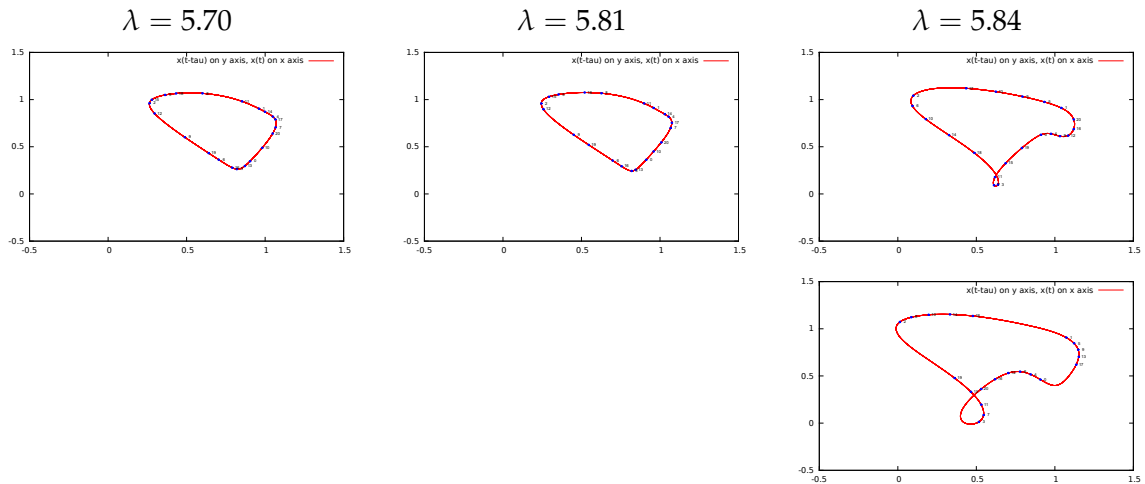


Table 4.3: Some apparently unstable periodic orbits (with one unstable direction) in (1.1) for various λ .

would be to increase the size of the (p, n) -representation, but for this we need to optimise our algorithms. We have already made some suggestions in [22].

Further, we do not have tools to prove that the existing orbits are in fact attracting. The tools for this are already present for ODEs [30] and in the future work we will try to translate them to the DDE setting.

An additional future goal is to establish existence of unstable periodic orbits. We have shown that our algorithms may be used to find good numerical approximations to such orbits

(examples in Table 4.3). The tools needed in the context of infinite dimensional flows to prove existence of unstable hyperbolic periodic solutions are available [31] and it should be easy to incorporate them in our algorithms.

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