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Non-monotone positive solutions of second-order linear differential equations: existence, nonexistence and criteria

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Abstract. We study non-monotone positive solutions of the second-order linear differential equations: $(p(t)x')' + q(t)x = e(t)$, with positive $p(t)$ and $q(t)$. For the first time, some criteria as well as the existence and nonexistence of non-monotone positive solutions are proved in the framework of some properties of solutions $\theta(t)$ of the corresponding integrable linear equation: $(p(t)\theta')' = e(t)$. The main results are illustrated by many examples dealing with equations which allow exact non-monotone positive solutions not necessarily periodic. Finally, we pose some open questions.

Keywords: non-monotonic behaviour, positive solutions, existence, nonexistence, criteria.

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1 Introduction

In recent years, mathematical models which admit non-monotone positive solutions pay attention in various disciplines of the applied sciences. For instance, non-monotonic behaviour of: the amplitude of harmonic oscillator driven with chirped pulsed force [9], the three-flavour oscillation probability [1,10], the particle density in Bose–Einstein condensates with attractive atom-atom interaction [2,5,14], the several kinds of cardiogenic oscillations [6], the structural analysis of blood glucosa [4], the response function in a delayed chemostat model [19].

In the paper, we consider the second-order linear differential equation:

$$(p(t)x')' + q(t)x = e(t), \quad t \geq t_0, \tag{1.1}$$

where $p, q, e \in C[t_0, \infty)$, $p(t) > 0$, $q(t) \geq 0$ for $t \geq t_0$, and $x = x(t)$. By a solution of (1.1), we mean a function $x \in C^1[t_0, \infty)$ which satisfies $p(t)x'(t) \in C^1[t_0, \infty)$ and (1.1) on $[t_0, \infty)$. We say

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that a function $x(t)$ is (eventually) positive if $x(t) > 0$ for all $t > t_1$ and some $t_1 \geq t_0$ (where it is not necessary, the word *eventually* is avoided). Also, a smooth $x(t)$ is a *non-monotone function* on $[t_0, \infty)$ (or shortly said, $x(t)$ is *non-monotonic on* $[t_0, \infty)$) if $x'(t)$ is a sign-changing function on $[t_0, \infty)$, that is, for each $t > t_0$, there exist $t_+, t_- \in [t, \infty)$ such that $x'(t_+) > 0$ and $x'(t_-) < 0$ (in the literature, such a function $x(t)$ is also called *weakly oscillatory*, see for instance [3,7]). It is easy to show that:

$$\liminf_{t\to\infty} x(t) < \limsup_{t\to\infty} x(t) \text{ implies } x(t) \text{ is non-monotonic on } [t_0, \infty), \tag{1.2}$$

which is used here as a criterion for the non-monotonic behaviour of continuous functions. The opposite claim to (1.2) in general does not hold, for instance: $x(t) = e^{-t}(\cos t + \sin t)$ is a non-monotone function but its limits inferior and superior are equal.

Many classes of homogeneous linear differential equations of second-order do not allow any non-monotone positive solution. For instance, equations with constant coefficients: $x'' + \mu x' + \lambda x = 0$, where $\mu, \lambda \in \mathbb{R}$, and the Euler equation $(E_{\mu\lambda})$: $t^2x'' + \mu t x' + \lambda x = 0$, because they only admit either oscillatory solutions ($\exists t_n \to \infty$ such that $x(t_n) = 0$) or monotone solutions ($x'(t) \geq 0$ or $x'(t) \leq 0$ on (t_0, ∞)). On the other hand, two simple constructions of the non-homogeneous term $e(t) \not\equiv 0$ are possible such that equation (1.1) allows non-monotone positive solutions on $[t_0, \infty)$:

1) for a given non-monotone positive function $x_0(t)$, let $e(t) = (p(t)x_0')' + q(t)x_0$; it means that $x_0(t)$ is a particular solution of (1.1) and thus, in such a case, (1.1) allows at least one non-monotone positive solution on $[t_0, \infty)$; for instance, letting $x_0(t) = 2 + \sin t$, then for $e(t) = 2\lambda + (\lambda - 1)\sin t + \mu \cos t$, the equation $x'' + \mu x' + \lambda x = e(t)$ admits $x_0(t)$ as a non-monotone positive solution;

2) let the homogeneous part of (1.1): $(p(t)x')' + q(t)x = 0$ admit infinitely many bounded oscillatory solutions $x_h(t)$ and let $x_0(t) \equiv c_0 > 0$ be a large enough particular solution of (1.1); then for $e(t) = (p(t)x_0')' + q(t)x_0 = q(t)c_0$, the equation (1.1) allows infinitely many non-monotone positive solutions $x(t) = x_h(t) + c_0$; for instance, if $\mu \geq 1$ and $D = (\mu - 1)^2 - 4\lambda < 0$, then equation $(E_{\mu\lambda})$ admits bounded oscillatory solutions $x_h(t) = t^{(1-\mu)/2}(c_1 \cos(\rho \ln t) + c_2 \sin(\rho \ln t))$, where $\rho = \sqrt{|D|}/2$, $c_1, c_2 \in \mathbb{R}$, $0 < c_1^2 + c_2^2 < 4$; if we now chose for $x_0(t) \equiv 2$ and $e(t) = 2\lambda$, then the corresponding non-homogeneous equation $(E_{\mu\lambda e})$: $t^2x'' + \mu t x' + \lambda x = e(t)$ allows infinitely many non-monotone positive solutions in the form $x(t) = x_h(t) + x_0(t)$; obviously such a construction of $e(t)$ from given $p(t)$, $q(t)$, and $x_0(t)$ does not hold if $\mu < 1$, $D < 0$ (unbounded oscillatory solutions) and $\mu \in \mathbb{R}$, $D \geq 0$ (monotone solutions).

However, in our main problems of the paper, the non-homogeneous part $e(t)$ is not a point of any construction, but $e(t)$ is an arbitrary given function just as $p(t)$ and $q(t)$.

Main problems. 1) *Find sufficient and necessary conditions on arbitrary given $p(t)$, $q(t)$, and $e(t)$, such that every positive solution of* (1.1) *is non-monotonic.* 2) *Prove the existence of at least one non-monotone positive solution of* (1.1).

Taking into account the preceding observation, we can positively answer to the *main problem* concerning the concrete Euler equation: $t^2x'' + \mu t x' + \lambda x = 2\lambda$, where $\mu \geq 1$ and $\lambda > (\mu - 1)^2/4$.

The purpose of this paper is to give some answers to the *main problem* in the framework of non-monotonic behaviour of the function $\theta = \theta(t)$, $\theta \in C^2(t_0, \infty)$, which is a solution of the next integrable second-order linear differential equation:

$$(p(t)\theta')' = e(t), \quad t \geq t_0. \tag{1.3}$$

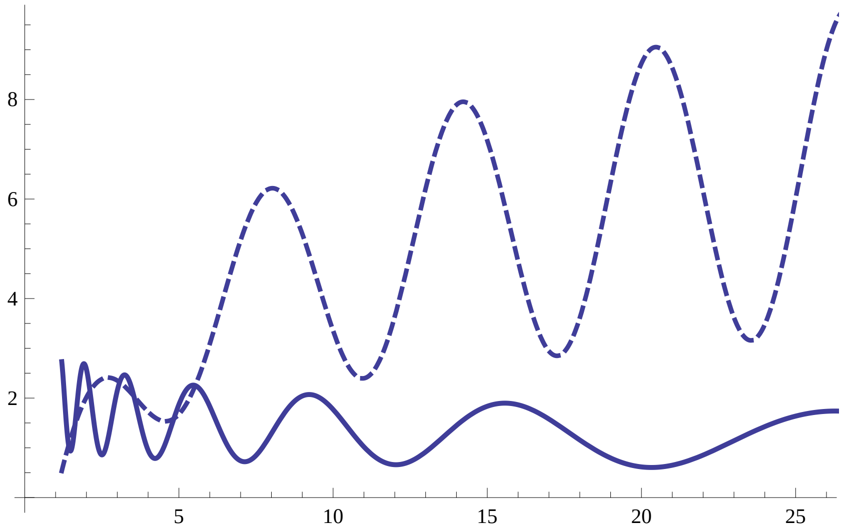

Figure 1.1: thick line: $x(t) = t^{\gamma}\big(d + \sin(\omega \ln t)\big)$ for $\gamma = -1/6$, $d = 2$ and $\omega = 12$, dashed line: $x(t) = \ln t\big(2 + \sin t\big)$, $t \geq t_0 > 1$.

The most simple model for the linear equation (1.1) having $p(t)$, $q(t)$, $e(t)$, and $x(t)$ that satisfy all required assumptions and conclusions of this paper is:

$$(t^a x')' + t^{-b} x = e(t), \quad t \geq t_0 > 0. \tag{1.4}$$

For some a, b and $e(t)$, the equation (1.4) allows exact non-monotone positive not necessarily periodic solutions $x(t)$, by which we can illustrate our main results below: two different cases $a > 1$, $a + b > 2$ (bounded $x(t)$) and $a \leq 1$, $a + b > 2$ (unbounded $x(t)$) are considered in Subsections 2.1 and 2.2. Figure 1.1 shows the graphs of two examples of non-monotone positive (non-periodic) functions $x(t) = \alpha(t)\big(d + S(\omega(t))\big)$, where the amplitude $\alpha(t)$ is positive, the frequency $\omega(t)$ goes to infinity as t goes to infinity, and $S(\tau)$ is a continuous periodic function.

In Section 2, we give some relations for lower and upper limits of $x(t)$ and $\theta(t)$ as the solutions of respectively (1.1) and (1.3), in two different cases: bounded and possible unbounded solutions. It will ensure some conditions on $\theta(t)$ which imply the non-monotonicity of positive solutions of $x(t)$. In Sections 3 and 4, some conditions on $\theta(t)$ are involved such that the main equation (1.1) allows or not the positive non-monotone solutions. Finally in Section 5, we suggest some open problems for further study on this subject.

Our approach here to non-monotone positive solutions of second-order differential equations is quiet different than in [13], where (without limits inferior and superior of $x(t)$) the sign-changing property of $x'(t)$ of positive solutions $x(t)$ of a class of nonlinear differential equations has been studied by means of a variational criterion. On the existence of positive periodic solutions as a particular case of non-monotonic behaviour of the second-order linear differential equations, see for instance [18, Section 2], [11, Lemma 2.2] and references cited therein.

2 Criteria for non-monotonicity of solutions

Since the right-hand side of both equations (1.1) and (1.3) are the same, we can derive the next relation between all their solutions.

Proposition 2.1. *Let $x(t)$ and $\theta(t)$ be two smooth functions on $[t_0,\infty)$ that satisfy the following equality:*

$$\theta(t) = x(t) + \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(r)x(r)drds + C_1 \int_{t_0}^{t} \frac{1}{p(s)}ds + C_2, \tag{2.1}$$

with arbitrary constants $C_1, C_2 \in \mathbb{R}$. Then, $\theta(t_0) = x(t_0)$ and $\theta'(t_0) = x'(t_0)$ if and only if $C_1 = C_2 = 0$. Moreover, $\theta(t)$ is a solution of equation (1.3) *if and only if $x(t)$ is a solution of equation* (1.1).

In what follows, we consider two rather different cases: the bounded and not necessarily bounded solutions of equation (1.1).

2.1 Non-monotone positive bounded solutions

In this subsection, the main assumption on $p(t)$ and $q(t)$ is:

$$\int_{t_0}^{\infty} \frac{1}{p(s)} \int_{t_0}^{s} q(r)drds < \infty. \tag{2.2}$$

According to (2.1)–(2.2), we easily derive:

Lemma 2.2. *Supposing* (2.2)*, let $x(t)$ and $\theta(t)$ be two smooth functions on $[t_0,\infty)$ that satisfy* (2.1) *with $C_1 = C_2 = 0$, and let $0 \le x(t) \le M$ on $[t_0,\infty)$ for some $M \in \mathbb{R}$, $M > 0$. Then $0 \le \theta(t) \le N$ on $[t_0,\infty)$ for some $N \in \mathbb{R}$, $N > 0$. Moreover:*

i) $\liminf_{t\to\infty} x(t) = \limsup_{t\to\infty} x(t) \iff \liminf_{t\to\infty} \theta(t) = \limsup_{t\to\infty} \theta(t)$;

ii) if $\theta'(t)$ is bounded and

$$\lim_{t\to\infty} \frac{1}{p(t)} \int_{t_0}^{t} q(r)dr = 0, \tag{2.3}$$

then $\liminf_{t\to\infty} x'(t) = \liminf_{t\to\infty} \theta'(t)$, $\limsup_{t\to\infty} x'(t) = \limsup_{t\to\infty} \theta'(t)$;

iii) the statement ii) still holds if (2.3) *is replaced with*

$$\frac{1}{p(t)} \int_{t_0}^{t} q(r)dr \quad \text{is decreasing on } [t_0,\infty). \tag{2.4}$$

In general, assumption (2.2) does not imply (2.3), but assumptions (2.2) and (2.4) together imply (2.3). It is easy to check that, for all $a, b \in \mathbb{R}$ such that $a > 1$ and $a + b > 2$, the coefficients $p(t) = t^a$ and $q(t) = t^{-b}$, $t \ge t_0 > 0$, satisfy both conditions (2.2) and (2.3).

If $\theta'(t)$ is a sign-changing function on $[t_0,\infty)$, then from equality (2.1) we cannot say anything about the sign of the function $x'(t)$. However, according to (1.2), from Lemma 2.2 we can derive the following criteria for non-monotonicity of positive bounded solutions of equation (1.1).

Theorem 2.3 (Criterion for non-monotonicity of solution)**.** *Let us assume* (2.2)*. If every solution $\theta(t)$ of equation* (1.3) *satisfies*

$$\liminf_{t\to\infty} \theta(t) < \limsup_{t\to\infty} \theta(t), \tag{2.5}$$

then every positive bounded solution $x(t)$ of equation (1.1) *satisfies*

$$\liminf_{t\to\infty} x(t) < \limsup_{t\to\infty} x(t). \tag{2.6}$$

In particular, $x(t)$ is non-monotonic on $[t_0,\infty)$. Moreover, if (2.3) *holds and $\theta'(t)$ is bounded, then*

$$\liminf_{t\to\infty}\theta'(t) < \limsup_{t\to\infty}\theta'(t) \quad \textit{implies} \quad \liminf_{t\to\infty} x'(t) < \limsup_{t\to\infty} x'(t),$$
$$\liminf_{t\to\infty}\theta'(t) = \limsup_{t\to\infty}\theta'(t) \quad \textit{implies} \quad \liminf_{t\to\infty} x'(t) = \limsup_{t\to\infty} x'(t).$$

We illustrate this result with the help of equation (1.4).

Example 2.4. Let $a = 2$, $b = 1$ and

$$e(t) = t^\gamma\left[2\gamma(\gamma+1) + (2\gamma+1)\cos(\ln t) + (\gamma^2+\gamma-1)\sin(\ln t) + \frac{1}{t}\big(2+\sin(\ln t)\big)\right],$$

where $\gamma \in (-\sqrt{3}/3, 0]$. Since $a = 2 > 1$ and $a + b = 3 > 2$, the assumption (2.2) is satisfied. By a direct integration of equation (1.3), we can see that the set of all solutions $\theta(t)$ of (1.3) is the next two parametric family of functions:

$$\theta(t) = \int_{t_0}^{t}\frac{1}{p(s)}\int_{t_0}^{s} e(r)drds + c_1\int_{t_0}^{t}\frac{1}{p(s)}ds + c_2, \tag{2.7}$$

where the parameters $c_1, c_2 \in \mathbb{R}$ satisfy: $c_1 = \theta'(t_0)p(t_0)$ and $c_2 = \theta(t_0)$. Now, from (2.7) it follows:

$$\theta(t) = c_1 + t^\gamma\big(2+\sin(\ln t)\big) + \frac{c_2}{t} + \frac{1}{t^{1-\gamma}}\left[C_1\cos(\ln t) + C_2\sin(\ln t) + C_3\right],$$

where $c_1, c_2 \in \mathbb{R}$ and the real constants C_1, C_2 and C_3 only depend on γ. Next, we have: if $\gamma < 0$, then $\liminf_{t\to\infty}\theta(t) = \limsup_{t\to\infty}\theta(t) = c_1$, and if $\gamma = 0$, then $\liminf_{t\to\infty}\theta(t) = c_1 + 1 < c_1 + 3 = \limsup_{t\to\infty}\theta(t)$. Thus, if $\gamma = 0$, then condition (2.5) is fulfilled, and by Theorem 2.3, every positive bounded solution $x(t)$ of equation (1.4) is non-monotonic on $[t_0,\infty)$. Next, since $a = 2$ and $b = 1$, we especially have

$$\frac{1}{p(t)}\int_{t_0}^{t} q(r)dr = \frac{1}{t^2}\ln\frac{t}{t_0},$$

and thus, the extra assumption (2.3) is also satisfied in this case. Finally, it is worth to mention that the function $x(t) = t^\gamma\big(2+\sin(\ln t)\big)$ is an exact non-monotone positive solution of equation (1.4) with such a, b and $e(t)$. We leave to the reader to make a related example in which the solution $x(t) = t^\gamma\big(d + \sin(\omega\ln t)\big)$ is considered, where $\gamma \in (-\sqrt{3}/3, 0]$, $d > 1$ and $\omega > 0$.

If $q(t) \not\equiv 0$, then assumption (2.2) implies $1/p \in L^1(t_0,\infty)$. By direct integration of equation (1.1), we obtain

$$x(t) + \int_{t_0}^{t}\frac{1}{p(s)}\int_{t_0}^{s} q(r)x(r)drds + c_1\int_{t_0}^{t}\frac{1}{p(s)}ds + c_2 = \int_{t_0}^{t}\frac{1}{p(s)}\int_{t_0}^{s} e(r)drds,$$

for some c_1, c_2 depending on t_0. Since in the subsection we are working with positive bounded solutions $x(t)$, from the previous equality and (2.2), we have:

$$\begin{aligned}\liminf_{t\to\infty} x(t) + \int_{t_0}^{\infty}\frac{1}{p(s)}\int_{t_0}^{s} q(r)x(r)drds + c_1\int_{t_0}^{\infty}\frac{1}{p(s)}ds + c_2 \\ = \liminf_{t\to\infty}\int_{t_0}^{t}\frac{1}{p(s)}\int_{t_0}^{s} e(r)drds,\end{aligned} \tag{2.8}$$

$$\limsup_{t\to\infty} x(t) + \int_{t_0}^{\infty} \frac{1}{p(s)} \int_{t_0}^{s} q(r)x(r)drds + c_1 \int_{t_0}^{\infty} \frac{1}{p(s)} ds + c_2$$
$$= \limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds. \quad (2.9)$$

Hence, from (2.8) and (2.9), we can easily prove the next two simple results.

Theorem 2.5. *Let $q(t) \not\equiv 0$ and assume (2.2).*

i) If
$$-\infty < \liminf_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds < \limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds < \infty, \quad (2.10)$$
then every positive bounded solution $x(t)$ of equation (1.1) satisfies (2.6), and so, $x(t)$ is non-monotonic on $[t_0, \infty)$.

ii) If there exists a (particular) positive bounded solution $x_0(t)$ of equation (1.1) satisfying (2.6), then every positive bounded solution $x(t)$ of (1.1) also satisfies (2.6), and so, $x(t)$ is non-monotonic on $[t_0, \infty)$.

As pointed out above, the coefficients $p(t) = t^a$ and $q(t) = t^{-b}$, $t \geq t_0 > 0$, satisfy condition (2.2) if $a > 1$ and $a + b > 2$. Moreover, we have $q(t) \not\equiv 0$ and so, we may use Theorem 2.5.

Example 2.6. Let $a > 1$, $a + b > 2$, $t_0 > 0$, and $\omega > 0$. If we chose for $e(t) = (t^a \cos(\omega t))'$ or $e(t) = (t^{a-1} \sin(\omega \ln t))'$ (non-periodic case), then the required condition (2.10) is fulfilled, because:

$$\int_{t_0}^{t} \frac{1}{s^a} \int_{t_0}^{s} \left(r^a \cos(\omega r)\right)' drds = \frac{1}{\omega} \sin(\omega t) + c_1 + c_2 t^{1-a},$$
$$\int_{t_0}^{t} \frac{1}{s^a} \int_{t_0}^{s} \left(r^{a-1} \sin(\omega \ln r)\right)' drds = -\frac{1}{\omega} \cos(\omega \ln t) + c_1 + c_2 t^{1-a},$$

for some $c_1, c_2 \in \mathbb{R}$, and in both cases of $e(t)$, we have:

$$\liminf_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds = -\frac{1}{\omega} + c_1$$
$$< \frac{1}{\omega} + c_1 = \limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds.$$

Therefore, by Theorem 2.5 (i) we conclude that in these cases of $e(t)$, all positive bounded solutions of equation (1.4) are non-monotonic on $[t_0, \infty)$.

The previous example can be generalised to the case when $e(t)$ is the first derivative of an oscillating (chirped) function with general frequency $\omega(t)$.

Example 2.7. Let us assume (2.2) and $1/p \in L^1(t_0, \infty)$. Let $\omega(t)$ be a positive increasing frequency such that $\lim_{t\to\infty} \omega(t) = \infty$ and $S(\tau)$ be a periodic smooth function on $\mathbb{R}$. For instance, $\omega(t) = \omega_0 t$, $\omega(t) = \omega_0 \ln t$, $\omega_0 > 0$ and $S(\tau) = \sin \tau$, $S(\tau) = \cos \tau$. Let us now choose $e(t) = (p(t)\omega'(t)S'(\omega(t)))'$. Then condition (2.10) is fulfilled, because:

$$\int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} \left(p(r)\omega'(r)S'(\omega(r))\right)' drds = S(\omega(t)) + c_1 + c_2 \int_{t_0}^{t} \frac{1}{p(s)} ds,$$

for some $c_1, c_2 \in \mathbb{R}$, and hence,

$$\begin{aligned}\liminf_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds &= \liminf_{\tau\to\infty} S(\tau) + c_1 + c_2 \int_{t_0}^{\infty} \frac{1}{p(s)} ds \\ &< \limsup_{\tau\to\infty} S(\tau) + c_1 + c_2 \int_{t_0}^{\infty} \frac{1}{p(s)} ds \\ &= \limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds.\end{aligned}$$

Now, by Theorem 2.5 (i) we conclude that for such a class of $e(t)$, all positive bounded solutions of equation (1.1) are non-monotonic on $[t_0, \infty)$.

Now, in the next two examples, we illustrate Theorem 2.5 (ii).

Example 2.8. Let $a = b = 2$ and $e(t)$ be given by

$$e(t) = \cos(\ln t) - \sin(\ln t) + \frac{1}{t^2}(2 + \sin(\ln t)). \tag{2.11}$$

Because of $\ln t$, the frequency in $e(t)$ is varying in time and hence, such $e(t)$ is often called as *oscillating chirped force*, see for instance in [9] and about the chirps, in [15, 16]. Since $a = 2 > 1$ and $a + b = 4 > 2$, the coefficients $p(t) = t^2$ and $q(t) = t^{-2}$ satisfy assumption (2.2) and $1/p \in L^1(t_0, \infty)$. Furthermore, the function $x(t) = 2 + \sin(\ln t)$ is an exact positive bounded solution of (1.4) satisfying required condition (2.6). Hence, by Theorem 2.5 (ii) we conclude that all positive bounded solutions of equation (1.4), with $e(t)$ from (2.11), are non-monotonic on $[t_0, \infty)$.

Example 2.9. Let assume (2.2) and $1/p \in L^1(t_0, \infty)$. Let the functions $\omega(t)$ and $S(\tau)$ be as in Example 2.7. If $e(t) = (p(t)\omega'(t)S'(\omega(t)))' + q(t)(d + S(\omega(t)))$, where $d \in \mathbb{R}$ such that $d > -\liminf_{\tau\to\infty} S(\tau)$, then $x_0(t) = d + S(\omega(t))$ is an exact positive bounded solution of equation (1.1) satisfying (2.6). Therefore, by Theorem 2.5 (ii) we conclude that all positive bounded solutions of equation (1.1), with such a class of $e(t)$, are non-monotonic on $[t_0, \infty)$.

Remark 2.10. An important particular class of equations (1.1) is the Euler nonhomogeneous equation:

$$x'' + \frac{\mu}{t}x' + \frac{\lambda}{t^2}x = f(t), \quad t > 0, \tag{2.12}$$

where $\mu \in \mathbb{R}$ and $\lambda > 0$. It can be easily rewritten in the form of equation (1.1): $\left(t^{\mu}x'\right)' + \lambda t^{\mu-2}x = t^{\mu}f(t)$, $t > 0$. If we set $a = \mu$ and $b = 2 - \mu$, by the same argument as for the coefficients of the equation (1.4), one can show that $p(t) = t^a$ and $q(t) = \lambda t^{-b}$ satisfy the assumption (2.2) provided $a > 1$ and $a + b > 2$. But, the last inequality is not possible in this case, because $a + b = \mu + 2 - \mu = 2$. Hence, the assumption (2.2) does not hold for all $\mu \in \mathbb{R}$ and $\lambda > 0$ and consequently, we cannot apply the criterion from Theorems 2.3 and 2.5 to equation (2.12), see an open problem in Section 5.1.

2.2 Non-monotone positive not necessarily bounded solutions

Since $p(t) > 0$, we can define the next function,

$$P(t) = \int_{t_0}^{t} \frac{1}{p(s)} ds, \quad t \geq t_0, \tag{2.13}$$

and we suppose that:

$$\lim_{t\to\infty} P(t) = \infty, \tag{2.14}$$

$$\int_{t_0}^{\infty} P(t)q(t)dt < \infty. \tag{2.15}$$

At the first, we prove the following technical result.

Proposition 2.11. *Let $x(t)$ be a continuous function such that $0 \le \frac{x(t)}{P(t)} \le M$ for all $t \ge t_0$ and some $M > 0$. If assumptions (2.14) and (2.15) hold, then there exists a constant $L \in [0, \infty)$ such that*

$$L = \lim_{t\to\infty} \frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(r)x(r)drds. \tag{2.16}$$

Moreover, if

$$\lim_{t\to\infty} [p(t)P(t)] = \infty, \tag{2.17}$$

then

$$\lim_{t\to\infty} \left(\frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(r)x(r)drds \right)' = 0. \tag{2.18}$$

Proof. We introduce two auxiliary functions $X_p(t)$ and $X_q(t)$ defined by:

$$X_p(t) = \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(r)x(r)drds \quad \text{and} \quad X_q(t) = \int_{t_0}^{t} q(r)x(r)dr.$$

If $q(t) \equiv 0$ or $x(t) \equiv 0$, then the conclusion of this proposition obviously holds. Thus, we may assume $q(t) \ge 0, q(t) \not\equiv 0$ and $x(t) \ge 0, x(t) \not\equiv 0$. Hence, the functions $X_p(t)$ and $X_q(t)$ are positive, $X_p(t)$ is increasing and $X_q(t)$ is nondecreasing. Moreover, with the help of assumptions $\frac{x(r)}{P(r)} \le M$ and (2.15), we have

$$X_q(t) = \int_{t_0}^{t} P(r)q(r)\frac{x(r)}{P(r)}dr \le M \int_{t_0}^{\infty} P(r)q(r)dr < \infty, \quad t \ge t_0.$$

Therefore, there exists $L_q \in (0, \infty)$ such that $L_q = \lim_{t\to\infty} X_q(t)$. In particular, $X_q(t) \ge L_q/2$ on $[t_1, \infty)$ for some $t_1 \ge t_0$, and hence

$$X_p(t) \ge \int_{t_1}^{t} \frac{1}{p(s)} X_q(s)ds \ge \frac{L_q}{2} \int_{t_1}^{t} \frac{1}{p(s)}ds = \frac{L_q}{2}[P(t) - P(t_1)],$$

which implies $\lim_{t\to\infty} X_p(t) = \infty$. Hence, the L'Hospital rule yields that:

$$\lim_{t\to\infty} \frac{X_p(t)}{P(t)} = \frac{\infty}{\infty} = \lim_{t\to\infty} \frac{X_p'(t)}{P'(t)} = \lim_{t\to\infty} X_q(t) = L_q,$$

and thus, the desired statement (2.16) is shown. Finally, from previous equality we especially conclude that $\lim_{t\to\infty} \left| X_q(t) - \frac{X_p(t)}{P(t)} \right| = |L_q - L_q| = 0$ and so,

$$\left| \left(\frac{X_p(t)}{P(t)} \right)' \right| = \frac{1}{P(t)p(t)} \left| X_q(t) - \frac{X_p(t)}{P(t)} \right| \to 0,$$

as $t \to \infty$, where (2.17) is used. It proves (2.18). □

A model equation for (1.1) with the coefficients $p(t)$ and $q(t)$ satisfying required assumptions (2.14), (2.15) and (2.17) is equation (1.4), which is shown in the next example.

Example 2.12. Let $p(t) = t^a$ and $q(t) = t^{-b}$, where $a \le 1$ and $a + b > 2$. If $a = 1$ then $P(t) = \ln t - \ln t_0 \to \infty$ as $t \to \infty$. If $a < 1$, then $P(t) = (t^{1-a} - t_0^{1-a})/(1-a) \to \infty$ as $t \to \infty$. Hence, (2.14) is satisfied for $a \le 1$. Since $a \le 1$ and $b > 2 - a$ imply $b > 1$, in both cases of $P(t)$, we have:

$$\int_{t_0}^{\infty} P(t)q(t)dt = \frac{t_0^{2-a-b}}{(a+b-2)(b-1)} < \infty,$$

and thus, (2.15) is also satisfied. Moreover, $p(t)P(t) = t(\ln t - \ln t_0) \to \infty$ (the case of $a = 1$) and $p(t)P(t) = (t - t^a t_0^{1-a})/(1-a) \to \infty$ as $t \to \infty$ (the case of $a < 1$), which show that (2.17) is satisfied too.

Lemma 2.13. *Supposing* (2.14) *and* (2.15)*, let $x(t)$ and $\theta(t)$ be two smooth functions on $[t_0, \infty)$ that satisfy* (2.1) *with $C_1 = C_2 = 0$, and $0 \le \frac{x(t)}{P(t)} \le M$ for all $t \ge t_0$ and some $M \in \mathbb{R}$, $M > 0$. Then $0 \le \frac{\theta(t)}{P(t)} \le N$ for all $t \ge t_0$ and some $N \in \mathbb{R}$, $N > 0$, and moreover:*

i) $\liminf_{t\to\infty} \frac{x(t)}{P(t)} = \limsup_{t\to\infty} \frac{x(t)}{P(t)} \Leftrightarrow \liminf_{t\to\infty} \frac{\theta(t)}{P(t)} = \limsup_{t\to\infty} \frac{\theta(t)}{P(t)}$*;*

ii) if $p(t)$ and $q(t)$ additionally satisfy (2.17)*, and $\left(\frac{\theta(t)}{P(t)}\right)'$ is bounded, then*

$$\liminf_{t\to\infty} \left(\frac{x(t)}{P(t)}\right)' = \liminf_{t\to\infty} \left(\frac{\theta(t)}{P(t)}\right)' \quad \text{and} \quad \limsup_{t\to\infty} \left(\frac{x(t)}{P(t)}\right)' = \limsup_{t\to\infty} \left(\frac{\theta(t)}{P(t)}\right)'.$$

The previous lemma plays an essential role in proof of the following main result of this subsection, which is a criterion for the non-monotonicity of positive not necessarily bounded solutions.

Theorem 2.14 (Criterion for non-monotonicity of solutions)**.** *Let us assume* (2.14) *and* (2.15)*. If every solution $\theta(t)$ of equation* (1.3) *satisfies*

$$\liminf_{t\to\infty} \frac{\theta(t)}{P(t)} < \limsup_{t\to\infty} \frac{\theta(t)}{P(t)}, \tag{2.19}$$

then every positive solution $x(t)$ of equation (1.1)*, for which $\frac{x(t)}{P(t)}$ is bounded, satisfies*

$$\liminf_{t\to\infty} \frac{x(t)}{P(t)} < \limsup_{t\to\infty} \frac{x(t)}{P(t)}. \tag{2.20}$$

In particular, $\frac{x(t)}{P(t)}$ is non-monotonic on $[t_0, \infty)$. Moreover, if we additionally suppose (2.17)*, and*

$$-\infty < \liminf_{t\to\infty} \left(\frac{\theta(t)}{P(t)}\right)' < 0 < \limsup_{t\to\infty} \left(\frac{\theta(t)}{P(t)}\right)' < \infty, \tag{2.21}$$

then $x(t)$ is non-monotonic on $[t_0, \infty)$.

Remark 2.15. In general, the condition (2.20) does not imply that the $x(t)$ is non-monotonic on $[t_0, \infty)$. For example, if $p(t) = e^{-t}$, then $P(t) = e^t - e^{t_0}$; it is clear that the function $x(t) = e^t(2 + \sin t)$ satisfies (2.20), because

$$0 \le \liminf_{t\to\infty} \frac{x(t)}{P(t)} = 1 < 3 = \limsup_{t\to\infty} \frac{x(t)}{P(t)} < \infty;$$

but, at the same time, we have $x'(t) = e^t(2 + \sin t + \cos t) > 0$ for all large enough t. Thus, $x(t)$ is not non-monotonic on $[t_0, \infty)$ even if (2.20) holds.

However, in the next lemma, we give an additional condition on $x(t)$ such that (2.20) implies the non-monotonicity of $x(t)$ on $[t_0, \infty)$, which is together with Lemma 2.13 used in the proof of Theorem 2.14.

Lemma 2.16. *Let assume (2.17). Let $x(t)$ be a positive smooth function for which $\frac{x(t)}{P(t)}$ is bounded. If $x(t)$ satisfies (2.20) and*

$$\liminf_{t\to\infty}\left(\frac{x(t)}{P(t)}\right)' < 0 < \limsup_{t\to\infty}\left(\frac{x(t)}{P(t)}\right)', \tag{2.22}$$

then $x(t)$ is non-monotonic on $[t_0, \infty)$.

We illustrate the preceding results with the help of equation (1.4).

Example 2.17. Let $a = 1$, $b = 2$, $t_0 > 0$, and

$$e(t) = 2\cos t + \ln t\cos t - t\ln t\sin t + \frac{\ln t}{t^2}(2+\sin t),\ t \ge t_0. \tag{2.23}$$

Since $p(t) = t$, $q(t) = t^{-2}$, and $a + b = 1 + 2 = 3 > 2$, by Example 2.12 it follows that assumptions (2.14), (2.15) and (2.17) are satisfied. Next, from equality (2.7) and (2.23), we derive that $\theta(t) = c_1 \ln t + c_2 + I_1(t) + I_2(t)$, where $c_1, c_2 \in \mathbb{R}$ and

$$I_1(t) = \int_{t_0}^{t}\frac{1}{s}\int_{t_0}^{s}(2\cos r + \ln r\cos r - r\ln r\sin r)drds,$$
$$I_2(t) = \int_{t_0}^{t}\frac{1}{s}\int_{t_0}^{s}\frac{\ln r}{r^2}(2+\sin r)drds.$$

Thus,

$$\begin{cases}\frac{\theta(t)}{P(t)} = \frac{c_1\ln t}{P(t)} + \frac{c_2}{P(t)} + \frac{I_1(t)}{P(t)} + \frac{I_2(t)}{P(t)},\\ \left(\frac{\theta(t)}{P(t)}\right)' = \left(\frac{c_1\ln t}{P(t)}\right)' + \left(\frac{c_2}{P(t)}\right)' + \left(\frac{I_1(t)}{P(t)}\right)' + \left(\frac{I_2(t)}{P(t)}\right)'.\end{cases} \tag{2.24}$$

Since: $2\cos r + \ln r\cos r - r\ln r\sin r = [r[\ln r(2+\sin r)]']'$, we have

$$I_1(t) = \int_{t_0}^{t}\frac{1}{s}\int_{t_0}^{s}\left[r[\ln r(2+\sin r)]'\right]'drds = \ln t(2+\sin t) + c_3\ln t + c_4,$$

where $c_3, c_4 \in \mathbb{R}$. Therefore,

$$\begin{cases}\liminf_{t\to\infty}\frac{I_1(t)}{P(t)} = 1 + c_3 < 3 + c_3 = \limsup_{t\to\infty}\frac{I_1(t)}{P(t)}\\ \liminf_{t\to\infty}\left(\frac{I_1(t)}{P(t)}\right)' = -1 < 0 < 1 = \limsup_{t\to\infty}\left(\frac{I_1(t)}{P(t)}\right)'.\end{cases} \tag{2.25}$$

Next, in particular for $x(t) = \ln t\,(2+\sin t)$, $p(t) = t$, $P(t) = \ln t - \ln t_0$, and $q(t) = t^{-2}$, from Proposition 2.11 we obtain the existence of an $L \in [0, \infty)$ such that

$$\lim_{t\to\infty}\frac{I_2(t)}{P(t)} = L \quad\text{and}\quad \lim_{t\to\infty}\left(\frac{I_2(t)}{P(t)}\right)' = 0. \tag{2.26}$$

Hence, from (2.24), (2.25) and (2.26) we derive:

$$\begin{cases}\liminf_{t\to\infty}\frac{\theta(t)}{P(t)} = c_1 + 1 + c_3 + L < c_1 + 3 + c_3 + L = \limsup_{t\to\infty}\frac{\theta(t)}{P(t)},\\ \liminf_{t\to\infty}\left(\frac{\theta(t)}{P(t)}\right)' = -1 < 0 < 1 = \limsup_{t\to\infty}\left(\frac{\theta(t)}{P(t)}\right)',\end{cases}$$

and thus, $\theta(t)$ satisfies the desired conditions (2.19) and (2.21). Therefore, we may apply Theorem 2.14 to equation (1.4) with $a = 1$, $b = 2$ and $e(t)$ from (2.23), and conclude that every its positive solution $x(t)$, for which $x(t)/P(t)$ is bounded, is a non-monotonic on $[t_0, \infty)$. Furthermore, one can check that the function $x(t) = \ln t\,(2 + \sin t)$ is an exact non-monotone positive unbounded solution of (1.4) satisfying (2.20).

The previous example could be generalized to

$$e(t) = \omega'(t)S'(\omega(t)) + \left[p(t)P(t)\omega'(t)S'(\omega(t))\right]' + q(t)P(t)[d + S(\omega(t))],$$

where $\omega(t)$ is a positive increasing frequency and $S(\tau)$ is a smooth periodic function such that $\liminf_{\tau\to\infty} S'(\tau) < 0 < \limsup_{\tau\to\infty} S'(\tau)$, $\lim_{t\to\infty} \omega(t) = \infty$ and $\lim_{t\to\infty} \omega'(t) \in (0, \infty)$. In this case, $x(t) = P(t)[d + S(\omega(t))]$ is a particular unbounded positive non-monotone solution of equation (1.1). The details are left to the reader.

Remark 2.18. In Remark 2.10 it is mentioned that the coefficients of the Euler type equation (2.12) do not satisfy the condition $a + b > 2$, which causes an impossibility to apply Theorem 2.3 to equation (2.12). This is the same with Theorem 2.14 and hence, an open problem in Section 5.1 is posed.

2.3 The proofs of main results of the previous subsections

Proof of Proposition 2.1. Differentiating equality (2.1), and multiplying with $p(t)$, and again differentiating such obtained equality, we derive equality: $(p(t)\theta'(t))' = (p(t)x'(t))' + q(t)x(t)$, which proves this proposition. □

Proof of Lemma 2.2. For arbitrary two functions $\theta(t)$ and $x(t)$, let equality (2.1) hold with $C_1 = C_2 = 0$. Let $G(t)$ be a new auxiliary function defined by:

$$G(t) := \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(r)x(r)drds \geq 0, \quad t \geq t_0.$$

From equality (2.1), the assumptions $0 \leq x(t) \leq M$ on $[t_0, \infty)$ and (2.2), we conclude that $G(t)$ is increasing on $[t_0, \infty)$ and:

$$\theta(t) = x(t) + G(t),\ 0 \leq G(t) \leq M \int_{t_0}^{\infty} \frac{1}{p(s)} \int_{t_0}^{s} q(r)drds,\ t \geq t_0. \tag{2.27}$$

In particular,

$$0 \leq \theta(t) \leq M\Big(1 + \int_{t_0}^{\infty} \frac{1}{p(s)} \int_{t_0}^{s} q(r)drds\Big),$$

that is, $\theta(t)$ is also a positive bounded function on $[t_0, \infty)$. Moreover, there exists $L \in \mathbb{R}$, $L > 0$, such that $L = \lim_{t\to\infty} G(t)$, and with $\theta(t) = x(t) + G(t)$, it shows that

$$\liminf_{t\to\infty} \theta(t) = \liminf_{t\to\infty} x(t) + L \text{ and } \limsup_{t\to\infty} \theta(t) = \limsup_{t\to\infty} x(t) + L.$$

Now, these equalities prove Lemma 2.2 (i).

Next, from (2.2), (2.3) and $0 \leq x(t) \leq M$ on $[t_0, \infty)$, we easily conclude that

$$0 \leq G'(t) \leq \frac{M}{p(t)} \int_{t_0}^{t} q(r)dr \leq M_1,\ t \geq t_0, \text{ and } \lim_{t\to\infty} G'(t) = 0. \tag{2.28}$$

From (2.27), it follows $\theta'(t) = x'(t) + G'(t)$. Since $x'(t) = \theta'(t) - G'(t)$ and $\theta'(t)$ is supposed to be bounded function, that is, $c_1 \le \theta'(t) \le c_2$ for some $c_1, c_2 \in \mathbb{R}$, we have: $c_1 - M_1 \le \theta'(t) - G'(t) = x'(t) \le \theta'(t) \le c_2$, and thus, $x'(t)$ is bounded too. Now, (2.28) proves Lemma 2.2 (ii).

Next, for the function $f(t) = \frac{M}{p(t)} \int_{t_0}^{t} q(r)dr$, from (2.2) and (2.4), we have $f \in C[t_0, \infty) \cap L^1(t_0, \infty)$, $f(t) \ge 0$ and $f(t)$ is decreasing on $[t_0, \infty)$. It shows that $\lim_{t\to\infty} f(t) = 0$, and thus, assumption (2.3) holds in this case too. Hence, Lemma 2.2 (ii) proves Lemma 2.2 (iii). □

Proof of Theorem 2.3. Let $x(t)$ be a positive bounded solution of equation (1.1). Let $\theta(t)$ be a function satisfying $\theta(t_0) = x(t_0)$, $\theta'(t_0) = x'(t_0)$, and equality (2.1). In such a case, by Proposition 2.1 we know that (2.1) holds with $C_1 = C_2 = 0$ and $\theta(t)$ is a solution of equation (1.3). Now, assumption (2.5) and Lemma 2.2 (i) prove that $x(t)$ satisfies the desired inequality (2.6). This together with (1.2) shows that $x(t)$ is non-monotonic on $[t_0, \infty)$. The rest of Theorem 2.3 immediately follows from Lemma 2.2 (ii). □

Proof of Theorem 2.5. The first conclusion of this theorem immediately follows from (2.8), (2.9), and (2.10). Next, let $x_0(t)$ be a positive bounded solution of equation (1.1) satisfying (2.6). Putting such $x_0(t)$ into (2.8) and (2.9), we conclude that the condition (2.10) is fulfilled. Hence, we may use Theorem 2.5 (i), which proves the second part of this theorem. □

Proof of Lemma 2.13. Firstly, from (2.1) with $C_1 = C_2 = 0$, we have:

$$\frac{\theta(t)}{P(t)} = \frac{x(t)}{P(t)} + \frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(r)x(r)drds. \tag{2.29}$$

Then from (2.29), $x(r) = \frac{x(r)}{P(r)} P(r)$, and $0 \le \frac{x(t)}{P(t)} \le M$, we derive:

$$0 \le \frac{\theta(t)}{P(t)} \le M(1 + M_1) < \infty, \quad t \in [t_0, \infty),$$

as well as by Proposition 2.11, there exists $L \in [0, \infty)$ such that

$$L = \lim_{t\to\infty} \frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(r)x(r)drds.$$

Now with the help of (2.29), we deduce:

$$\liminf_{t\to\infty} \frac{\theta(t)}{P(t)} = \liminf_{t\to\infty} \frac{x(t)}{P(t)} + L \quad \text{and} \quad \limsup_{t\to\infty} \frac{\theta(t)}{P(t)} = \limsup_{t\to\infty} \frac{x(t)}{P(t)} + L,$$

from which the proof of Lemma 2.13 (i) immediately follows. Also, from (2.29) we have:

$$\left(\frac{\theta(t)}{P(t)}\right)' = \left(\frac{x(t)}{P(t)}\right)' + \left(\frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(r)x(r)drds\right)'.$$

According to (2.18) and since $\left(\frac{\theta(t)}{P(t)}\right)'$ is supposed to be bounded, we conclude that $\left(\frac{x(t)}{P(t)}\right)'$ is also bounded and

$$\liminf_{t\to\infty} \left(\frac{\theta(t)}{P(t)}\right)' = \liminf_{t\to\infty} \left(\frac{x(t)}{P(t)}\right)' \quad \text{and} \quad \limsup_{t\to\infty} \left(\frac{\theta(t)}{P(t)}\right)' = \limsup_{t\to\infty} \left(\frac{x(t)}{P(t)}\right)',$$

which prove Lemma 2.13 (ii). □

Proof of Lemma 2.16. Let $P(t)$ be defined in (2.13) and $x(t)$ be arbitrary function satisfying all assumptions of this lemma. We define $\varphi(t) = x(t)/P(t)$. Then the assumptions (2.20) and (2.22) can be rewritten in the form:

$$\begin{cases} 0 \le \liminf_{t\to\infty} \varphi(t) < \limsup_{t\to\infty} \varphi(t) < \infty, \\ \liminf_{t\to\infty} \varphi'(t) < 0 < \limsup_{t\to\infty} \varphi'(t). \end{cases} \tag{2.30}$$

Since $x'(t) = P'(t)\varphi(t) + P(t)\varphi'(t) = \frac{\varphi(t)}{p(t)} + P(t)\varphi'(t)$, we have

$$\frac{x'(t)}{P(t)} = \frac{\varphi(t)}{p(t)P(t)} + \varphi'(t). \tag{2.31}$$

Therefore, from (2.30), (2.31), and assumption (2.17), we obtain

$$\liminf_{t\to\infty} \frac{x'(t)}{P(t)} = \liminf_{t\to\infty} \varphi'(t) < 0 < \limsup_{t\to\infty} \varphi'(t) = \limsup_{t\to\infty} \frac{x'(t)}{P(t)},$$

and hence $x'(t)$ is a sign-changing function, which shows that $x(t)$ is a non-monotone positive function on $[t_0, \infty)$. □

Proof of Theorem 2.14. The first part of this theorem is very similar to Theorem 2.3 and so, its proof is leaved to the reader. Next, according to the assumptions of the second part of this theorem, we my apply Lemma 2.13 (ii) which together with assumption (2.21) ensure that every positive solution $x(t)$ of equation (1.1) satisfies the required condition (2.22). Now, Lemma 2.16 proves that $x(t)$ is non-monotonic on $[t_0, \infty)$. □

3 Existence of positive non-monotone solutions

Next, on the coefficients $p(t)$ and $q(t)$ we involve the following conditions:

$$\int_{t_0}^{\infty} q(t)dt < \infty, \tag{3.1}$$

$$\int_{t_0}^{\infty} \frac{1}{p(s)} \int_s^{\infty} q(r)drds < \infty. \tag{3.2}$$

Remark 3.1. Assumption (3.1) and $1/p \in L^1(t_0, \infty)$ imply (3.2). However, we can work here also with $1/p \notin L^1(t_0, \infty)$.

Theorem 3.2 (Existence of solution). *Assume* (3.1) *and* (3.2), *and let $\theta(t)$ be a solution of equation* (1.3). *If*

$$-\infty < \liminf_{t\to\infty} \theta(t) \le \limsup_{t\to\infty} \theta(t) < \infty, \tag{3.3}$$

then the main equation (1.1) *has a positive solution $x(t)$ such that*

$$0 < \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) < \infty. \tag{3.4}$$

Moreover,

$$\liminf_{t\to\infty} \theta(t) < \limsup_{t\to\infty} \theta(t) \quad \textit{implies} \quad \liminf_{t\to\infty} x(t) < \limsup_{t\to\infty} x(t).$$

Example 3.3. The coefficients $p(t) = t^a$ and $q(t) = t^{-b}$ of the equation (1.4) also satisfy required conditions (3.1) and (3.2) provided $b > 1$ and $a + b > 2$. Moreover, if

$$e(t) = t^{-2+a+\gamma}\Big[2(a\gamma + \gamma^2 - \gamma) + \omega(a + 2\gamma - 1)\cos(\omega \ln t) \\ + (a\gamma - \omega^2 + \gamma^2 - \gamma)\sin(\omega \ln t)\Big] + \frac{1}{t^{b-\gamma}}(2 + \sin(\omega \ln t)),$$

where $\omega > 0$, $-\frac{\sqrt{3}}{3}\omega < \gamma \le 0$, $a > 1$ and $a + b > 2 + \gamma$, then by (2.7),

$$\theta(t) = c_1 + t^{\gamma}\big(2 + \sin(\omega \ln t)\big) + \frac{c_2}{t^{a-1}} + \frac{1}{t^{a+b-2-\gamma}}\big[C_1\cos(\omega \ln t) + C_2 \sin(\omega \ln t) + C_3\big],$$

where the real constants C_1, C_2 and C_3 only depend on parameters ω, γ, a and b. It follows that (3.3) is satisfied. On the other hand, $x(t) = t^{\gamma}\big(2 + \sin(\omega \ln t)\big)$ is an exact non-monotone non-periodic positive bounded solution of the equation (1.4) with above $e(t)$ such that $x(t)$ satisfies (3.4).

We can observe now that the coefficients $p(t) = t^a$ and $q(t) = t^{-b}$ of equation (1.4) simultaneously satisfy the required assumptions (2.2), (3.1) and (3.2) provided $a > 1$ and $b > 1$. In fact, in Section 2 it is mentioned that (2.2) holds if $a > 1$ and $a + b > 2$, and in the previous example, it is mentioned that (3.1) and (3.2) hold if $b > 1$ and $a + b > 2$. These together imply $a > 1$ and $b > 1$.

Proof of Theorem 3.2. According to (3.3), there exist $t_1 \ge t_0$, $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$-\delta_1 \le \theta(t) \le \delta_2, \quad t \ge t_1.$$

Because of (3.1), we can take $t_2 \ge t_1$ so large that

$$\int_{t_2}^{\infty} \frac{1}{p(s)} \int_s^{\infty} q(r)drds \le \frac{1}{\delta_1 + \delta_2 + 2}. \tag{3.5}$$

Let

$$Y = \{y \in C[t_2, \infty) : \delta_1 + 1 \le y(t) \le \delta_1 + 2 \text{ for } t \ge t_2\}.$$

Define the mapping $\mathcal{F} : Y \longrightarrow C[t_2, \infty)$ by

$$(\mathcal{F}y)(t) = \delta_1 + 1 + \int_{t_2}^{t} \frac{1}{p(s)} \int_s^{\infty} q(r)[y(r) + \theta(r)]drds, \quad t \ge t_2.$$

If $y \in Y$, then

$$1 \le y(t) + \theta(t) \le \delta_1 + \delta_2 + 2, \quad t \ge t_2. \tag{3.6}$$

Hence, by (3.5), we find that

$$\delta_1 + 1 \le (\mathcal{F}y)(t) \le \delta_1 + 1 + (\delta_1 + \delta_2 + 2)\int_{t_2}^{t} \frac{1}{p(s)} \int_s^{\infty} q(r)drds \le \delta_1 + 2$$

for $t \ge t_2$, which implies that $\mathcal{F}$ is well defined on Y and maps Y into itself. Here and hereafter, $C[t_2, \infty)$ is regarded as the Fréchet space of all continuous functions on $[t_2, \infty)$ with the topology of uniform convergence on every compact subinterval of $[t_2, \infty)$. Lebesgue's dominated convergence theorem shows that $\mathcal{F}$ is continuous on Y.

Now we claim that $\mathcal{F}(Y)$ is relatively compact. We note that $\mathcal{F}(Y)$ is uniformly bounded on every compact subinterval of $[t_2, \infty)$, because of $\mathcal{F}(Y) \subset Y$. By the Ascoli–Arzelà theorem, it suffices to verify that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[t_2, \infty)$. From (3.6) it follows that

$$|(\mathcal{F}y)'(t)| \leq \frac{1}{p(t)} \int_t^\infty q(s)[y(s) + \theta(s)]ds \leq \frac{\delta_1 + \delta_2 + 2}{p(t)} \int_t^\infty q(s)ds$$

for $t \geq t_2$. Let I be an arbitrary compact subinterval of $[t_2, \infty)$. Then we see that $\{(\mathcal{F}y)'(t) : y \in Y\}$ is uniformly bounded on I, because of (3.1) and Remark 3.1. The mean value theorem implies that $\mathcal{F}(Y)$ is equicontinuous on I.

Now we are ready to apply the Schauder–Tychonoff fixed point theorem to the mapping $\mathcal{F}$. Then there exists a $y_* \in Y$ such that $y_* = \mathcal{F}y_*$. Therefore, $\lim_{t\to\infty} y_*(t) = \lim_{t\to\infty}(\mathcal{F}y_*)(t) = c$ for some $c \in [\delta_1 + 1, \delta_1 + 2]$. Set

$$x_*(t) = y_*(t) + \theta(t), \quad t \geq t_2.$$

Then it is easy to check that x_* is a solution of (1.1) on $[t_2, \infty)$ and (3.6) implies

$$1 \leq x_*(t) \leq \delta_1 + \delta_2 + 2, \quad t \geq t_2$$

and hence

$$0 < \liminf_{t\to\infty} x_*(t) < \limsup_{t\to\infty} x_*(t) < \infty,$$

provided $\liminf_{t\to\infty} \theta(t) \neq \limsup_{t\to\infty} \theta(t)$. The proof is complete. □

Theorem 3.4. *Assume that* (2.14), (3.1), *and* (3.2) *hold and let* $\theta(t)$ *be a solution of equation* (1.3) *such that*

$$-\infty < \liminf_{t\to\infty} \frac{\theta(t)}{P(t)} < \limsup_{t\to\infty} \frac{\theta(t)}{P(t)} < \infty. \tag{3.7}$$

Then equation (1.1) *has a positive solution* $x(t)$ *such that*

$$0 < \liminf_{t\to\infty} \frac{x(t)}{P(t)} < \limsup_{t\to\infty} \frac{x(t)}{P(t)} < \infty. \tag{3.8}$$

Moreover, if additionally assume (2.17) *and* $\theta(t)$ *satisfies* (2.21), *then* $x(t)$ *is a non-monotone positive solution of equation* (1.1).

Proof of Theorem 3.4. There exist $t_1 > t_0$, $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$-\delta_1 \leq \frac{\theta(t)}{P(t)} \leq \delta_2, \quad t \geq t_1.$$

We take $t_2 \geq t_1$ so large that

$$\int_{t_2}^\infty \frac{1}{p(s)} \int_s^\infty q(r)drds \leq \frac{\delta_1 + 1}{(\delta_1 + \delta_2 + 3)(\delta_1 + 2)}. \tag{3.9}$$

Set

$$P_2(t) = \int_{t_2}^t \frac{1}{p(s)}ds.$$

By L'Hospital's rule, we have

$$\lim_{t\to\infty} \frac{(\delta_1+2)P_2(t)}{(\delta_1+1)P(t)} = \frac{\delta_1+2}{\delta_1+1} > 1.$$

Hence there exists $t_3 > t_2$ such that

$$(\delta_1+1)P(t) < (\delta_1+2)P_2(t), \quad t \geq t_3. \tag{3.10}$$

Let

$$Y = \{y \in C[t_3,\infty) : (\delta_1+1)P(t) \leq y(t) \leq (\delta_1+3)P_2(t) \text{ for } t \geq t_3\}.$$

Define the mapping $\mathcal{F} : Y \longrightarrow C[t_3,\infty)$ by

$$(\mathcal{F}y)(t) = (\delta_1+1)P(t) + \int_{t_2}^{t} \frac{1}{p(s)} \int_s^\infty q(r)[y(r)+\theta(r)]drds, \quad t \geq t_3.$$

If $y \in Y$, then

$$y(t)+\theta(t) \geq P(t) > 0, \quad t \geq t_3 \tag{3.11}$$

and, by (3.10),

$$\begin{aligned} y(t)+\theta(t) &\leq (\delta_1+3)P_2(t) + \delta_2 P(t) \\ &\leq (\delta_1+3)P_2(t) + \frac{\delta_2(\delta_1+2)}{\delta_1+1}P_2(t) \\ &\leq \frac{(\delta_1+3)(\delta_1+1)+\delta_2(\delta_1+2)}{\delta_1+1}P_2(t) \\ &\leq \frac{(\delta_1+3)(\delta_1+2)+\delta_2(\delta_1+2)}{\delta_1+1}P_2(t) \\ &= \frac{(\delta_1+\delta_2+3)(\delta_1+2)}{\delta_1+1}P_2(t), \quad t \geq t_3. \end{aligned} \tag{3.12}$$

Hence we have

$$(\mathcal{F}y)(t) \geq (\delta_1+1)P(t) \tag{3.13}$$

and

$$(\mathcal{F}y)(t) \leq (\delta_1+1)P(t) + \frac{(\delta_1+\delta_2+3)(\delta_1+2)}{\delta_1+1} \int_{t_2}^{t} \frac{1}{p(s)} \int_s^\infty q(r)P_2(r)drds \tag{3.14}$$

for $t \geq t_3$. From (3.9) it follows that

$$\begin{aligned} \int_s^\infty q(r)P_2(r)dr &= \int_s^\infty q(r) \int_{t_2}^{r} \frac{1}{p(u)}dudr \\ &\leq \int_{t_2}^\infty q(r) \int_{t_2}^{r} \frac{1}{p(u)}dudr \\ &= \int_{t_2}^\infty \frac{1}{p(u)} \int_u^\infty q(r)drdu \\ &\leq \frac{\delta_1+1}{(\delta_1+\delta_2+3)(\delta_1+2)}, \quad s \in [t_2,t]. \end{aligned}$$

Therefore, (3.10) and (3.14) imply that

$$\begin{aligned} (\mathcal{F}y)(t) &\leq (\delta_1+1)P(t) + P_2(t) \\ &\leq (\delta_1+2)P_2(t) + P_2(t) = (\delta_1+3)P_2(t), \quad t \geq t_3. \end{aligned}$$

Therefore, $\mathcal{F}$ is well defined on Y and maps Y into itself. By the same argument as in the proof of Theorem 3.2, we can conclude that $\mathcal{F}$ is continuous on Y and $\mathcal{F}(Y)$ is relatively compact. By applying the Schauder–Tychonoff fixed point theorem to the mapping $\mathcal{F}$, there exists a $y_* \in Y$ such that $y_* = \mathcal{F}y_*$. By L'Hospital's rule, we observe that

$$\begin{aligned}\lim_{t\to\infty}\frac{y_*(t)}{P(t)} &= \lim_{t\to\infty}\frac{(\mathcal{F}y_*)(t)}{P(t)}\\ &= \lim_{t\to\infty}\frac{(\mathcal{F}y_*)'(t)}{P'(t)}\\ &= \lim_{t\to\infty}\left(\delta_1+1+\int_t^\infty q(r)[y_*(r)+\theta(r)]dr\right) = c\end{aligned}$$

for some constant $c \ge \delta_1 + 1$. Set

$$x_*(t) = y_*(t) + \theta(t), \quad t \ge t_2. \tag{3.15}$$

Then it is easy to check that x_* is a solution of (1.1). From (3.11) and (3.12) it follows that

$$P(t) \le x_*(t) \le \frac{(\delta_1+\delta_2+3)(\delta_1+2)}{\delta_1+1}P_2(t), \quad t \ge t_3$$

and

$$0 < \liminf_{t\to\infty}\frac{x_*(t)}{P(t)} \le \limsup_{t\to\infty}\frac{x_*(t)}{P(t)} < \infty, \tag{3.16}$$

since

$$\lim_{t\to\infty}\frac{P_2(t)}{P(t)} = \lim_{t\to\infty}\frac{P_2'(t)}{P'(t)} = 1.$$

From assumption (3.7) and inequality (3.16), we easily derive the desired inequality (3.8).

Finally we assume (2.21). Since

$$x_*(t) = P(t)\left(\frac{y_*(t)}{P(t)} + \frac{\theta(t)}{P(t)}\right),$$

we have

$$x_*'(t) = P'(t)\left(\frac{y_*(t)}{P(t)} + \frac{\theta(t)}{P(t)}\right) + P(t)\left(\frac{y_*(t)}{P(t)} + \frac{\theta(t)}{P(t)}\right)'$$

and hence

$$\frac{x_*'(t)}{P(t)} = \frac{1}{p(t)P(t)}\frac{x_*(t)}{P(t)} + \left(\frac{y_*(t)}{P(t)}\right)' + \left(\frac{\theta(t)}{P(t)}\right)'. \tag{3.17}$$

Since $x_*(t)/P(t)$ is bounded and $\lim_{t\to\infty} p(t)P(t) = \infty$, we have

$$\lim_{t\to\infty}\frac{1}{p(t)P(t)}\frac{x_*(t)}{P(t)} = 0. \tag{3.18}$$

We claim that

$$\lim_{t\to\infty}\left(\frac{y_*(t)}{P(t)}\right)' = 0. \tag{3.19}$$

We observe that

$$\begin{aligned}\left(\frac{y_*(t)}{P(t)}\right)' &= \left(\frac{(\mathcal{F}y_*)(t)}{P(t)}\right)'\\ &= \frac{1}{p(t)P(t)}\left(\int_t^\infty q(r)x_*(r)dr - \frac{1}{P(t)}\int_{t_2}^t\frac{1}{p(s)}\int_s^\infty q(r)x_*(r)drds\right).\end{aligned}$$

L'Hospital's rule implies

$$\lim_{t\to\infty} \frac{1}{P(t)} \int_{t_2}^{t} \frac{1}{p(s)} \int_{s}^{\infty} q(r)x_*(r)drds = \lim_{t\to\infty} \int_{t}^{\infty} q(r)x_*(r)dr = 0.$$

Since $\lim_{t\to\infty} p(t)P(t) = \infty$, we have (3.19) as claimed. Combining (3.17)–(3.19) with (2.21), we conclude that

$$\liminf_{t\to\infty} \frac{x_*'(t)}{P(t)} = \liminf_{t\to\infty} \left(\frac{\theta(t)}{P(t)}\right)' < 0 < \limsup_{t\to\infty} \left(\frac{\theta(t)}{P(t)}\right)' = \limsup_{t\to\infty} \frac{x_*'(t)}{P(t)},$$

which means that $x_*'(t)$ is a sign-changing function, and thus, $x_*(t)$ is a non-monotone positive solution of (1.1). The proof is complete. □

4 Nonexistence of positive non-monotone solutions

Theorem 4.1. *Assume that*

$$\liminf_{t\to\infty} \frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds = -\infty.$$

Then (1.1) *has no any positive solution. In particular,* (1.1) *has no any positive non-monotone solution.*

Proof. Assume, to the contrary, that there exists a solution $x(t)$ of (1.1) such that $x(t) > 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$. Integrating (1.1) on $[t_0, t]$, we have

$$\begin{aligned} p(t)x'(t) &= p(t_0)x'(t_0) - \int_{t_0}^{t} q(r)x(r)dr + \int_{t_0}^{t} e(r)dr \\ &= C_1 - \int_{t_1}^{t} q(r)x(r)dr + \int_{t_0}^{t} e(r)dr, \quad t \geq t_0, \end{aligned}$$

where

$$C_1 = p(t_0)x'(t_0) - \int_{t_0}^{t_1} q(r)x(r)dr.$$

Therefore,

$$x'(t) = \frac{C_1}{p(t)} - \frac{1}{p(t)} \int_{t_1}^{t} q(r)x(r)dr + \frac{1}{p(t)} \int_{t_0}^{t} e(r)dr, \quad t \geq t_0. \tag{4.1}$$

Integrating (4.1) on $[t_0, t]$, we have

$$\begin{aligned} 0 < x(t) &= x(t_0) + C_1 P(t) - \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_1}^{s} q(r)x(r)drds \\ &\quad + \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds \\ &= C_2 + C_1 P(t) - \int_{t_1}^{t} \frac{1}{p(s)} \int_{t_1}^{s} q(r)x(r)drds \\ &\quad + \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds \\ &\leq C_2 + C_1 P(t) + \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds, \quad t \geq t_1, \end{aligned}$$

where

$$C_2 = x(t_0) - \int_{t_0}^{t_1} \frac{1}{p(s)} \int_{t_1}^{s} q(r)x(r)drds.$$

Hence we obtain

$$\frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds \geq -\frac{C_2}{P(t)} - C_1, \quad t \geq t_1.$$

Since $1/P(t)$ is positive and decreasing on (t_0, ∞), there exists the limit

$$\lim_{t\to\infty} \frac{1}{P(t)} \in [0, \infty),$$

which implies that

$$\frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds \geq C_3, \quad t \geq t_1$$

for some constant C_3. This is a contradiction. □

As a consequence of Theorem 4.1, we derive two useful criteria for the nonexistence of non-monotone positive solution.

Corollary 4.2. *Let $\theta(t)$ be a solution of equation (1.3).*

i) If

$$\liminf_{t\to\infty} \frac{\theta(t)}{P(t)} = -\infty,$$

then (1.1) has no positive solution.

ii) If

$$\lim_{t\to\infty} \frac{1}{P(t)} > 0 \quad \text{and} \quad \liminf_{t\to\infty} \theta(t) = -\infty,$$

then (1.1) has no positive solution.

In particular, in both cases, (1.1) has no positive non-monotone solution.

Proof. Since $1/P(t)$ is decreasing, we have $1/P(t)$ is bounded from above and according to (1.3), we obtain:

$$\begin{aligned}
\frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds &= \frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} \left(p(r)\theta'(r)\right)' drds \\
&= \frac{\theta(t)}{P(t)} - \frac{\theta(t_0)}{P(t)} - p(t_0)\theta'(t_0) \\
&\leq \frac{\theta(t)}{P(t)} + C_2 \;\; \text{for all } t \geq t_0,
\end{aligned}$$

where the constant $C_2 > 0$. Taking the limit inferior on both sides of previous inequality and using the assumption in i), we obtain:

$$\begin{aligned}
\liminf_{t\to\infty} \frac{1}{P(t)} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} e(r)drds &\leq \liminf_{t\to\infty} \left(\frac{\theta(t)}{P(t)} + C_2\right) \\
&= \liminf_{t\to\infty} \frac{\theta(t)}{P(t)} + C_2 \\
&= -\infty.
\end{aligned}$$

Now, we see that the main assumption of Theorem 4.1 is satisfied and therefore, Theorem 4.1 proves the first part of this corollary. Next, according to the assumptions in ii), we have:

$$\liminf_{t\to\infty} \frac{\theta(t)}{P(t)} = \lim_{t\to\infty} \frac{1}{P(t)} \liminf_{t\to\infty} \theta(t) = -\infty,$$

and hence, i) proves ii), that is, ii) is a particular case of i). □

Both cases of the previous corollary will be illustrated in the next two examples.

Example 4.3. Let $a = 1$, $b = 2$ and

$$e(t) = \frac{4\ln t}{t} \cos(\ln t) + \left(\frac{2}{t} + \frac{\ln^2 t}{t^2} - \frac{\ln^2 t}{t}\right) \sin(\ln t). \tag{4.2}$$

Since $p(t) = t$ and $P(t) = \ln t - \ln t_0$, we have $1/P(t) \to 0$ as $t \to \infty$ and thus, we are dealing with the first case of Corollary 4.2. Next, from (2.7) and (4.2), we obtain

$$\theta(t) = \ln^2 t \sin(\ln t) + c_1 \ln t + c_2 + \frac{1}{2t}\big[\ln t(\ln t + 2)\cos(\ln t) - (3 + 2\ln t)\sin(\ln t)\big],$$

and

$$\liminf_{t\to\infty} \frac{\theta(t)}{P(t)} = c_1 + \liminf_{t\to\infty} \big[\ln t \sin(\ln t)\big] = -\infty.$$

Therefore, we may use Corollary 4.2 (i) and conclude that equation (1.4) with such a, b and $e(t)$, has no any positive solution. Moreover, it is clear that the function $x(t) = \ln^2 t\, \sin(\ln t)$ is an exact oscillatory (non-positive) solution of (1.4).

Example 4.4. Let $a = 2$, $b = 1$ and

$$e(t) = t^\gamma \Big[(2\gamma + 1)\cos(\ln t) + \Big(\gamma^2 + \gamma - 1 + \frac{1}{t}\Big)\sin(\ln t)\Big], \tag{4.3}$$

where $\gamma > 0$. From (2.7) and (4.3), we have

$$\theta(t) = t^\gamma \sin(\ln t) + c_1 + \frac{c_2}{t} + t^{\gamma-1}\big[C_1 \cos(\ln t) + C_2 \sin(\ln t)\big], \tag{4.4}$$

where $c_1, c_2 \in \mathbb{R}$ and the real constants C_1, C_2 only depend on γ. Since $p(t) = t^2$, it is clear that

$$P(t) = \frac{t - t_0}{t_0 t} \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{P(t)} = t_0 > 0. \tag{4.5}$$

Since $\gamma > 0$ and $\gamma > \gamma - 1$, from (4.4), it follows:

$$\liminf_{t\to\infty} \theta(t) = c_1 + \liminf_{t\to\infty} \Big[t^\gamma \sin(\ln t) + \sqrt{C_1^2 + C_2^2}\, t^{\gamma-1} \sin(\ln t + C_3)\Big] = -\infty,$$

which together with (4.5) and Corollary 4.2 (ii) prove that equation (1.4) has no any positive solution. Moreover, the function $x(t) = t^\gamma \sin(\ln t)$ is an exact oscillatory (non-positive) solution of (1.4) with such a, b and $e(t)$.

Theorem 4.5. *Assume that* (2.14) *holds and* θ *be a solution of equation* (1.3) *such that* $\theta(t)$ *is not eventually positive and*

$$\liminf_{t\to\infty} \theta(t) = 0. \tag{4.6}$$

Assume moreover that there exists $\lambda \in (0, 1)$ *such that every solution of the equation*

$$(p(t)x')' + \lambda q(t)x = 0 \tag{4.7}$$

is oscillatory. Then (1.1) *has no positive solution. In particular,* (1.1) *has no positive non-monotone solution.*

To prove Theorem 4.5, we need the following well-known result.

Lemma 4.6. *If*

$$(p(t)y')' + q(t)y \le 0 \tag{4.8}$$

has an eventually positive solution, then so is

$$(p(t)x')' + q(t)x = 0. \tag{4.9}$$

For the proof of Lemma 4.6, see for example Onose [12].

Proof of Theorem 4.5. Assume, to the contrary, that there exists a solution $x(t)$ of (1.1) such that $x(t) > 0$ on $[t_1, \infty)$ for some $t_1 \ge t_0$. Set $y(t) = x(t) - \theta(t)$. Then

$$(p(t)y'(t))' = -q(t)x(t) < 0, \quad t \ge t_1,$$

which implies that $p(t)y'(t)$ is decreasing on $[t_1, \infty)$. Hence, either the following (i) or (ii) holds: (i) $p(t)y'(t) \ge 0$ on $[t_1, \infty)$; (ii) $p(t)y'(t) < 0$ on $[t_2, \infty)$ for some $t_2 \ge t_1$. Assume that (ii) holds. Since $p(t)y'(t)$ is decreasing and negative on $[t_2, \infty)$, we find that

$$p(t)y'(t) \le p(t_2)y'(t_2) < 0, \quad t \ge t_2,$$

that is,

$$y'(t) \le \frac{p(t_2)y'(t_2)}{p(t)}, \quad t \ge t_2. \tag{4.10}$$

Integrating (4.10) on $[t_2, t]$, we have

$$y(t) \le y(t_2) + p(t_2)y'(t_2) \int_{t_2}^{t} \frac{1}{p(s)} ds.$$

Letting $t \to \infty$, by (2.14), we have $\lim_{t\to\infty} y(t) = -\infty$. On the other hand, since $y(t) = x(t) - \theta(t) > -\theta(t)$ on $[t_1, \infty)$, and hence

$$\limsup_{t\to\infty} y(t) \ge -\liminf_{t\to\infty} \theta(t) = 0, \tag{4.11}$$

which is a contradiction, and hence (i) holds.

From (i) it follows that $y'(t) \ge 0$ for $t \ge t_1$, which means that $y(t)$ is nondecreasing on $[t_1, \infty)$. Therefore, either $y(t) > 0$ on $[t_3, \infty)$ for some $t_3 \ge t_1$ or $y(t) \le 0$ on $[t_1, \infty)$. If $y(t) \le 0$ on $[t_1, \infty)$, then

$$\theta(t) \ge y(t) + \theta(t) = x(t) > 0, \quad t \ge t_1,$$

which contradicts the fact that $\theta(t)$ is not positive, and hence $y(t) > 0$ on $[t_3, \infty)$ for some $t_3 \ge t_1$. Since $\liminf_{t\to\infty} \theta(t) = 0$, there exists $t_4 \ge t_3$ such that

$$\theta(t) \ge -(1-\lambda)y(t_3), \quad t \ge t_4.$$

Since $y(t) \ge y(t_3)$ for $t \ge t_4$, we have

$$\theta(t) \ge -(1-\lambda)y(t), \quad t \ge t_4,$$

which implies

$$x(t) = y(t) + \theta(t) \ge y(t) - (1-\lambda)y(t) = \lambda y(t), \quad t \ge t_4.$$

Therefore $y(t)$ is a positive solution of

$$(p(t)y'(t))' + \lambda q(t)y(t) \le 0, \quad t \ge t_4.$$

Lemma 4.6 implies that (4.7) also has a positive solution. This is a contradiction. □

The following result is well-known as the Leighton–Winter oscillation criterion and the Hille oscillation criterion. See, for example, [17].

Lemma 4.7. *Assume that* (2.14) *holds. Then every solution of* (4.9) *is oscillatory if either*

$$\int_{t_0}^{\infty} q(t)dt = \infty \tag{4.12}$$

or (3.1) *holds and*

$$\liminf_{t\to\infty} P(t) \int_t^{\infty} q(s)ds > \frac{1}{4}. \tag{4.13}$$

Corollary 4.8. *Let* (2.14) *hold and θ be a solution of equation* (1.3) *such that $\theta(t)$ is not eventually positive and satisfies* (4.6)*. Assume that either* (4.12) *holds or* (3.1) *and* (4.13) *hold. Then* (1.1) *has no eventually positive solution. In particular,* (1.1) *has no positive non-monotone solution.*

Proof. Assume that

$$\lim_{t\to\infty} f(t) > \frac{1}{4}.$$

for some $f \in C(t_0, \infty)$. Then there exists a constant c such that

$$\lim_{t\to\infty} f(t) > c > \frac{1}{4}.$$

We can take $\lambda \in (0, 1)$ such that $\lambda c > 1/4$, and hence

$$\lim_{t\to\infty} \lambda f(t) > \lambda c > \frac{1}{4}.$$

Therefore, Theorem 4.5 and Lemma 4.7 imply Corollary 4.8. □

5 Some open questions

5.1 Euler type equations

According to considerations from Remarks 2.10 and 2.18 about the Euler type equation (2.12), we are able to pose the next question:

Open Question 5.1. *Find sufficient conditions on $\mu \in \mathbb{R}$, $\lambda > 0$ and continuous function $f(t)$ such that every positive solution $x(t)$ of the Euler type equation* (2.12) *is a non-monotone function on* $[t_0, \infty)$.

5.2 Non-monotone positive solutions and upper-lower solutions technique

We start this subsection with the next classic definition: arbitrary two functions $\alpha = \alpha(t)$, $\alpha \in C^2$ and $\beta = \beta(t)$, $\beta \in C^2$ are said to be respectively the lower and upper solutions of equation (1.1) if the following inequalities are satisfied:

$$\big(p(t)\alpha'\big)' + q(t)\alpha \geq e(t), \quad t > t_0, \tag{5.1}$$

$$\big(p(t)\beta'\big)' + q(t)\beta \leq e(t), \quad t > t_0. \tag{5.2}$$

Here we suppose that lower and upper solutions of equation (1.1) are well-ordered, that is,

$$\alpha(t) \leq \beta(t), \quad t \geq t_0.$$

About the method of lower and upper solutions method in the second-order differential equations we refer reader to [8]. The next principle gives the relation between the well-ordered lower and upper solutions with the reverse-ordered first derivatives.

Lemma 5.1. *If $\alpha(t)$ and $\beta(t)$ are well-ordered lower and upper solutions of equation (1.1) such that*

$$\alpha'(t_0) \geq \beta'(t_0), \tag{5.3}$$

then $\alpha'(t)$ and $\beta'(t)$ are reverse-ordered, that is,

$$\alpha'(t) \geq \beta'(t), \quad t \geq t_0.$$

Proof. From (5.1) and (5.2) we derive:

$$\big(p(t)\alpha'\big)' + q(t)\alpha \geq \big(p(t)\beta'\big)' + q(t)\beta,$$

which together with (5.2) gives:

$$\big(p(t)(\alpha'(t) - \beta'(t))\big)' \geq q(t)(\beta(t) - \alpha(t)) \geq 0, \quad t > t_0.$$

Integrating this inequality and using (5.3), we obtain

$$p(t)(\alpha'(t) - \beta'(t)) \geq p(t_0)(\alpha'(t_0) - \beta'(t_0)) \geq 0, \quad t > t_0,$$

which proves that $\alpha'(t) \geq \beta'(t)$, $t \geq t_0$. □

Such a comparison principle can be proved for solutions of equation (1.1).

Theorem 5.2. *Let $\alpha(t)$ and $\beta(t)$ be the well-ordered lower and upper solutions of equation (1.1) satisfying (5.3). If a solution $x(t)$ of equation (1.1) satisfies*

$$\begin{cases} \alpha(t) \leq x(t) \leq \beta(t),\ t \geq t_0, \\ \alpha'(t_0) \geq x'(t_0) \geq \beta'(t_0), \end{cases} \tag{5.4}$$

then

$$\alpha'(t) \geq x'(t) \geq \beta'(t), \quad t \geq t_0. \tag{5.5}$$

As a consequence we easily derive the following criterion for non-monotonicity of solutions.

Corollary 5.3 (Criterion for non-monotonicity of a solution). *Let $\alpha(t)$ and $\beta(t)$ be the well-ordered lower and upper solutions of equation (1.1) satisfying (5.3). If $\alpha(t)$ and $\beta(t)$ are non-monotonic on $[t_0, \infty)$, then every solution $x(t)$ of equation (1.1) that satisfies (5.4) is also non-monotonic on $[t_0, \infty)$.*

Proof of Theorem 5.2. Since every solution $x(t)$ of equation (1.1) is an upper solution of (1.1), Lemma 5.1 and assumption (5.4) imply $\alpha'(t) \geq x'(t)$. Since $x(t)$ is also a lower solution of (1.1), Lemma 5.1 and (5.4) again give $x'(t) \geq \beta'(t)$. □

Proof of Corollary 5.3. From assumption that $\alpha(t)$ and $\beta(t)$ are two non-monotone functions, there exist two sequences s_n and t_n, $s_n \to \infty$ and $t_n \to \infty$ as $t \to \infty$, and $n_0 \in \mathbb{N}$ such that

$$\alpha'(s_n) < 0 \quad \text{and} \quad \beta'(t_n) > 0, \quad n \geq n_0.$$

Now, taking into account the conclusion (5.5), from previous we derive that

$$x'(s_n) \leq \alpha'(s_n) < 0 \quad \text{and} \quad x'(t_n) \geq \beta'(t_n) > 0, \quad n \geq n_0.$$

It verifies that $x'(t)$ is a sign-changing function, that is, $x(t)$ is a non-monotone function on $[t_0, \infty)$. □

According to the preceding observation, we can pose the following question.

Open Question 5.2. *Find concrete classes of functions $p(t)$, $q(t)$ and $e(t)$ such that equation* (1.1) *admits exact well-ordered lower and upper non-monotone solutions $\alpha(t)$ and $\beta(t)$ satisfying* (5.3) *as well as an exact non-monotone solution $x(t)$ satisfying* (5.4) *and* (5.5).

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