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Stability and instability of solutions of semilinear problems with Dirichlet boundary condition on surfaces of revolution

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Abstract. We consider the equation $\Delta u + f(u) = 0$ on a surface of revolution with Dirichlet boundary conditions. We obtain conditions on f, the geometry of the surface and the maximum value of a positive solution in order to ensure its stability or instability. Applications are given for our main results.

Keywords: surface of revolution, stability or instability of solutions.

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1 Introduction

In [13], P. Korman provides results of stability and instability for positive solutions of the problem

$$\Delta u(x) + \lambda f(u(x)) = 0, \quad \text{for } |x| < 1, \quad u = 0, \quad \text{when } |x| = 1,$$

with $x \in \mathbb{R}^n$ $(n = 1, 2)$, $f \in C^1(\mathbb{R}^+)$ and λ a positive parameter. As application, some multiplicity results are obtained.

In this paper, we extend some of these results for certain classes of surfaces of revolution in $\mathbb{R}^3$. We consider the positive solutions of

$$\left.\begin{array}{ll} \Delta_g u(x) + f(u(x)) = 0 & x \in \mathcal{S} \\ u(x) = 0 & x \in \partial\mathcal{S} \end{array}\right\} \tag{1.1}$$

where $\mathcal{S} \subset \mathbb{R}^3$ is a surface of revolution with metric g, Δ_g stands for the Laplace–Beltrami operator in $\mathcal{S}$ and $f \in C^1(\mathbb{R}^+)$. Since $\mathcal{S}$ satisfies certain conditions we proceeded as in [13] to prove as the stability or instability can often be determined by the maximum value of $u(x)$. Basically, we studied the sign of $h(u) - h(\alpha)$ in $(0, \alpha)$ where $h(u) = 2\int_0^u f(t)dt - uf(u)$ and α is the maximum value of positive solution $u(x)$ in $\mathcal{S}$. Under some conditions, we conclude that if $h(u) - h(\alpha) > 0$ $(h(u) - h(\alpha) < 0)$ then the solution is stable (unstable). It is common to consider the function h in such matters, we cite [10, 12–14] and references therein.

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Recently it has been considered by some authors the question of stability in problems on surfaces of revolution or, in a more general setting, on compact Riemannian manifold. For example, see [1,6,7,16,18] for problems on surfaces without boundary or with Neumann boundary conditions and [2] for a problem with Robin boundary conditions. This work seems to be the first to consider the problem with Dirichlet boundary conditions.

In the final section, we explained how our results can be applied to obtain the multiplicity of solutions, in addition we present two simple examples. More specifically, we introduced a positive parameter λ in (1.1) (i.e. $\Delta_g u(x) + \lambda f(u(x)) = 0$) and, if $u(x)$ is a positive solution, then $\|u\|_{L^\infty(\mathcal{S})}$ uniquely identifies the solution pair $(\lambda, u(x))$ [5]. Hence, the solution set of (1.1) can be depicted by planar curves in $(\lambda, \|u\|_{L^\infty(\mathcal{S})})$ plane and our stability and instability results indicate the turning points of this curve. For more detail on this subject, see [12,14] for instance.

This paper is divided as follows. In Section 2 we recall some material from differential geometry and stability of solution. Moreover we prove two essential propositions to our approach. In Section 3 we present a result of instability for a class of surfaces of revolution that has only one pole. In Section 4 we consider $\mathcal{S}$ a cylindrical surface to obtain conditions for stability and instability while Section 5 is devoted to applications.

2 Preliminaries

We begin with some definitions and known results from differential geometry which will be used in the following sections.

2.1 Surface of revolution

Consider $\mathcal{M} = (\mathcal{M}, g)$ a 2-dimensional Riemannian manifold with a metric given in local coordinates $x = (x^1, x^2)$ given by (using Einstein summation convention)

$$dr^2 = g_{ij} dx^i dx^j, \quad (g^{ij}) = (g_{ij}^{-1}), \quad |g| = \det(g_{ij}).$$

Given a smooth vector field X on $\mathcal{M}$, the divergence operator of X is defined as

$$\operatorname{div}_g X = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} X^i \right)$$

and the Riemannian gradient, denoted by ∇_g, of a sufficiently smooth real function ϕ defined on $\mathcal{M}$, as the vector field

$$(\nabla_g \phi)^i = g^{ij} \partial_j \phi.$$

We will see how the operator Δ_g can be expressed for the particular case where $\mathcal{M}$ is a surface of revolution. Let C be the curve of $\mathbb{R}^3$ parametrized by

$$\begin{cases} x_1 = \psi(s) \\ x_2 = 0 \\ x_3 = \chi(s) \end{cases} \qquad (s \in I := [0, l])$$

where $\psi, \chi \in C^2(I)$, $\psi > 0$ in $(0, l)$ and $(\psi')^2 + (\chi')^2 = 1$ in I. Moreover,

$$\psi(0) = \psi(l) = 0, \tag{2.1}$$

and

$$\psi'(0) = -\psi'(l) = 1. \tag{2.2}$$

Let $\mathcal{M}$ be the surface of revolution parametrized by

$$\begin{cases} x_1 = \psi(s)\cos(\theta) \\ x_2 = \psi(s)\sin(\theta) \qquad (s,\theta) \in [0,l] \times [0,2\pi). \\ x_3 = \chi(s) \end{cases} \tag{2.3}$$

Setting $x^1 = s, x^2 = \theta$ then a surface of revolution in $\mathbb{R}^3$ with the above parametrization is a 2-dimensional Riemannian manifold with metric

$$g = ds^2 + \psi^2(s)d\theta^2.$$

By (2.1) and (2.2) $\mathcal{M}$ has no boundary and we always assume that $\mathcal{M}$ and the Riemannian metric g on it are smooth (see [3], for instance). The area element on $\mathcal{M}$ is $d\sigma = \psi d\theta ds$ and the gradient of u with respect to the metric g is given by

$$\nabla_g u = \left(\partial_s u, \frac{1}{\psi^2}\partial_\theta u\right).$$

Hence,

$$\Delta_g u = u_{ss} + \frac{\psi_s}{\psi}u_s + \frac{1}{\psi^2}u_{\theta\theta}. \tag{2.4}$$

2.2 Stability analysis

Consider a solution $u(x)$ of (1.1) with $\mathcal{S} \subset \mathcal{M}$ a surface of revolution with boundary. The eigenvalue problem for the corresponding linearized equation is

$$\begin{cases} \Delta_g\phi(x) + f'(u)\phi(x) + \mu\phi = 0, & x \in \mathcal{S} \\ \phi(x) = 0 & x \in \partial\mathcal{S}. \end{cases} \tag{2.5}$$

It is well know that if the principal eigenvalue μ_1 is positive then $u(x)$ is *stable* and if μ_1 is negative then $u(x)$ is *unstable*. In the case $\mu_1 = 0$, $u(x)$ is sometimes called *neutrally stable*.

This is so called linear stability and, roughly speaking, means that solutions of the corresponding parabolic equation,

$$\left.\begin{array}{ll} u_t = \Delta_g u + f(u) & (t,x) \in \mathbb{R}^+ \times \mathcal{S} \\ u = 0 & (t,x) \in \mathbb{R}^+ \times \partial\mathcal{S} \end{array}\right\} \tag{2.6}$$

with the initial data near u will tend to u, as $t \to \infty$.

Some properties on the principal eigenpair (μ_1, ϕ_1) of (2.5) have fundamental role in this work. Namely, μ_1 is a simple eigenvalue (i.e. the eigenspace corresponding to μ_1 is one-dimensional); ϕ_1 can be assumed positive in $\mathcal{S}$ and $\int_{\mathcal{S}} \phi_1^2 d\sigma = 1$. We outline the proof of the first one below. The others we omitted since classical argument of linearized stability can be applied to the present situation (e.g., see [9])

Let us start with a simple observation concerning solutions of (1.1). This result was observed in [1, 6, 18] for Neumann boundary condition, in [2] for Robin boundary conditions, and for convenience of the reader we will prove it in our case.

Proposition 2.1. *Every solution $u(x)$ of problem (1.1), which depends on the angular variable θ, is unstable.*

Proof. We have that u satisfies the equation

$$u_{ss} + \frac{\psi_s}{\psi} u_s + \frac{1}{\psi^2} u_{\theta\theta} + f(u) = 0.$$

Now, if we differentiate this equation with respect to θ we see that u_θ is an eigenfunction of (2.5) with corresponding eigenvalue $\mu = 0$. Since u_θ must change sign it cannot be the eigenfunction corresponding to the lowest eigenvalue. Hence $\mu_1 < 0$. □

Proposition 2.2. *If ϕ_1 is an eigenfunction corresponding to the principal eigenvalue μ_1 of problem (2.5) then ϕ_1 is independent of θ.*

Proof. We first observe that for any $\theta_0 > 0$, $\phi_1(s, \theta + \theta_0)$ is also an eigenfunction corresponding to μ_1. Moreover we have that ϕ_1 is 2π-periodic in θ and

$$\int_0^1 \int_0^{2\pi} \phi_1^2(s,\theta)\psi d\theta ds = 1. \tag{2.7}$$

It is well known that μ_1 is a simple eigenvalue. We outline the proof for the reader's convenience. We suppose that ϕ_2 $(= \phi_1(s, \theta + \theta_0)$, for instance) is also an eigenfunction corresponding to μ_1 and then ϕ_1 and ϕ_2 satisfy the equation

$$\Delta_g \phi + f_u(u_\epsilon, x)\phi + \mu_1 \phi = 0 \quad \text{in } \mathcal{S}. \tag{2.8}$$

We can assume $\phi_1 > 0$, $\phi_2 > 0$ and is not difficult to see that

$$\begin{aligned} 0 &= \phi_1 \Delta_g \phi_2 - \phi_2 \Delta_g \phi_1 \\ &= \nabla_g(\phi_1 \nabla_g \phi_2 - \phi_2 \nabla_g \phi_1) \\ &= \nabla_g(\phi_1^2 \nabla_g(\phi_2/\phi_1)). \end{aligned}$$

Using Green's theorem it follows that

$$\begin{aligned} 0 &= \int_{\mathcal{S}} (\phi_2/\phi_1) \nabla_g \left[\phi_1^2 \nabla_g(\phi_2/\phi_1)\right] d\sigma \\ &= \int_{\mathcal{S}} \phi_2\phi_1 \Delta(\phi_2/\phi_1) d\sigma + \int_{\mathcal{S}} (\phi_2/\phi_1)\nabla_g(\phi_2)\nabla_g(\phi_2/\phi_1) d\sigma \\ &= \int_{\mathcal{S}} \phi_2\phi_1 \Delta(\phi_2/\phi_1) d\sigma - \int_{\mathcal{S}} \phi_1^2 \nabla_g[(\phi_2/\phi_1)\nabla_g(\phi_2/\phi_1)] d\sigma \\ &= -\int_{\mathcal{S}} \phi_1^2 \left|\nabla_g(\phi_2/\phi_1)\right|^2 d\sigma. \end{aligned}$$

To use Green's theorem we define (ϕ_2/ϕ_1) and each component of $\nabla_g(\phi_2/\phi_1)$, as well as its derivatives, on $\partial\mathcal{S}$, using a limit process so as to make it functions of $H^1(\mathcal{S})$. Therefore, we prove that ϕ_2 differs from ϕ_1 by a multiplicative constant and our claim follows.

Hence, there exists a constant $k > 0$ such that

$$\phi_1(s,\theta) = k\phi_1(s, \theta + \theta_0),$$

and by (2.7)

$$\int_0^1 \int_0^{2\pi} \phi_1^2(s,\theta+\theta_0)\psi d\theta ds = \int_0^1 \int_{\theta_0}^{2\pi+\theta_0} \phi_1^2(s,\theta)\psi d\theta ds$$
$$= \int_0^1 \int_0^{2\pi} \phi_1^2(s,\theta)\psi d\theta ds = 1,$$

then

$$1 = \int_0^1 \int_0^{2\pi} \phi_1^2(s,\theta)\psi d\theta ds = k^2 \int_0^1 \int_0^{2\pi} \phi_1^2(s,\theta+\theta_0)\psi d\theta ds = k^2.$$

It follows that $k = 1$ for any $\theta_0 > 0$, $0 \le s \le 1$ and $0 < \theta < 2\pi$ which proves the proposition. $\square$

3 A result of instability

Consider $\mathcal{S} = \mathcal{D} \subset \mathcal{M}$ a surface of revolution with boundary such that $\mathcal{D}$ has one of the poles but not the other. For example, consider $\psi(1) > 0$ and $\partial\mathcal{D} = C_1$ where $0 < 1 < l$. Then C_1 is parametrized in the local coordinates (s,θ)

$$C_1 : \begin{cases} s(t) = 1 \\ \theta(t) = t \end{cases}$$

with $t \in [0, 2\pi)$ and $\mathcal{D}$ is parametrized by

$$\begin{cases} x_1 = \psi(s)\cos(\theta) \\ x_2 = \psi(s)\sin(\theta) \\ x_3 = \chi(s) \end{cases} \qquad (s,\theta) \in [0,1] \times [0,2\pi). \tag{3.1}$$

By (2.4), the problem (1.1) on $\mathcal{D}$ reduces to

$$\left.\begin{array}{ll} u_{ss} + \dfrac{\psi_s}{\psi}u_s + \dfrac{1}{\psi^2}u_{\theta\theta} + f(u) = 0, & (s,\theta) \in (0,1) \times [0,2\pi) \\ u(1,\theta) = 0 & \theta \in [0,2\pi) \end{array}\right\} \tag{3.2}$$

and the eigenvalue problem for the corresponding linearized equation is

$$\left.\begin{array}{ll} \phi_{ss} + \dfrac{\psi_s}{\psi}\phi_s + \dfrac{1}{\psi^2}\phi_{\theta\theta} + f'(u)\phi + \mu\phi = 0, & (s,\theta) \in (0,1) \times [0,2\pi) \\ \phi(1,\theta) = 0 & \theta \in [0,2\pi). \end{array}\right\} \tag{3.3}$$

In order to state our main result, we define

$$h(u) = 2F(u) - uf(u) \tag{3.4}$$

where $F(u) = \int_0^u f(t)dt$.

We use the notation $v'(s)$ instead of $v_s(s)$ when it is convenient.

Theorem 3.1. *Assume that*

(h_1) $\psi'(s) > 0$ *for* $s \in (0,l)$;

(h_2) $\psi''(s) \leq 0$ *for* $s \in (0,l)$;

(h_3) $f(u) > 0$ *for* $u > 0$;

(h_4) $f'(u) > 0$ *for* $u > 0$ *and*

(h_5) $h(u) > h(\alpha)$ *for* $u < \alpha$.

Then the positive solution of (1.1) *($\mathcal{S} = \mathcal{D}$), with* $\|u\|_{L^\infty(\mathcal{D})} = \alpha$, *is unstable.*

Proof. As noted above, we can consider the positive solutions of (3.2). By Proposition 2.1, the problem (3.2) reduces to

$$\left.\begin{aligned} &u_{ss} + \frac{\psi_s}{\psi} u_s + f(u) = 0, \quad s \in (0,1) \\ &u_s(0) = u(1) = 0. \end{aligned}\right\} \tag{3.5}$$

It should be remarked that the boundary condition $u_s(0) = 0$ follows from a simple computation taking into consideration the hypothesis on ψ and χ.

Now note that

$$u'(s) \leq 0, \quad \text{for all } s \in [0,1).$$

Indeed, assuming otherwise, $u(s)$ would have a point of local minimum in $(0,1)$, at which the left hand side of (3.5) is positive (see (h_3)), a contradiction. Thus, we can conclude that $u(0)$ is the maximum value of solution, i.e., $\|u\|_{L^\infty(\mathcal{D})} = u(0)$.

Let μ_1 be the principal eigenvalue of (2.5) (i.e. of (3.3)) and ϕ_1 the corresponding eigenfunction. We have that ϕ_1 can be assumed positive on $(0,1)$ and moreover, by Proposition 2.2, ϕ_1 is independent of θ. Hence, the pair (μ_1, ϕ_1) satisfies

$$\left.\begin{aligned} &\phi_1'' + \frac{\psi'}{\psi}\phi_1' + f'(u)\phi_1 + \mu_1\phi_1 = 0, \quad s \in (0,1) \\ &\phi_1'(0) = \phi_1(1) = 0. \end{aligned}\right\} \tag{3.6}$$

We need to prove that $\mu_1 < 0$. Assume on the contrary that $\mu_1 \geq 0$. In this case,

$$\phi_1'(s) \leq 0, \quad \text{for all } s \in [0,1)$$

and the argument is the same as above for $u(s)$, but now we use (h_4).

We claim that

$$p(s) := \psi(s)\left[\phi_1'(s)\psi(s)u'(s) + \phi_1(s)\psi'(s)u'(s) - u''(s)\phi_1(s)\psi(s)\right] > 0, \tag{3.7}$$

for all $s \in (0,1)$.

Indeed, $p(0) = 0$ and $p(s)$ is increasing in $(0,1)$ since expressing u'' and ϕ_1'' from the corresponding equations, we have that

$$p'(s) = 2\psi(s)\psi'(s)u'(s)\phi_1'(s) + 2\psi(s)\phi_1(s)\psi''(s)u'(s) - \mu_1\phi_1(s)\psi^2(s)u'(s) > 0$$

for all $s \in (0,1)$. Note that here we use the hypothesis (h_1) and (h_2).

Now, from the equations (1.1) and (2.5) we have

$$\int_{\mathcal{D}}[f(u)-uf'(u)]\phi_1 d\sigma=\int_{\mathcal{D}}\mu_1 u\phi_1 d\sigma>0.$$

On the other hand, in view of (h_5) and (3.7),

$$\begin{aligned}\int_{\mathcal{D}}[f(u)-uf'(u)]\phi_1 d\sigma &= 2\pi\int_0^1 \frac{d}{ds}[h(u)-h(\alpha)]\frac{\phi_1\psi}{u'}ds\\ &= -2\pi\int_0^1 [h(u)-h(\alpha)]\frac{[(\phi_1\psi)'u'-u''\phi_1\psi]}{(u')^2}ds\\ &= -2\pi\int_0^1 [h(u)-h(\alpha)]\frac{p(s)}{\psi(u')^2}ds<0,\end{aligned}$$

which is a contradiction. □

Remark 3.2.

(i) It is not difficult to get $\mathcal{D}$ with ψ satisfying (h_1) and (h_2). A simple example is $\psi(s)=(2/\pi)\sin(\pi s/2)$, $\chi(s)=(2/\pi)\cos(\pi s/2)$ with $s\in(0,1)$. In this case $\mathcal{D}$ is the north hemisphere of a sphere of radius $2/\pi$. Obviously, surfaces that have the south pole and not the north pole can also be obtained. However, a careful analysis of the proof above shows that symmetry conditions on ψ are required if $\mathcal{S}$ has no poles. Such conditions reduce $\mathcal{S}$ to a cylindrical surface and this is the subject of the next section.

(ii) For the hypothesis (h_3) and (h_4), $f(u)=e^{\frac{u}{1+\epsilon u}}$, $\epsilon>0$, is an important example since it is related to perturbed Gelfand problem.

(iii) The Gaussian curvature of $\mathcal{S}$ is given by $K(s)=(-\psi''/\psi)(s)$ whereas $K_g(s)=(\psi'/\psi)(s)$ represents the geodesic curvature of the parallel circles $s=$ constant on $\mathcal{S}$ (see e.g. [1,2,7]). Hence, by (h_1) and (h_2), $K_g(s)>0$ and $K(s)\geq 0$ for $s\in(0,l)$.

4 Stability and instability on cylindrical surfaces

In this section we consider the problem (1.1) with $\mathcal{S}=\mathcal{C}$,

$$\left.\begin{array}{ll}\Delta_g u(x)+f(u(x))=0 & x\in\mathcal{C}\\ u(x)=0 & x\in\partial\mathcal{C}\end{array}\right\}\tag{4.1}$$

where $\mathcal{C}\subset\mathcal{M}$ is a cylindrical surface parametrized by

$$\begin{cases}x_1=a\cos(\theta)\\ x_2=a\sin(\theta) \qquad\qquad (s,\theta)\in[0,1]\times[0,2\pi).\\ x_3=s\end{cases}\tag{4.2}$$

Here $\psi(s)=a$ in $[0,1]$ and $a>0$ is a constant. By (2.4), the problem (4.1) reduces to

$$\left.\begin{array}{ll}u_{ss}+\dfrac{1}{a^2}u_{\theta\theta}+f(u)=0, & (s,\theta)\in(0,1)\times[0,2\pi)\\ u(0,\theta)=u(1,\theta)=0 & \theta\in[0,2\pi).\end{array}\right\}\tag{4.3}$$

As before, denote $h(u)=2F(u)-uf(u)$, where $F(u)=\int_0^u f(t)dt$.

Theorem 4.1.

(i) If

$$h(\alpha) < h(u), \ \text{for all } u < \alpha, \tag{4.4}$$

then the positive solution of (4.1), with $\|u\|_{L^\infty(\mathcal{C})} = \alpha$, is unstable.

(ii) On the other hand, if

$$h(\alpha) > h(u), \ \text{for all } u < \alpha, \tag{4.5}$$

then the positive solution of (4.1), with $\|u\|_{L^\infty(\mathcal{C})} = \alpha$, is stable.

Proof. By Proposition 2.1, instead of (4.1), we can consider the problem (see (4.3))

$$\left.\begin{array}{ll} u_{ss} + f(u) = 0, & s \in (0,1) \\ u(0) = u(1) = 0. & \end{array}\right\} \tag{4.6}$$

It is well known that positive solutions of (4.6) are symmetric functions about $1/2$ (see [8]), with

$$u'(s) > 0 \ \text{ for } s \in (0,1/2) \ \text{ and } \ u'(s) < 0 \ \text{ for } s \in (1/2,1).$$

Therefore, we conclude that $\|u\|_{L^\infty(\mathcal{C})} = u(1/2)$.

Again, let μ_1 be the principal eigenvalue of

$$\begin{cases} \Delta_g \phi(x) + f'(u)\phi(x) + \mu\phi = 0, & x \in \mathcal{C} \\ \phi(x) = 0 & x \in \partial\mathcal{C} \end{cases} \tag{4.7}$$

and ϕ_1 the corresponding eigenfunction. By (2.8) and Proposition 2.2, ϕ_1 is a solution of

$$\left.\begin{array}{ll} \phi'' + f'(u)\phi + \mu_1\phi = 0, & s \in (0,1) \\ \phi(0) = \phi(1) = 0. & \end{array}\right\} \tag{4.8}$$

Observe that $\phi_1(s)$ is also symmetric about $1/2$ since, assuming otherwise, $\phi_1(1-s)$ would give us another solution to the problem (4.8), contradicting the simplicity of the principal eigenvalue. Hence,

$$\phi_1'(s) > 0 \ \text{ for } s \in (0,1/2) \ \text{ and } \ \phi_1'(s) < 0 \ \text{ for } s \in (1/2,1).$$

In order to prove (i) assume on the contrary that $\mu_1 \geq 0$. We claim that

$$p(s) := \psi(s)[\phi_1'(s)u'(s) - \phi_1(s)u''(s)] > 0, \ \text{ for } s \in (0,1). \tag{4.9}$$

Indeed, note that $u'(1/2) = 0$, $u''(1/2) < 0$ and so

$$p(1/2) = -\psi(1/2)\phi_1(1/2)u''(1/2) > 0.$$

As

$$p'(s) = -\mu_1\psi(s)\phi_1(s)u'(s), \tag{4.10}$$

we have that $p(s)$ is increasing for $s \in (1/2,1)$ and decreasing for $s \in (0,1/2)$ which proves our claim.

Now, from the equations (4.6) and (4.8),

$$\begin{aligned}\int_{\mathcal{C}} h'(u(x))\phi_1(x)d\sigma &= \int_{\mathcal{C}}[f(u(x)) - u(x)f'(u(x))]\phi_1(x)d\sigma \\ &= \int_{\mathcal{C}} \mu_1 u(x)\phi_1(x)d\sigma \geq 0.\end{aligned}$$

On the other hand, from (4.4) and (4.9),

$$\begin{aligned}\int_{\mathcal{C}} h'(u(x))\phi_1(x)d\sigma &= 2\pi\int_0^1 \frac{d}{ds}[h(u(s)) - h(\alpha)]\frac{\psi(s)w(s)}{u'(s)}ds \\ &= -2\pi\int_0^1 [h(u(s)) - h(\alpha)]\frac{p(s)}{(u'(s))^2}ds \\ &< 0,\end{aligned}$$

which is a contradiction.

To prove (ii) assume $\mu_1 \leq 0$. Again, we have $p(s) = \psi(s)[\phi_1'(s)u'(s) - \phi_1(s)u''(s)] > 0$ since $p(0) = \psi(0)\phi_1'(0)u'(0) \geq 0$, $p(1) = \psi(1)\phi_1'(1)u'(1) \geq 0$ and (4.10) implies that $p(s)$ is increasing in $(0, 1/2)$ and decreasing in $(1/2, 1)$. Similarly to item (i) we have a contradiction,

$$\begin{aligned}\int_{\mathcal{C}} h'(u(x))\phi_1(x)d\sigma &= \int_{\mathcal{C}}[f(u(x)) - u(x)f'(u(x))]\phi_1(x)d\sigma \\ &= \int_{\mathcal{C}} \mu_1 u(x)\phi_1(x)d\sigma \leq 0\end{aligned}$$

and from (4.5)

$$\begin{aligned}\int_{\mathcal{C}} h'(u(x))\phi_1(x)d\sigma &= 2\pi\int_0^1 \frac{d}{ds}[h(u(s)) - h(\alpha)]\frac{\psi(s)w(s)}{u'(s)}ds \\ &= -2\pi\int_0^1 [h(u(s)) - h(\alpha)]\frac{p(s)}{(u'(s))^2}ds \\ &> 0.\end{aligned}$$

The theorem is proved. □

Remark 4.2. Unlike our results of instability (Theorem 3.1), Theorem 4.1 occurs for any $f(u)$. It is easy to see that the symmetry of $\mathcal{C}$ makes it possible.

5 Applications

In this section consider $\mathcal{S}$ a surface of revolution which can be either $\mathcal{D}$ or $\mathcal{C}$. Let u be a positive solution of (1.1) with a positive parameter λ introduced. Moreover, suppose that u is independent of θ, i.e., $u = u(s)$ is solution of

$$\left.\begin{array}{l}(\psi u')' + \psi\lambda f(u) = 0, \quad s \in (0,1) \\ u'(0) = u(1) = 0 \ \text{ if } \mathcal{S} = \mathcal{D} \ \text{ or } u(0) = u(1) = 0 \ \text{ if } \mathcal{S} = \mathcal{C}.\end{array}\right\} \tag{5.1}$$

If $\mathcal{S} = \mathcal{D}$ we have that $u(0)$ is the maximum value of u and, as mentioned in the Introduction, $\alpha = u(0)$ uniquely identifies the solution pair (λ, u). Hence the solution set of (5.1) can be depicted by planar curves in (λ, α) plane. The same is true if $\mathcal{S} = \mathcal{C}$, with $\alpha = u(1/2)$ the maximum value of u.

The behavior of the solution curves has been extensively studied for many different types of equations, for instance, see [11, 12, 14, 19] or the recent work [15] and references therein. In [13] a simple and general method was presented which we apply here to surfaces of revolution.

For the next result, we write $u = u(s, \alpha)$ and $\lambda = \lambda(\alpha)$. Recall that $\alpha = u(0)$ (or $\alpha = u(1/2)$) uniquely identify the pair (λ, u). The result below appears in [13] (Proposition 1) and the same proof, which is based on Sturm comparison theorem, can be used to our case (the presence of the function ψ adds no significant additional difficulty).

Proposition 5.1. *Let $u(s, \alpha)$ be a positive solution of (5.1), with $u(0, \alpha) = \alpha$ (or $u(1/2, \alpha) = \alpha$). Assume that $u_x(1, \alpha) < 0$. Then $\mu_1 < 0$ ($\mu_1 > 0$) if and only if $\lambda'(\alpha) < 0$ ($\lambda'(\alpha) > 0$).*

Remark 5.2. The hypothesis $u_x(1, \alpha) < 0$ always occurs when $\mathcal{S} = \mathcal{D}$ since, in this case, we assume $f(0) \geq 0$ (see p. 116 of [17] for instance). When $\mathcal{S} = \mathcal{C}$ is natural to assume that $u_x(1, \alpha) < 0$ since, if $u_x(1, \alpha) = 0$, we have symmetry breaking [12, p. 31].

Proposition 5.1 allows to apply the Theorems 3.1 and 4.1 in order to obtain the solution curve behavior in the (λ, α) plane and, consequently, multiplicity results. In particular, some problems presented in [11–14, 19] can be extended to surfaces of revolution considered here. For example, the Theorem 3.1 can be used to extend Theorem 5.3 of [13] where the perturbed Gelfand problem was considered on a 2-dimensional unit ball. In short, there is a interval (λ_1, λ_2) so that for any $\lambda \in (\lambda_1, \lambda_2)$ the problem $\Delta_g u(x) + \lambda e^{\frac{u}{1+\epsilon u}} = 0$ for $x \in \mathcal{D}$ (with $\epsilon > 0$ small, see Remark 3.2) and $u(x) = 0$ when $x \in \partial\mathcal{D}$, has at least three positive solutions that are independent of θ. In fact, in a future paper we consider only this case in order to prove that the solution curve is S-shaped.

For the reader's convenience, we detail another simple case. Take $\psi(s) = 1$, $\chi(s) = s$ for $s \in [0, 1]$ and $f(u) = au - u\sin(u)$ with $a > 1$. Then $\mathcal{S} = \mathcal{C}$ (i.e. $\mathcal{S}$ is a cylindrical surface) and we consider the problem

$$\left.\begin{array}{ll} \Delta_g u(x) + \lambda f(u(x)) = 0 & x \in \mathcal{C} \\ u(x) = 0 & x \in \partial\mathcal{C}, \end{array}\right\} \tag{5.2}$$

where $\lambda > 0$ is a parameter (the same problem on a interval was considered in [19] when $a = 2$).

The solutions that are independent of θ satisfies

$$\left.\begin{array}{ll} u'' + \lambda u(a - \sin(u)) = 0, & s \in (0, 1) \\ u(0) = u(1) = 0. & \end{array}\right\} \tag{5.3}$$

First, we note that the positive solutions of (5.3) lie in a bounded in λ strip. We follow the steps of [11, Lemma 3] to conclude that if (5.3) has a positive solution, then

$$\frac{\lambda_1}{a+1} < \lambda < \frac{\lambda_1}{a-1}, \tag{5.4}$$

where λ_1 is the principal eigenvalue of $-u''$ on the interval $(0, 1)$, with Dirichlet boundary conditions. We have that $\lambda_1 = \pi^2$ and $\phi_1 = \sin(\pi s)$ is the corresponding eigenfunction.

Observe that $0 < f(u) < (a+1)u$ for all $u > 0$. Multiplying the equation (5.3) by u, integrating by parts, and using the Poincaré inequality

$$(a+1)\lambda \int_0^1 u^2 ds > \lambda \int_0^1 u f(u) ds = \int_0^1 (u')^2 ds \geq \lambda_1 \int_0^1 u^2 ds,$$

from which the left inequality in (5.4) follows. Now, multiplying the equation (5.3) by $\phi_1 = \sin(\pi s)$ and integrating by parts twice over (0,1), we obtain

$$\int_0^1 [-\lambda_1 + a\lambda - \lambda \sin(u)] u \phi_1 ds = 0, \tag{5.5}$$

which is a contradiction if

$$-\lambda_1 + a\lambda - \lambda \sin(u) \geq 0,$$

for all $u > 0$. This would happen if $(-\lambda_1 + a\lambda)/\lambda \geq 1$. Thus, $\lambda < \lambda_1/(a-1)$.

Now, as $h(u)$ is given by

$$h(u) = -2\sin(u) + 2u\cos(u) + u^2 \sin(u)$$

we have that

- $\alpha_n = 3\pi/2 + 2\pi n$ is a sequence such that $h(u) > h(\alpha_n)$ for all $u \in (0, \alpha_n)$ and
- $\beta_n = \pi/2 + 2\pi n$ is a sequence such that $h(u) < h(\beta_n)$ for all $u \in (0, \beta_n)$.

By Theorem 4.1, solutions with $u(1/2) = \alpha_n$ are unstable and the ones with $u(1/2) = \beta_n$ are stable.

There is a curve of positive independent of θ solutions of (5.2) (i.e. positive solutions of (5.3)) in the $(\lambda, u(1/2))$ plane, which bifurcates from the trivial one at $\lambda_1/2$ ([4]). Finally, by Proposition 5.1 and (5.4), we can conclude that this curve has infinitely many turns. This occurs because there are infinitely many changes of stability to $u(1/2)$ increasing.

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