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Existence of homoclinic orbits for unbounded time-dependent p-Laplacian systems

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Abstract. In this paper, we consider the following ordinary p-Laplacian system

$$\frac{d}{dt}\big(|\dot{u}(t)|^{p-2}\dot{u}(t)\big) - \nabla K(t,u(t)) + \nabla W(t,u(t)) = f(t), \qquad (HS)$$

where $t \in \mathbb{R}$ and $p > 1$. Using the Mountain Pass Theorem, we establish the existence of a nontrivial homoclinic solution for (HS) under new assumptions on the growth of the potential which allow $W(t,x)$ to be either super p-linear or asymptotically p-linear at infinity. Also, contrary to previous works, $W(t,x)$ will be neither periodic nor bounded with respect to the variable t. Recent results in the literature are generalized even if $p = 2$.

Keywords: homoclinic solutions, Hamiltonian systems, Mountain Pass Theorem, p-Laplacian systems.

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1 Introduction

Consider the ordinary p-Laplacian system

$$\frac{d}{dt}\big(|\dot{u}(t)|^{p-2}\dot{u}(t)\big) - \nabla K(t,u(t)) + \nabla W(t,u(t)) = f(t) \qquad (HS)$$

where $t \in \mathbb{R}$, $p > 1$, K, $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are C^1-maps and $f : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a continuous and bounded function. We will say that a solution u of (HS) is a nontrivial homoclinic (to 0) if $u \not\equiv 0$ and $u(t) \longrightarrow 0$ as $t \longrightarrow \pm\infty$.

When $p = 2$, (HS) reduces to the following second order Hamiltonian system

$$\ddot{u}(t) - \nabla K(t,u(t)) + \nabla W(t,u(t)) = f(t). \qquad (1.1)$$

Homoclinic orbits were introduced by Poincaré more than a century ago, and since then, they became a fundamental tool in the study of chaos. Their existence has been extensively

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investigated in the last two decades in many papers via critical point theory. In particular, the following second-order systems were considered in many works (see [1, 3–8, 11, 13, 16, 17, 19, 23, 26])

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0 \tag{1.2}$$

where $L(t)$ is a symmetric matrix valued function. Later, the authors of [7] introduced the more general system (1.1) where the quadratic function $(L(t)x, x)$ is replaced by $K(t, x)$.

Most of the previous works treat the superquadratic case under the global Ambrosetti–Rabinowitz condition, i.e., *there exists* $\mu > 2$ *such that*

$$0 < \mu W(t, x) \leq (\nabla W(t, x), x), \quad \textit{for all } (t, x) \in \mathbb{R} \times \mathbb{R}^N \backslash \{0\}. \tag{AR}$$

Moreover, they suppose that $L(t)$ and $W(t, x)$ are either periodic in t or independent of t.

In the case where $W(t, x)$ and $L(t)$ or $K(t, x)$ are neither autonomous nor periodic, $\nabla W(t, x)$ is usually bounded with respect to the first variable. Indeed, a variant of the following condition is used:

There is a function $\overline{W} \in C(\mathbb{R}^N, \mathbb{R})$ *such that*

$$|W(t, x)| + |\nabla W(t, x)| \leq |\overline{W}(x)|, \quad \textit{for all } (t, x) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.3}$$

In a recent paper the authors of [9] studied the problem (HS) under new superquadratic conditions which allow W to be neither periodic nor bounded in t. Particularly, they suppose

$$K(t, x) = a(t)|x|^p, \quad \text{with } a(t) \longrightarrow +\infty \text{ as } |t| \longrightarrow +\infty$$

and

$$W = W_1 - W_2 \in C^1(\mathbb{R}^N, \mathbb{R}),$$

where the functions W_1, W_2 satisfy some (AR)-type conditions to be either increasing or decreasing (see [9], Lemma 2.5).

Motivated by the above mentioned works, in the present paper we study the existence of homoclinic solutions for (HS) under more general conditions which cover the case of unbounded potentials with respect to the variable t. Here, to overcome the difficulty due to the unboundedness of the domain, a homoclinic solution will be obtained as a limit of a sequence of solutions of some nil-boundary-value problem. The existence of such sequence of solutions is guaranteed through a standard version of the Mountain Pass Theorem. Furthermore, the forcing term f satisfies an easier condition compared to that given in [12, 20] mainly. Our results complete and improve recent works in the literature even in the case $p = 2$.

Precisely, we suppose:

(H_1) there exist $\gamma \in (1, p]$ and $a > 0$ such that

$$a|x|^\gamma \leq K(t, x) \leq (x, \nabla K(t, x)) \leq pK(t, x), \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

(H_2) $W(t, 0) = 0$ and $\nabla W(t, x) = o(|x|^{p-1})$, as $|x| \longrightarrow 0$ uniformly in $t \in \mathbb{R}$,

(H_3) there exists $T_0 > 0$ such that

$$\liminf_{|x| \longrightarrow \infty} \frac{W(t, x)}{|x|^p} > \frac{\pi^p}{pT_0^p} + m_1, \quad \text{uniformly in } t \in [-T_0, T_0],$$

where $m_1 = \sup\{K(t, x) | t \in [-T_0, T_0], |x| = 1\}$,

(H_4) there exist constants $\mu > p$, $0 \le b < \mu - p$ and $\beta \in L^1(\mathbb{R}, \mathbb{R}+)$ such that

$$\mu W(t,x) - (\nabla W(t,x), x) \le bK(t,x) + \beta(t), \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Remark 1.1. Note that, by (H_2), there exists $0 < \rho_0 < 1$ such that

$$|W(t,x)| \le \frac{a}{p}|x|^p, \qquad \forall\ |x| \le \rho_0,\ t \in \mathbb{R}. \tag{1.4}$$

Now, we state our main results.

Theorem 1.2. *Assume that W and K satisfy (H_1)–(H_4) and*

(H_5) $0 < \int_{\mathbb{R}} |f(t)|^q dt < \left(\frac{\min\{1, a(p-1)\}}{p}\right)^q (\rho_0/2)^p$, *where* $\frac{1}{q} + \frac{1}{p} = 1.$

Then (HS) possesses a nontrivial homoclinic solution $u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$.

Example 1.3. Consider the functions

$$W(t,x) = \frac{|x|^3[e^{t^2(|x|^2-1)} - 1]}{t^2+1}, \qquad K(t,x) = (2 + \sin t)|x|^2 + \cos^2 t\ |x|^{5/2}.$$

A straightforward computation shows that W and K satisfy the assumptions of Theorem 1.2, with $\gamma = 2, p = 5/2, \mu = 3$ but W does not satisfy neither (*AR*) nor (1.3). Moreover, contrary to [6,24], we have $\inf_{t\in\mathbb{R},|x|=1} W(t,x) = \sup_{t\in\mathbb{R},|x|=1} W(t,x) = 0$. Note also that W changes sign near the origin. Hence, Theorem 1.2 extends and completes the results in [3,6,12,16,19,24,25].

Corollary 1.4. *Assume that W and K satisfy (H_1)–(H_3), (H_5) and*

(H_4') *there exist constants $\mu > p, 0 \le c < a(\mu - p)$ such that*

$$\mu W(t,x) \le (\nabla W(t,x), x) + c|x|^\gamma, \qquad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Then (HS) possesses a nontrivial homoclinic solution $u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$.

Remark 1.5. It is easy to see that (H_4') implies (H_4). However, the condition (H_4') is weaker than (H_4) in [12]. So, Corollary 1.4 significantly improves Theorem 1.1 in [12].

Corollary 1.6. *Assume that W and K satisfy (H_1), (H_2), (H_4), (H_5) and*

(H_3') $$\liminf_{|x|\longrightarrow\infty} \frac{W(t,x)}{|x|^p} > m_1, \quad \textit{uniformly in } t \in \mathbb{R}.$$

Then (HS) possesses a nontrivial homoclinic solution $u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$.

2 Preliminary results

Consider for each $T > 0$ the following problem

$$\begin{cases} \frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) - \nabla K(t,u(t)) + \nabla W(t,u(t)) = f_T(t), & \text{for } t \in [-T,T] \\ u(-T) = u(T) = 0, \end{cases} \tag{2.1}$$

where f_T is the function defined on $\mathbb{R}$ by

$$f_T(t) := \begin{cases} f(t) & \text{for } t \in [-T,T], \\ 0 & \text{for } t \in \mathbb{R} \setminus [-T,T]. \end{cases}$$

Let

$$E_T := W^{1,p}([-T,T],\mathbb{R}^N) = \Big\{ u : [-T,T] \longrightarrow \mathbb{R}^N \text{ is absolutely continuous function,} \\ u(-T) = u(T) = 0 \text{ and } \dot{u} \in L^p([-T,T],\mathbb{R}^N) \Big\}$$

equipped with the norm

$$\|u\|_{E_T} = \left[\int_{-T}^{T} \Big(|\dot{u}(t)|^p + |u(t)|^p \Big) dt \right]^{\frac{1}{p}}.$$

Furthermore, for $\alpha \geq 1$, let $L_T^\alpha = L^\alpha([-T,T],\mathbb{R}^N)$ and $L_T^\infty = L^\infty([-T,T],\mathbb{R}^N)$ with their usual norms. Let $\eta_T : E_T \longrightarrow [0,+\infty)$ given by

$$\eta_T(u) = \left[\int_{-T}^{T} \Big(|\dot{u}(t)|^p + pK(t,u(t)) \Big) dt \right]^{1/p},$$

and $I_T : E_T \longrightarrow \mathbb{R}$, be defined by

$$I_T(u) = \frac{1}{p}\eta_T^p(u) - \int_{-T}^{T} W(t,u(t))dt + \int_{-T}^{T} (f_T(t),u(t))dt. \tag{2.2}$$

Then $I_T \in C^1(E_T,\mathbb{R})$ and it's easy to show that for all $u,v \in E_T$, we have

$$I_T'(u)v = \int_{-T}^{T} \Big[(|\dot{u}|^{p-2}\dot{u}(t),\dot{v}(t)) + (\nabla K(t,u(t)),v(t)) - (\nabla W(t,u(t)),v(t)) \Big] dt \\ + \int_{-T}^{T} (f_T(t),v(t))dt.$$

By (H_1), we obtain, for all $u \in E_T$

$$I_T'(u)u \leq \eta_T^p(u) - \int_{-T}^{T} (\nabla W(t,u(t)),u(t))dt + \int_{-T}^{T} (f_T(t),u(t))dt. \tag{2.3}$$

It is well known that critical points of I_T are classical solutions of the problem (2.1), (see [2,14]). We will obtain a critical point of I_T by using a standard version of the Mountain Pass Theorem. It provides the minimax characterization for the critical value which is important for what follows. For completeness, we give this theorem.

Theorem 2.1 ([18]). *Let E be a real Banach space and $I : E \longrightarrow \mathbb{R}$ be a C^1-smooth functional satisfies the Palais–Smale condition. If I satisfies the following conditions:*

(I_1) $I(0) = 0$,

(I_2) *there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$,*

(I_3) *there exists $e \in E \backslash \bar{B}_\rho(0)$ such that $I(e) \leq 0$, where $B_\rho(0)$ is an open ball in E of radius ρ centered at 0,* *then I possesses a critical value $c \geq \alpha$ given by*

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s))$$

where

$$\Gamma = \{g \in C([0,1], E); g(0) = 0, g(1) = e\}.$$

Next we need an extension to the p-case of the following proposition first proved by Rabinowitz in [16].

Lemma 2.2 ([12, 22])**.** *Let $u : \mathbb{R} \longrightarrow \mathbb{R}^N$ be a continuous map such that $\dot{u} \in L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$. Then, for all $t \in \mathbb{R}$, we have*

$$|u(t)| \leq 2^{\frac{p-1}{p}} \left[\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|u(s)|^p + |\dot{u}(s)|^p) ds \right]^{\frac{1}{p}}. \tag{2.4}$$

Corollary 2.3. *For all $u \in E_T$ the following inequality holds:*

$$\|u\|_{L^\infty_T} \leq 2^{\frac{p-1}{p}} \left(1 + \left[\frac{1}{2T}\right]\right)^{1/p} \|u\|_{E_T}. \tag{2.5}$$

Remark 2.4. Note that for $T \geq \frac{1}{2}$ we have $2^{\frac{p-1}{p}} (1 + [\frac{1}{2T}])^{1/p} \leq 2$. So, from (2.5), we get

$$\|u\|_{L^\infty_T} \leq 2\|u\|_{E_T}, \quad \text{for all } u \in E_T. \tag{2.6}$$

Subsequently, we may assume this condition fulfilled.

Lemma 2.5. *Assume that (H_1) holds, then for all $t \in \mathbb{R}$, we have*

$$K(t,x) \leq K\left(t, \frac{x}{|x|}\right) |x|^p, \quad \text{if } |x| \geq 1. \tag{2.7}$$

The proof of Lemma 2.5 is a routine so we omit it.

3 Proof of Theorem 1.2

Lemma 3.1. *Under the assumptions of Theorem 1.2, the problem (2.1) possesses a nontrivial solution $u_T \in E_T$.*

Proof. It suffices to prove that the functional I_T satisfies all the assumptions of the Mountain Pass Theorem.

Step 1. The functional I_T satisfies the (PS)-condition, i.e., for every constant c and sequence $\{u_n\} \subset E$ such that $I_T(u_n) \longrightarrow c$ and $I'_T(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$, $\{u_n\}$ has a convergent subsequence. Indeed, let $\{u_n\} \subset E_T$ is a (PS)-sequence of I_T. By (2.2) and (2.3) there exists $M_T > 0$ such that

$$\begin{aligned} M_T(1 + \|u_n\|_{E_T}) &\geq \mu I_T(u_n) - I'_T(u_n)u_n \\ &\geq \left(\frac{\mu}{p} - 1\right) \eta_T^p(u_n) + \int_{-T}^{T} \Big[(\nabla W(t, u_n(t)), u_n(t)) - \mu W(t, u_n(t))\Big] dt \\ &\quad + (\mu - 1) \int_{-T}^{T} (f_T(t), u_n(t)) dt. \end{aligned} \tag{3.1}$$

Using (H_4) and Hölder's inequality, from (3.1), we get

$$\begin{aligned}\left(\frac{\mu}{p}-1\right)\eta_T^p(u_n) &\le M_T(1+\|u_n\|_{E_T})+b\int_{-T}^{T}K(t,u_n(t))dt+\int_{-T}^{T}\beta(t)dt\\ &\quad+(\mu-1)\|f\|_{L^q(\mathbb{R},\mathbb{R}^N)}\|u_n\|_{E_T}\\ &\le M_T(1+\|u_n\|_{E_T})+\frac{b}{p}\eta_T^p(u_n)+\int_{-T}^{T}\beta(t)dt+(\mu-1)\|f\|_{L^q(\mathbb{R},\mathbb{R}^N)}\|u_n\|_{E_T},\end{aligned}$$

which yields

$$\left(\frac{\mu-b}{p}-1\right)\eta_T^p(u_n)\le M_T(1+\|u_n\|_{E_T})+\int_{-\infty}^{\infty}\beta(t)dt+(\mu-1)\|f\|_{L^q(\mathbb{R},\mathbb{R}^N)}\|u_n\|_{E_T}. \tag{3.2}$$

Without loss of generality, we can assume that $\|u_n\|_{E_T}\neq 0$. Then from (H_1) and (2.6) we get

$$\begin{aligned}\eta_T^p(u_n) &= \int_{-T}^{T}\Big[|\dot{u}_n(t)|^p+pK(t,u_n(t))\Big]dt\\ &\ge \int_{-T}^{T}|\dot{u}(t)|^pdt+pa\int_{-T}^{T}|u_n(t)|^\gamma dt\\ &\ge \int_{-T}^{T}|\dot{u}(t)|^pdt+pa(2\|u_n\|_{E_T})^{\gamma-p}\int_{-T}^{T}|u_n(t)|^pdt\\ &\ge \min\Big\{1,pa(2\|u_n\|_{E_T})^{\gamma-p}\Big\}\|u_n\|_{E_T}^p\\ &\ge \min\Big\{\|u_n\|_{E_T}^p,pa2^{\gamma-p}\|u_n\|_{E_T}^\gamma\Big\}.\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) we obtain

$$\begin{aligned}&\left(\frac{\mu-b-p}{p}\right)\min\{\|u_n\|_{E_T}^p,pa2^{\gamma-p}a\|u_n\|_{E_T}^\gamma\}\\ &\qquad\le M_T(1+\|u_n\|_{E_T})+\int_{-\infty}^{\infty}\beta(t)dt+(\mu-1)\|f\|_{L^q(\mathbb{R},\mathbb{R}^N)}\|u_n\|_{E_T}.\end{aligned}$$

From (H_4), we know that $\mu-b-p>0$, then the sequence $\{u_n\}$ is bounded in E_T. In a similar way to Lemma 2 in [19], we can prove that $\{u_n\}$ has a convergent subsequence in E_T. Hence I_T satisfies the (PS)-condition.

Step 2. The functional I_T satisfies the condition (I_2) of the Mountain Pass Theorem.

Let $\rho=\frac{\rho_0}{2}$ and $q\in E_T$, such that $\|u\|_{E_T}=\rho$, then $0<\|u\|_{L_T^\infty}\le\rho_0$. By (1.4) we have

$$\int_{-T}^{T}W(t,u(t))dt\le\frac{a}{p}\int_{-T}^{T}|u(t)|^pdt. \tag{3.4}$$

On the other hand, since $\gamma\le p$, by (H_1), we have

$$\int_{-T}^{T}K(t,u(t))dt\ge a\int_{-T}^{T}|u(t)|^\gamma dt\ge a\int_{-T}^{T}|u(t)|^pdt. \tag{3.5}$$

Then, by (2.2), (3.4), (3.5) and Hölder's inequality it follows that

$$\begin{aligned} I_T(u) &\geq \frac{1}{p}\eta_T^p(u) - \frac{a}{p}\int_{-T}^{T}|u(t)|^p dt - \|f\|_{L^q(\mathbb{R},\mathbb{R}^N)}\|u\|_{E_T} \\ &\geq \frac{1}{p}\int_{-T}^{T}\Big[|\dot{u}(t)|^p + pK(t,u(t))\Big]dt - \frac{a}{p}\int_{-T}^{T}|u(t)|^p dt - \|f\|_{L^q(\mathbb{R},\mathbb{R}^N)}\|u\|_{E_T} \\ &\geq \frac{1}{p}\int_{-T}^{T}|\dot{u}(t)|^p dt + \frac{a(p-1)}{p}\int_{-T}^{T}|u(t)|^p dt - \|f\|_{L^q(\mathbb{R},\mathbb{R}^N)}\|u\|_{E_T} \\ &\geq \min\left\{\frac{1}{p}, \frac{a(p-1)}{p}\right\}\|u\|_{E_T}^p - \|f\|_{L^q(\mathbb{R},\mathbb{R}^N)}\|u\|_{E_T} \\ &\geq \frac{\min\{1, a(p-1)\}}{p}\rho^p - \|f\|_{L^q(\mathbb{R},\mathbb{R}^N)}\rho =: \alpha. \end{aligned} \tag{3.6}$$

From (H_5), it follows that $\alpha > 0$ and (3.6) shows that $\|u\|_{E_T} = \rho$ implies $I_T(u) \geq \alpha$.

Step 3. The functional I satisfies the condition (I_3) of the Mountain Pass Theorem.

Let $m_2 = \sup\big\{K(t,x) | t \in [-T_0, T_0], |x| \leq 1\big\}$. From (2.7), it is easy to see that

$$K(t,x) \leq m_1|x|^p + m_2, \quad \text{for all } t \in [-T_0, T_0],\ x \in \mathbb{R}^N. \tag{3.7}$$

Furthermore, by (H_3), there exists $\varepsilon_0 > 0$ and $r > 0$ such that

$$\frac{W(t,x)}{|x|^p} \geq \frac{\pi^p + \varepsilon_0}{pT_0^p} + m_1, \quad \text{for all } t \in [-T_0, T_0],\ |x| > r.$$

Let $\delta = \max\big\{(\frac{\pi^p+\varepsilon_0}{pT_0^p} + m_1)|x|^p - W(t,x)\ | t \in [-T_0, T_0], |x| \leq r\big\}$, hence we have

$$W(t,x) \geq \left(\frac{\pi^p + \varepsilon_0}{pT_0^p} + m_1\right)|x|^p - \delta, \quad \text{for all } t \in [-T_0, T_0],\ x \in \mathbb{R}^N. \tag{3.8}$$

For $T \geq T_0$, define

$$e(t) = \begin{cases} \xi|\sin(\omega t)|e_1 & \text{if } t \in [-T_0, T_0] \\ 0 & \text{if } t \in [-T, T] \setminus [-T_0, T_0]. \end{cases} \tag{3.9}$$

where $\omega = \frac{\pi}{T_0}$ and $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$. Then by (2.2) and (3.7)–(3.9), we obtain

$$\begin{aligned} I_T(e) &= \int_{-T}^{T}\left[\frac{1}{p}|\dot{e}(t)|^p + K(t,e(t)) - W(t,e(t))\right]dt + \int_{-T}^{T}(f_T(t), e(t))dt \\ &= \int_{-T_0}^{T_0}\left[\frac{1}{p}|\dot{e}(t)|^p + K(t,e(t)) - W(t,e(t))\right]dt + \int_{-T_0}^{T_0}(f_{T_0}(t), e(t))dt \\ &\leq \frac{|\xi|^p\omega^p}{p}\int_{-T_0}^{T_0}|\cos(\omega t)|^p dt + m_1|\xi|^p\int_{-T_0}^{T_0}|\sin(\omega t)|^p dt \\ &\quad - \left(\frac{\pi^p + \varepsilon_0}{pT_0^p} + m_1\right)|\xi|^p\int_{-T_0}^{T_0}|\sin(\omega t)|^p dt \\ &\quad + |\xi|\|f\|_{L^q}\left[\int_{-T_0}^{T_0}|\sin(\omega t)|^p dt\right]^{1/p} + 2T_0(\delta + m_2) \\ &\leq -\frac{\varepsilon_0}{pT_0^p}|\xi|^p\int_{-T_0}^{T_0}|\sin(\omega t)|^p dt + |\xi|\|f\|_{L^q}\left[\int_{-T_0}^{T_0}|\sin(\omega t)|^p dt\right]^{1/p} \\ &\quad + 2T_0(\delta + m_2) \longrightarrow -\infty, \quad \text{as } \xi \longrightarrow \infty. \end{aligned}$$

Thus, we can choose ξ large enough such that $\|e\|_{E_T} > \rho$ and $I_T(e) < 0$.

For our setting, clearly $I_T(0) = 0$, then, by application of the Mountain Pass Theorem, there exists a critical point $u_T \in E_T$ of I_T such that $I_T(u_T) \geq \alpha$ for all $T \geq T_0$. □

Lemma 3.2. *u_T is bounded uniformly for $T \geq T_0$.*

Proof. Define the set of paths

$$\Gamma_T = \{g \in C([0,1], E_T) \mid g(0) = 0,\ g(1) = e\}.$$

By Lemma 3.1, we know that there is a solution u_T of (2.1) at which

$$\inf_{g\in\Gamma_T} \max_{s\in[0,1]} I_T(g(s)) \equiv N_T$$

is achieved. Let now $\widetilde{T} > T$, then $\Gamma_T \subset \Gamma_{\widetilde{T}}$, since any function in E_T can be regarded as belonging to $E_{\widetilde{T}}$ if one extends it by zero in $[-\widetilde{T}, \widetilde{T}] \setminus [-T, T]$. Therefore, for all solution u_T of (2.1), we get

$$I_T(u_T) = N_T \leq N_{T_0} \quad \text{uniformly in } T \geq T_0. \tag{3.10}$$

Using the fact that $I_T'(u_T) = 0$ and (3.10), the rest of the proof is identical to Step 1 in Lemma 3.1. Hence there exists a constant $M_0 > 0$, independent of T such that

$$\|u_T\|_{E_T} \leq M_0, \quad \text{for all } T \geq T_0.$$

This ends the proof of Lemma 3.2. □

Now, take an increasing sequence $T_n \longrightarrow \infty$ with $T_1 > T_0$ and consider the problem (2.1) on the interval $[-T_n, T_n]$. By the conclusion of Lemma 3.1 and Lemma 3.2, there exists a nontrivial solution $u_n := u_{T_n}$ of (2.1) satisfying

$$\|u_n\|_{E_{T_n}} \leq M_0, \quad \text{for all } n \in \mathbb{N}. \tag{3.11}$$

Lemma 3.3. *Let $(u_n)_{n\in\mathbb{N}}$ be the sequence given above. Then there exists a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ convergent to a certain function u_0 in $C^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$.*

Proof. First of all from (2.6) and (3.11), we have

$$\|u_n\|_{L^\infty_{T_n}} \leq 2M_0 \equiv M_1 \tag{3.12}$$

and

$$\left(\int_{-T_n}^{T_n} |\dot{u}_n(t)|^p\right)^{1/p} \leq \|u_n\|_{E_{T_n}} \leq M_0 \tag{3.13}$$

for all $n \in \mathbb{N}$. By Hölder's inequality and (3.11), for $t_1, t_2 \in [-T_1, T_1]$ with $t_1 < t_2$,

$$|u_n(t_2) - u_n(t_1)| = \left|\int_{t_1}^{t_2} \dot{u}_n(t)dt\right| \leq (t_2 - t_1)^{1/q} \left(\int_{-T_1}^{T_1} |\dot{u}_n(t)|^p\right)^{1/p} \leq M_0(t_2 - t_1)^{1/q}. \tag{3.14}$$

From (3.12) and (3.14) the sequence $\{u_n\}_{n\in\mathbb{N}}$ is equicontinuous and uniformly bounded on $[-T_1, T_1]$. By the Arzelà–Ascoli Theorem, there exists a uniformly convergent subsequence $\{u^1_{n_j}\}_{j\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ on $[-T_1, T_1]$ and we can choose $n_1 > 1$.

Consider $\{u^1_{n_j}\}_{j\in\mathbb{N}}$ on $[-T_2, T_2]$. By the Arzelà–Ascoli Theorem, there exists again a uniformly convergent subsequence $\{u^2_{n_j}\}_{j\in\mathbb{N}}$ of $\{u^1_{n_j}\}_{j\in\mathbb{N}}$ on $[-T_2, T_2]$ with n_1 in $u^2_{n_1}$ satisfies

$n_1 > 2$. Repeat this procedure for all $i \geq 1$ and take the diagonal subsequence of $\{u^i_{n_j}\}_{j\in\mathbb{N}}$, $i \geq 1$, which consists of $u^1_{n_1}, u^2_{n_2}, u^3_{n_3}, \ldots$ It follows that this diagonal subsequence converges uniformly on any bounded interval to a certain function u_0.

Next, we denote u_n instead of $u^j_{n_j}$. Let I be a bounded interval, there exists n_0 such that $I \subset [-T_{n_0}, T_{n_0}]$. Using (2.1), for all $t \in I$, we have

$$
\begin{aligned}
|\frac{d}{dt}(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t)| &\leq |\nabla K(t, u_n(t))| + |\nabla W(t, u_n(t))| + |f_n(t)| \\
&\leq |\nabla K(t, u_n(t))| + |\nabla W(t, u_n(t))| + |f(t)|,
\end{aligned}
$$

for $n \geq n_0$ where here and subsequently $f_n = f_{T_n}$. Since f is bounded , by (3.12) there exists $M_2 > 0$ (dependent on I) such that

$$
\sup_{t\in I} \left| \frac{d}{dt}(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t)) \right| \leq M_2, \qquad \forall n \geq n_0. \tag{3.15}
$$

From the Mean Value Theorem it follows that for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$ there exists $\tau_n \in [t-1, t]$ such that

$$
\dot{u}_n(\tau_n) = \int_{t-1}^{t} \dot{u}_n(s)ds = u_n(t) - u_n(t-1).
$$

Combining the above with (3.12) and (3.15) we obtain

$$
\begin{aligned}
\big||\dot{u}_n(t)|^{p-2}\dot{u}_n(t)\big| &= \left| \int_{\tau_n}^{t} \frac{d}{dt}(|\dot{u}_n(s)|^{p-2}\dot{u}_n(s)ds + |\dot{u}_n(\tau_n)|^{p-2}\dot{u}_n(\tau_n) \right| \\
&\leq \int_{t-1}^{t} \left| \frac{d}{dt}(|\dot{u}_n(s)|^{p-2}\dot{u}_n(s) \right| ds + |\dot{u}_n(\tau_n)|^{p-1} \leq M_2 + (2M_1)^{p-1} \equiv M_3^{p-1},
\end{aligned}
$$

and hence

$$
\sup_{t\in I} |\dot{u}_n(t)| \leq M_3, \qquad \forall\, n \geq n_0. \tag{3.16}
$$

Now we prove that the sequence $\{\dot{u}_n\}_{n\in\mathbb{N}}$ is equicontinuous on I. If not, there exist $\epsilon > 0$, two sequences $\{t^1_i\}_{i\in\mathbb{N}} \subset I$, $\{t^2_i\}_{i\in\mathbb{N}} \subset I$ and a sequence $\{n_i\}_{i\in\mathbb{N}}$ of integers such that

$$
0 < t^2_i - t^1_i < \frac{1}{i}, \qquad |u_{n_i}(t^2_i) - u_{n_i}(t^1_i)| \geq \epsilon, \quad \text{and} \quad n_i \geq n_0, \quad i \in \mathbb{N}. \tag{3.17}
$$

Since the sequences $\{u_{n_i}(t^1_i)\}$ and $\{u_{n_i}(t^1_i)\}$ are bounded, passing, if necessary, to subsequences, one can assume that

$$
u_{n_i}(t^1_i) \longrightarrow \alpha_1, \quad \text{and} \quad u_{n_i}(t^2_i) \longrightarrow \alpha_2, \quad \text{as } i \longrightarrow \infty. \tag{3.18}
$$

Combining (3.17) and (3.18), we get

$$
|\alpha_2 - \alpha_1| \geq \epsilon. \tag{3.19}
$$

On the other hand, from (3.15) and (3.17), we have

$$
\begin{aligned}
\left| |u_{n_i}(t^2_i)|^{p-2}u_{n_i}(t^2_i) - |u_{n_i}(t^1_i)|^{p-2}u_{n_i}(t^1_i) \right| &= \left| \int_{t^1_i}^{t^2_i} \frac{d}{dt}(|\dot{u}_n(s)|^{p-2}\dot{u}_n(s)ds \right| \\
&\leq \int_{t^1_i}^{t^2_i} \left| \frac{d}{dt}(|\dot{u}_n(s)|^{p-2}\dot{u}_n(s) \right| ds \\
&\leq M_2(t^2_i - t^1_i) \leq \frac{M_2}{i}, \qquad i \in \mathbb{N}.
\end{aligned} \tag{3.20}
$$

Passing to the limit in (3.20) and using (3.18), we obtain

$$\left||\alpha_2|^{p-2}\alpha_2 - |\alpha_1|^{p-2}\alpha_1\right| = 0$$

and consequently $\alpha_1 = \alpha_2$, which contradicts (3.19). Thus, $\{\dot{u}_n\}_{n\in\mathbb{N}}$ is equicontinuous. By (3.16), $\{\dot{u}_n\}_{n\in\mathbb{N}}$ is also uniformly bounded on I, the Arzelà–Ascoli Theorem proves the existence of a subsequence convergent to a certain function v. Since the interval I is arbitrary we conclude that according to a subsequence

$$u_{n_j} \longrightarrow u, \quad \text{as} \quad j \longrightarrow \infty \quad \text{in } C^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N).$$

Lemma 3.3 is proved. □

Lemma 3.4. *Let $u_0 : \mathbb{R} \longrightarrow \mathbb{R}^N$ be the function given by Lemma 3.3. Then u_0 is the desired homoclinic solution of (HS).*

Proof. The first step is to show that u_0 is a solution of (*HS*). Let $(u_{n_j})_{j\in\mathbb{N}}$ be the sequence given by Lemma 3.3, then

$$\frac{d}{dt}(|\dot{u}_{n_j}(t)|^{p-2}\dot{u}_{n_j}(t)) - \nabla K(t, u_{n_j}(t)) + \nabla W(t, u_{n_j}(t)) = f_{n_j}(t)$$

for every $j \in \mathbb{N}$, and $t \in [-T_{n_j}, T_{n_j}]$. Take $a, b \in \mathbb{R}$ with $a < b$. There exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$ one has $[a, b] \subset [-T_{n_j}, T_{n_j}]$ and

$$\frac{d}{dt}(|\dot{u}_{n_j}(t)|^{p-2}\dot{u}_{n_j}(t)) = \nabla K(t, u_{n_j}(t)) - \nabla W(t, u_{n_j}(t)) + f(t), \qquad \forall\, t \in [a, b]. \tag{3.21}$$

Integrating (3.21) from a to $t \in [a, b]$, we obtain

$$|\dot{u}_{n_j}(t)|^{p-2}\dot{u}_{n_j}(t) - |\dot{u}_{n_j}(a)|^{p-2}\dot{u}_{n_j}(a) = \int_a^t \Big[\nabla K(s, u_{n_j}(s)) - \nabla W(s, u_{n_j}(s)) + f(s)\Big] ds, \qquad \forall\, t \in [a, b]. \tag{3.22}$$

Since $u_{n_j} \longrightarrow u_0$ uniformly on $[a, b]$, and $\dot{u}_{n_j} \longrightarrow \dot{u}_0$ uniformly on $[a, b]$ as $j \longrightarrow \infty$, then from (3.22), we get

$$|\dot{u}_0(t)|^{p-2}\dot{u}_0(t) - |\dot{u}_0(a)|^{p-2}\dot{u}_0(a) = \int_a^t \Big[\nabla K(s, u_0(s)) - \nabla W(s, u_0(s)) + f(s)\Big] ds, \qquad \forall\, t \in [a, b]. \tag{3.23}$$

Since a and b are arbitrary, we receive from (3.23) that u_0 satisfies (HS).

Now we prove that $u_0(t) \longrightarrow 0$, as $|t| \longrightarrow \infty$. First of all note that, from (3.11), for $l \in \mathbb{N}$ there exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$, we have

$$\int_{-T_{n_l}}^{T_{n_l}} (|u_{n_j}(t)|^p + |\dot{u}_{n_j}(t)|^p) dt \le \|u_{n_j}\|^p_{E_{T_{n_j}}} \le M_0^p.$$

Letting $j \longrightarrow \infty$, we get

$$\int_{-T_{n_l}}^{T_{n_l}} (|u_0(t)|^p + |\dot{u}_0(t)|^p) dt \le M_0^p,$$

and now, letting $l \longrightarrow \infty$, we obtain

$$\int_{-\infty}^{+\infty} (|u_0(t)|^p + |\dot{u}_0(t)|^p)dt \leq M_0^p,$$

and so

$$\int_{|t|\geq r} (|u_0(t)|^p + |\dot{u}_0(t)|^p)dt \longrightarrow 0, \quad \text{as } r \longrightarrow \infty. \tag{3.24}$$

From (2.4) and (3.24), we receive our claim. □

Finally, it is obvious that u_0 is nontrivial since, from (H_5), we have $f \not\equiv 0$ and the proof of Theorem 1.2 is complete.

Remark 3.5. Under the assumptions of Theorem 1.2 and
(H_6) there is $R > 0$ such that

$$\nabla K(t,x) \longrightarrow 0 \text{ as } |x| \longrightarrow 0 \quad \text{uniformly in } t \in (-\infty, -R] \cup [R, +\infty),$$

the homoclinic solution u_0 obtained above satisfies $\dot{u}_0(t) \longrightarrow 0$, as $|t| \longrightarrow \infty$. The proof is analogous to [21].

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