



On positive solutions of the Dirichlet problem involving the extrinsic mean curvature operator

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Abstract. In this paper, we are concerned with necessary conditions for the existence of positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space

$$\begin{aligned} -\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) &= f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

whose supremum norm bears a certain relationship to zeros of the nonlinearity f .

Keywords: Dirichlet problem, Minkowski-curvature operator, positive solutions, lower and upper solutions.

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1 Introduction

Hypersurfaces of prescribed mean curvature in Minkowski space, with coordinates (x_1, \dots, x_N, t) and metric $\sum_{i=1}^N (dx_i)^2 - (dt)^2$, are of interest in differential geometry and in general relativity (see e.g., [2, 14]). In this paper we are concerned with necessary conditions for the existence of such a kind of hypersurfaces which are graphs of solutions of the Dirichlet problem

$$\begin{aligned} -\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) &= f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

We assume throughout that Ω is a bounded domain in \mathbb{R}^N , with a boundary $\partial\Omega$ of class C^2 , and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -function satisfying the assumption

(H1) There exist $s_0, s_1, s_2 \in \mathbb{R}$ with $0 < s_0 < s_1 < s_2$ such that $f(s_0) \leq 0$, $f(s_j) = 0$ for $j = 1, 2$, $f(s) < 0$ in (s_0, s_1) and $f(s) > 0$ in (s_1, s_2) .

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Note that relatively little is known about the existence of positive solutions of problem (1.1) when Ω is a general bounded domain, see [11]. Yet, for one-dimensional cases and radial cases of (1.1), the existence and multiplicity of positive solutions have been widely considered in recent years, see e.g., [3, 4, 8, 9, 21, 22] and the references therein. Most of them allowed the nonlinearity f to be positive. It is worthy to point out that the result in [22], Ma and Lu used the quadrature arguments to show that if $s_2 < \frac{1}{2\sqrt{\kappa}}$ ($\kappa > 0$ is a constant) and

$$\int_{s_0}^{s_2} f(s)ds > 0, \quad (1.2)$$

then the nonlinear Dirichlet problem with one-dimension Minkowski-curvature operator

$$\begin{aligned} \left(\frac{u'}{\sqrt{1-\kappa u'^2}} \right)' + \lambda f(u) &= 0 \quad \text{in } (0, 1), \\ u(0) = u(1) &= 0 \end{aligned} \quad (1.3)$$

has at least two positive solutions u satisfying

$$u_{\max} = \max_{x \in [0,1]} u(x) \in (s_1, s_2) \quad (1.4)$$

for sufficiently large $\lambda > 0$. Their result is an analogous of the well-known result due to Brown and Budin [7], who studied the problem (1.3) with $\kappa = 0$ by using a generalization of the quadrature technique of Laetsch [18]. One is lead to show whether (1.2) is in fact a necessary condition for the existence of any positive solution of problem (1.3) satisfying (1.4). We shall answer this question in the affirmative employing the method of lower and upper solutions.

The existence and multiplicity of positive solutions for the analogous problem associated with the Laplacian operator

$$\begin{aligned} \Delta u + \lambda f(u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (1.5)$$

have been extensively studied in [1, 12, 13, 17] in the case when the nonlinearity f is allowed to change sign. In [17], Hess showed that, for all λ sufficiently large, (1.2) is a sufficient condition for the existence of any positive solution u of (1.5) satisfying

$$\|u\|_{\infty} = \max_{x \in \Omega} u(x) \in (s_1, s_2). \quad (1.6)$$

If the domain Ω satisfied a certain symmetry condition, it was proved by Cosner and Schmitt [12] that there exist lower bounds on the $C(\bar{\Omega})$ norm for certain positive solutions of (1.5). And then, Dancer and Schmitt [13] showed that (1.2) is in fact a necessary condition for the existence of any positive solution u of (1.5) satisfying (1.6), and that

$$\|u\|_{\infty} \geq r \quad (1.7)$$

holds for arbitrary domains, where $r \in (s_1, s_2)$ is given by

$$\int_{s_0}^r f(s)ds = 0. \quad (1.8)$$

Also recently, the above results are generalized by Loc and Schmitt in [19], who established (1.2) is a necessary and sufficient condition for the existence of positive solutions of quasilinear problem involving the p -Laplace operator.

Motivated by above papers [12,13,17,19,22], we shall attempt to show that (1.2) is in fact a necessary condition for the existence of any positive solution of problem (1.1) satisfying (1.6) and that (1.7) holds on the general bounded domain. To wit, we have

Theorem 1.1. *Assume (H1) and*

(H2) $s_2 < \frac{d(\Omega)}{2}$, *with $d(\Omega)$ the diameter of Ω .*

Let

$$\int_{s_0}^{s_2} f(s) ds \leq 0. \quad (1.9)$$

If problem (1.1) has a positive solution u , u cannot satisfy (1.6).

Theorem 1.2. *Assume (H1) and (H2). Let r be defined by (1.8). If u is a positive solution of (1.1) satisfying (1.6). Then $\|u\|_\infty \geq r$.*

Remark 1.3. Notice that (H2) is sharp. In fact, if u is a solution of (1.1), then $|\nabla u(x)| < 1$ and hence

$$\|u\|_\infty < \frac{d(\Omega)}{2}.$$

So, to get the multiplicity of solutions of (1.1), we only need to work with the function $f(s)$ in the interval $[0, \frac{d(\Omega)}{2})$.

The contents of this paper have been distributed as follows. In Section 2, we construct a linking local lower solution defined on different subdomains. Finally, Section 3 is devoted to the proof of our main results by applying the method of lower and upper solutions as developed in [10] and a result of [9] about the radial symmetry of positive solutions of (1.1) if Ω is a ball.

For other results concerning the Neumann problems associated with the prescribed mean curvature equation in Minkowski space we refer the reader to [5,20]. The basic tools concerning Sobolev spaces and maximum principles which we employ in this paper can be found in [16,23].

2 Linking local lower solution

By a similar argument from [6, Lemma 1.1] with obvious changes, we give the following lemma on linking local lower solution defined on different subdomains.

Lemma 2.1. *Assume that u is a positive solution of (1.1) which satisfies (1.6). Assume $f(0) \geq 0$. Let B denote a ball in \mathbb{R}^N , centered at the origin such that $\bar{\Omega} \subseteq B$. We denote by ν the unit outward normal to Ω and $\frac{\partial u}{\partial \nu} \leq 0$ on $\partial\Omega$. Let $\alpha(x)$ be defined by*

$$\alpha(x) = \begin{cases} u(x), & x \in \bar{\Omega}, \\ 0, & x \in \bar{B} \setminus \Omega. \end{cases} \quad (2.1)$$

Then α is a lower solution of the problem

$$\begin{aligned} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) + f(u) &= 0 && \text{in } B, \\ u &= 0 && \text{on } \partial B. \end{aligned} \quad (2.2)$$

Proof. For any $\varphi \in W_0^{1,1}(B)$ with $\varphi \geq 0$. Since

$$\begin{aligned} \int_{\Omega} \left[\frac{\nabla u \cdot \nabla \varphi}{\sqrt{1-|\nabla u|^2}} + \operatorname{div} \left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) \varphi \right] dx &= \int_{\Omega} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \varphi \right) dx \\ &= \int_{\partial\Omega} \varphi \frac{\nabla u \cdot \nu}{\sqrt{1-|\nabla u|^2}} dS \end{aligned} \quad (2.3)$$

and $\frac{\partial u}{\partial \nu} \leq 0$ on $\partial\Omega$. Then it follows from $f(0) \geq 0$ that

$$\begin{aligned} \int_B \left(\frac{\nabla \alpha}{\sqrt{1-|\nabla \alpha|^2}} \right) \cdot \nabla \varphi dx &= \int_{\Omega} \left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) \cdot \nabla \varphi dx \\ &= - \int_{\Omega} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) \varphi dx + \int_{\partial\Omega} \varphi \frac{\nabla u \cdot \nu}{\sqrt{1-|\nabla u|^2}} dS \\ &= - \int_{\Omega} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) \varphi dx + \int_{\partial\Omega} \varphi \frac{1}{\sqrt{1-|\nabla u|^2}} \frac{\partial u}{\partial \nu} dS \\ &\leq \int_{\Omega} f(u) \varphi dx \\ &\leq \int_B f(\alpha) \varphi dx. \end{aligned}$$

Thus α is a lower solution of (2.2). □

3 Proof of main results

We start with the following simple consequence of the strong maximum principle.

Lemma 3.1. *Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a C^1 -function, $a_0 > 0$ a number such that $g(a_0) \leq 0$, and u a classical positive solution of*

$$\begin{aligned} -\operatorname{div} \left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) &= g(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.1)$$

Then $\|u\|_{\infty} \neq a_0$.

Proof. Suppose, to the contrary, that $\|u\|_{\infty} = a_0$. Then $0 \leq u \leq a_0$ for all $x \in \Omega$. Note that there exists $m \geq 0$ such that $g(s) + ms$ is monotone increasing in s for $s \in [0, a_0]$. Then

$$-\operatorname{div} \left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) + mu = g(u) + mu,$$

and, since $-\operatorname{div} \left(\frac{\nabla a_0}{\sqrt{1-|\nabla a_0|^2}} \right) = 0 \geq g(a_0)$,

$$-\operatorname{div} \left(\frac{\nabla a_0}{\sqrt{1-|\nabla a_0|^2}} \right) + ma_0 \geq g(a_0) + ma_0.$$

Subtracting, we get

$$\begin{aligned} -\operatorname{div} \left(\frac{\nabla(a_0 - u)}{\sqrt{1 - |\nabla(a_0 - u)|^2}} \right) + m(a_0 - u) &\geq 0 && \text{in } \Omega, \\ a_0 - u &> 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.2)$$

The maximum principle implies that $a_0 - u > 0$ in $\bar{\Omega}$ and hence $\|u\|_\infty < a_0$, a contradiction. \square

Let $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. Denote

$$a(s) = \frac{1}{\sqrt{1-s}}. \quad (3.3)$$

Then the Dirichlet problem

$$\begin{aligned} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) + f(u) &= 0 && \text{in } B_R, \\ u &= 0 && \text{on } \partial B_R \end{aligned} \quad (3.4)$$

can be rewritten as

$$\begin{aligned} -\operatorname{div}(a(|\nabla u|^2)\nabla u) &= f(u) && \text{in } B_R, \\ u &= 0 && \text{on } \partial B_R. \end{aligned} \quad (3.5)$$

The following result is a simple modification of [9, Appendix], but we include the proof for the sake of completeness.

Lemma 3.2. *Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is of class C^1 . Then any positive solution $u \in C^2(\bar{B}_R)$ of (3.5) is radially symmetric. Moreover, $u'(r) < 0$ for $r \in (0, R)$.*

Proof. To prove u is radially symmetric, we use the result stated in the claim of [9, Appendix]. Notice that $u \in C^2(\bar{B}_R)$ given positive solution of (3.5) guarantees that there exists a constant $L \in (0, 1)$, such that

$$\|\nabla u\|_\infty < L. \quad (3.6)$$

Now by using the same truncation technique in [9, Appendix] and [15, Corollary 1], we may deduce that $u'(r) < 0$ for $r \in (0, R)$.

Indeed, let

$$\bar{a}(s) = \begin{cases} \alpha_1(s), & \text{if } s < 0, \\ a(s), & \text{if } 0 \leq s \leq L^2, \\ \alpha_2(s), & \text{if } L^2 < s < 1, \\ c, & \text{if } s \geq 1, \end{cases} \quad (3.7)$$

where the functions $\alpha_1, \alpha_2 : \mathbb{R} \rightarrow \mathbb{R}$ and the constant c are such that $\bar{a} \in C^\infty(\mathbb{R})$, \bar{a} is increasing and positive. Thus u is a positive solution of the modified problem

$$-\operatorname{div}(\bar{a}(|\nabla u|^2)\nabla u) = f(u) \quad \text{in } B_R, \quad u = 0 \quad \text{on } \partial B_R. \quad (3.8)$$

It is easy to check that the second order differential operator

$$\bar{a}(|\nabla u|^2) \sum_{i=1}^N \partial_{x_i x_i} u + 2\bar{a}'(|\nabla u|^2) \sum_{i,j=1}^N \partial_{x_i} u \partial_{x_j} u \partial_{x_i x_j} u + f(u)$$

associated with (3.8) is uniformly elliptic and satisfies all the assumptions in [15, Corollary 1], and consequently, u is radially symmetric and $u'(r) < 0$ for $r \in (0, R)$. \square

Proof of Theorem 1.1. Assume (1.1) has a positive solution u which satisfies (1.6). We assume first that $f(0) \geq 0$, this restriction will be removed later.

By Lemma 2.1, α defined by (2.1) is a lower solution of (2.2). Obviously, $\beta(x) = s_2$ is an upper solution. Hence it follows from [10, Proposition 1] that (2.2) has a solution $v(x)$, such that

$$\alpha(x) \leq v(x) \leq \beta(x), \quad x \in \bar{\Omega},$$

which together with Lemma 3.1 imply that (2.2) has a positive solution v such that

$$\|v\|_\infty \in (s_1, s_2). \quad (3.9)$$

Moreover, it follows from Lemma 3.2 that v is radially symmetric and $v'(r) < 0$ for $r \in (0, R)$, where R is the radius of B and $\bar{\Omega} \subset B$. In particular, v has a unique maximum at $r = 0$. Hence v is a positive solution of the ordinary differential equation

$$\begin{aligned} \left(\frac{v'}{\sqrt{1-(v')^2}} \right)' + \frac{N-1}{r} \frac{v'}{\sqrt{1-(v')^2}} + f(v) &= 0, \quad r \in (0, R), \\ v'(0) = v(R) &= 0. \end{aligned} \quad (3.10)$$

Multiplying (3.10) by v' and integrating over $(0, r)$, we obtain

$$H(v'(r)) + (N-1) \int_0^r \frac{(v')^2(s)}{s\sqrt{1-(v')^2(s)}} ds = F(v(0)) - F(v(r)), \quad r \in (0, R),$$

i.e.

$$H(v'(r)) + [F(v(r)) - F(v(0))] = -(N-1) \int_0^r \frac{(v')^2(s)}{s\sqrt{1-(v')^2(s)}} ds, \quad (3.11)$$

where

$$H(t) = \frac{1 - \sqrt{1-t^2}}{\sqrt{1-t^2}}. \quad (3.12)$$

Choose r_0 so that $v(r_0) = s_0$. If $N > 1$, then we get

$$H(v'(r_0)) + [F(s_0) - F(v(0))] < 0, \quad (3.13)$$

and so

$$\int_{s_0}^{\|v\|_\infty} f(s) ds > 0. \quad (3.14)$$

On the other hand, it follows from (3.9) that f is nonnegative in $[\|v\|_\infty, s_2]$. Combining this with (1.9) imply that

$$\int_{s_0}^{\|v\|_\infty} f(s) ds \leq \int_{s_0}^{s_2} f(s) ds \leq 0,$$

which contradicts (3.14).

If $N = 1$, then it follows from (3.11) that

$$H(v'(r_0)) + [F(s_0) - F(v(0))] = 0.$$

Since $\|v\|_\infty = v(0)$ and $v'(r) < 0$ for $r \in (0, R)$, then $v'(r_0) \neq 0$ and $H(v'(r_0)) > 0$. Hence (3.14) holds. Thus, by the same argument to treat the case $N > 1$, we also get a contradiction.

We note that the assumption that $f(0) \geq 0$ is needed in order to conclude that $\alpha(x)$ is a lower solution of (2.2).

Next assume that $f(0) < 0$. Again assume that (1.1) has a positive solution v satisfying (1.6). Define \tilde{f} so that

$$\begin{aligned} \tilde{f}(s) &\geq f(s), & 0 \leq s \leq \|v\|_\infty, \\ \tilde{f}(0) &\geq 0, & \int_{s_0}^{s_2} \tilde{f}(s) ds < 0. \end{aligned} \quad (3.15)$$

Here we use that $\|v\|_\infty < s_2$. Then

$$\operatorname{div} \left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + \tilde{f}(v) \geq \operatorname{div} \left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + f(v) = 0. \quad (3.16)$$

Hence v is a lower solution of

$$\begin{aligned} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) + \tilde{f}(u) &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (3.17)$$

and as before $\beta(x) = s_2$ is an upper solution. Thus, it follows from [10, Proposition 1] that (3.17) has a solution u satisfying $v(x) \leq u(x) \leq s_2$, i.e., u satisfies (1.6).

Let $\tilde{\alpha}(x)$ be defined by

$$\tilde{\alpha}(x) = \begin{cases} u(x), & x \in \bar{\Omega}, \\ 0, & x \in \bar{B} \setminus \Omega. \end{cases} \quad (3.18)$$

Then, by Lemma 2.1, $\tilde{\alpha}$ is a lower solution of the problem

$$\begin{aligned} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) + \tilde{f}(u) &= 0 & \text{in } B, \\ u &= 0 & \text{on } \partial B. \end{aligned} \quad (3.19)$$

Clearly, $\tilde{\beta}(x) = s_2$ is an upper solution of (3.19). Hence it follows from [10, Proposition 1] that (3.19) has a solution $\tilde{v}(x)$, such that

$$\tilde{\alpha}(x) \leq \tilde{v}(x) \leq \tilde{\beta}(x), \quad x \in \bar{\Omega},$$

which together with Lemma 3.1 imply that (3.19) has a positive solution \tilde{v} such that

$$\|\tilde{v}\|_\infty \in (s_1, s_2). \quad (3.20)$$

Moreover, it follows from Lemma 3.2 that \tilde{v} is radially symmetric and $\tilde{v}'(r) < 0$ for $r \in (0, R)$. In particular, \tilde{v} has a unique maximum at $r = 0$. Hence \tilde{v} is a positive solution of the ordinary differential equation

$$\begin{aligned} \left(\frac{\tilde{v}'}{\sqrt{1 - (\tilde{v}')^2}} \right)' + \frac{N-1}{r} \frac{\tilde{v}'}{\sqrt{1 - (\tilde{v}')^2}} + \tilde{f}(\tilde{v}) &= 0, & r \in (0, R), \\ \tilde{v}'(0) = \tilde{v}(R) &= 0. \end{aligned} \quad (3.21)$$

Multiplying (3.21) by \tilde{v}' and integrating over $(0, r)$, we obtain

$$H(\tilde{v}'(r)) + [F(\tilde{v}(r)) - F(\tilde{v}(0))] = -(N-1) \int_0^r \frac{(\tilde{v}')^2(s)}{s\sqrt{1-(\tilde{v}')^2(s)}} ds, \quad (3.22)$$

where H is defined by (3.12). Choose \tilde{r}_0 so that $\tilde{v}(\tilde{r}_0) = s_0$. We now proceed as in the first part of the proof with \tilde{v} in place of v , and so we get a contradiction. \square

According to Theorem 1.1, we show that (1.2) is in fact a necessary condition for the existence of positive solution u of problem (1.3) satisfying (1.4).

Corollary 3.3. *Let $\kappa > 0$ and $s_2 < \frac{1}{2\sqrt{\kappa}}$. Assume that f satisfies (H1) and (1.9). If problem (1.3) has a positive solution u , u can not satisfy (1.4).*

Proof. Note that (1.3) is the one-dimensional version of the Dirichlet problem associated with the Minkowski-curvature equation

$$\begin{aligned} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1-\kappa|\nabla u|^2}} \right) + \lambda f(u) &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (3.23)$$

Since the parameter λ does not play a role in our consideration we shall replace λf by f . Then, arguing as in the proof of Theorem 1.1, the conclusion can be proved. \square

Next, we give the proof of lower bounds on the $C(\bar{\Omega})$ norm for certain positive solutions of (1.1).

Proof of Theorem 1.2. Assume, to the contrary, that $\|u\|_\infty < r$. Let \tilde{f} be defined as follows:

$$\tilde{f} = \begin{cases} f(s), & 0 \leq s \leq \|u\|_\infty, \\ g(s), & s > \|u\|_\infty, \end{cases} \quad (3.24)$$

where $g(s)$ is chosen such that $\tilde{f} > 0$ in (s_1, s_2) , $\tilde{f}(s_2) = 0$, and $\int_{s_0}^{s_2} \tilde{f}(s) ds \leq 0$. This clearly can be done since $\int_{s_0}^{\|u\|_\infty} f(s) ds < 0$. Note that u also solves the Dirichlet problem

$$\begin{aligned} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right) + \tilde{f}(u) &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (3.25)$$

Nevertheless, it follows from Theorem 1.1 that (3.25) cannot have a positive solution satisfying (1.6), a contradiction. \square

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