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Existence for semilinear equations on exterior domains

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Abstract. In this paper we study radial solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ where f is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞), and f superlinear. The function $K(r)$ is assumed to be positive and $K(r) \to 0$ as $r \to \infty$. We prove existence of an infinite number of radial solutions with $u \to 0$ as $r \to \infty$ when $K(r) \sim r^{-\alpha}$ with $N < \alpha < 2(N - 1)$.

Keywords: exterior domains, semilinear, superlinear, radial.

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1 Introduction

In this paper we study radial solutions of:

$$\Delta u + K(r)f(u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

$$u \to 0 \quad \text{as } |x| \to \infty \tag{1.3}$$

where $x \in \Omega = \mathbb{R}^N \setminus B_R(0)$ is the complement of the ball of radius $R > 0$ centered at the origin.

Since we are interested in radial solutions of (1.1)–(1.3) we assume that $u(x) = u(|x|) = u(r)$ where $x \in \mathbb{R}^N$ and $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$ so that u solves:

$$u''(r) + \frac{N-1}{r}u'(r) + K(r)f(u(r)) = 0 \quad \text{on } (R, \infty), \text{ where } R > 0, \tag{1.4}$$

$$u(R) = 0, \qquad u'(R) = b > 0. \tag{1.5}$$

Throughout this paper we denote $'$ as differentiation with respect to r.

We make the following assumptions on f and K. Let f be odd and locally Lipschitz with:

$$f'(0) < 0, \ \exists \beta > 0 \text{ s.t. } f(u) < 0 \text{ on } (0, \beta) \text{ and } f(u) > 0 \text{ on } (\beta, \infty). \tag{H1}$$

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In addition, let:

$$f(u) = |u|^{p-1}u + g(u), \text{ where } p > 1 \text{ and } \lim_{|u|\to\infty} \frac{|g(u)|}{|u|^p} = 0. \tag{H2}$$

Denoting $F(u) = \int_0^u f(s)\,ds$ we assume:

$$\exists\gamma > 0 \text{ with } 0 < \beta < \gamma \text{ s.t. } F < 0 \text{ on } (0,\gamma) \text{ and } F > 0 \text{ on } (\gamma,\infty). \tag{H3}$$

Further we also assume K and K' are continuous on $[R,\infty)$ and:

$$K(r) > 0,\ \exists\alpha \in (0, 2(N-1)) \text{ s.t. } \lim_{r\to\infty} \frac{rK'}{K} = -\alpha \text{ and} \tag{H4}$$

$$\exists \text{ positive } d_1, d_2 \text{ s.t. } 2(N-1) + \frac{rK'}{K} > 0,\ \ d_1 r^{-\alpha} \le K(r) \le d_2 r^{-\alpha} \text{ for } r \ge R. \tag{H5}$$

Theorem 1.1. *Let $N > 2$ and $N < \alpha < 2(N-1)$. Assuming (H1)–(H5) then for every nonnegative integer n there exists a solution, u_n, of (1.4)–(1.5) such that $\lim_{r\to\infty} u_n(r) = 0$ and u_n has n zeros on (R,∞).*

Note: The model case for this theorem is $f(u) = |u|^{p-1}u - u$ for $p > 1$ (and thus $F(u) = \frac{1}{p+1}|u|^{p+1} - \frac{1}{2}u^2$) and $K(r) = r^{-\alpha}$ with $N < \alpha < 2(N-1)$.

Note: when $\Omega = \mathbb{R}^N$, $K(r) \equiv 1$, and f grows superlinearly at infinity – i.e. $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, then the problem (1.1), (1.3) has been extensively studied [1–3, 9, 11, 13].

Interest in the topic for this paper comes from recent papers [5, 10, 12] about solutions of semilinear equations on exterior domains. In [5] the authors use variational methods to prove the existence of a positive solution. In this paper we examine a similar differential equation and use ordinary differential equation methods to prove the existence of an infinite number of solutions – one with n zeros for each nonnegative integer n.

In [8] we studied (1.1)–(1.3) under the assumptions (H1)–(H5) with $K(r) \sim r^{-\alpha}$ where $0 < \alpha < N$ and $\Omega = \mathbb{R}^N \setminus B_R(0)$ and (H1)–(H5). In that paper we proved existence of an infinite number of solutions – one with exactly n zeros for each nonnegative integer n such that $u \to 0$ as $|x| \to \infty$. In earlier papers [6, 7] we have also studied (1.1), (1.3) when $\Omega = \mathbb{R}^N$ and $K(r) \equiv 1$ where f is odd, $f < 0$ on $(0,\beta)$, $f > 0$ on (β,δ), and $f \equiv 0$ on (δ,∞).

2 Preliminaries

For $R > 0$ existence of solutions of (1.4)–(1.5) on a small interval $[R, R+\epsilon)$ with $\epsilon > 0$ and continuous dependence of solutions with respect to b follows from the standard existence-uniqueness-continuous dependence theorem of ordinary differential equations [4].

Recall that $K(r) > 0$, $K(r)$ is differentiable, and that $N > 2$. We define the "energy" of a solution of (1.4) as follows:

$$E(r,b) = \frac{1}{2}\frac{u'^2(r,b)}{K(r)} + F(u(r,b)) \tag{2.1}$$

where u solves (1.4)–(1.5). Then it is straightforward to show:

$$E'(r,b) = -\frac{u'^2}{2rK}\left(\frac{rK'}{K} + 2(N-1)\right) = -\frac{u'^2}{2r^{2(N-1)}K^2}\left(r^{2(N-1)}K\right)'. \tag{2.2}$$

Thus we see that $E(r,b)$ is non-increasing precisely when $r^{2(N-1)}K$ is non-decreasing. In particular, if $K(r) = c_0 r^{-\alpha}$ with $c_0 > 0$ and $\alpha > 0$ then we see $E' \leq 0$ if and only if $\alpha \leq 2(N-1)$.

Lemma 2.1. *Let u satisfy (1.4)–(1.5) and suppose (H1)–(H5) hold. If $b > 0$ and b is sufficiently small then $u(r,b) > 0$ for all $r > R$.*

Proof. The proof of this lemma is similar to the one we used in [8]. First, we see that if $u'(r,b) > 0$ for $r \geq R$ then $u(r,b) > 0$ for $r > R$ and so we are done in this case. Otherwise, $u(r,b)$ has a first local maximum, M_b, with $u'(r,b) > 0$ on $[R, M_b)$. Thus $u'(M_b,b) = 0$ and $u''(M_b,b) \leq 0$. In fact, $u''(M_b,b) < 0$ for if $u''(M_b,b) = 0$ then by uniqueness of solutions of initial value problems this would imply that $u(r,b)$ is constant contradicting that $u'(R,b) = b > 0$. It then follows that $f(u(M_b,b)) > 0$ and therefore $u(M_b,b) > \beta$. So there is an r_b with $R < r_b < M_b$ such that $u(r_b,b) = \beta$. Next we note that since $N < \alpha < 2(N-1)$ then $E' \leq 0$ thus:

$$\frac{1}{2}\frac{u'^2(r,b)}{K(r)} + F(u(r,b)) = E(r,b) \leq E(R,b) = \frac{1}{2}\frac{b^2}{K(R)} \quad \text{for } r \geq R. \tag{2.3}$$

After rewriting (2.3) and using (H5) we obtain:

$$\frac{|u'(r,b)|}{\sqrt{\frac{b^2}{K(R)} - 2F(u(r,b))}} \leq \sqrt{K} \leq \sqrt{d_2}\, r^{-\frac{\alpha}{2}} \quad \text{for } r \geq R. \tag{2.4}$$

Integrating (2.4) on (R, r_b) where $u' > 0$ and using (H5) as well as $\alpha > 2$ gives:

$$\begin{aligned}\int_0^\beta \frac{dt}{\sqrt{\frac{b^2}{K(R)} - 2F(t)}} &= \int_R^{r_b} \frac{u'(r,b)\,dr}{\sqrt{\frac{b^2}{K(R)} - 2F(u(r,b))}} \\ &\leq \int_R^{r_b} \sqrt{K}\,dr \leq \int_R^{r_b} \sqrt{d_2}\, r^{-\frac{\alpha}{2}}\,dr = \frac{\sqrt{d_2}}{\frac{\alpha}{2}-1}\left(R^{1-\frac{\alpha}{2}} - r_b^{1-\frac{\alpha}{2}}\right).\end{aligned}$$

Thus:

$$\int_0^\beta \frac{dt}{\sqrt{\frac{b^2}{K(R)} - 2F(t)}} \leq \frac{\sqrt{d_2}}{\frac{\alpha}{2}-1} R^{1-\frac{\alpha}{2}}. \tag{2.5}$$

Next we observe by (H1) and the definition of F that there is a $t_0 > 0$ such that:

$$\sqrt{\frac{b^2}{K(R)} - 2F(t)} \leq \sqrt{\frac{b^2}{K(R)} + 2|f'(0)|t^2} \quad \text{for } 0 < t < t_0 < \beta \tag{2.6}$$

and therefore combining (2.5)–(2.6) gives:

$$\frac{\sqrt{d_2}}{\frac{\alpha}{2}-1} R^{1-\frac{\alpha}{2}} \geq \int_0^\beta \frac{dt}{\sqrt{\frac{b^2}{K(R)} - 2F(t)}} \geq \int_0^{t_0} \frac{dt}{\sqrt{\frac{b^2}{K(R)} + 2|f'(0)|t^2}} \to \infty \quad \text{as } b \to 0^+.$$

This is a contradiction since the left-hand side is bounded but the right-hand side is not. Thus we see that $u(r,b) > 0$ if $b > 0$ is sufficiently small. □

Lemma 2.2. *Let u satisfy (1.4)–(1.5) and suppose (H1)–(H5) hold. Then $\max_{[R,2R]} u(r,b) \to \infty$ as $b \to \infty$.*

Proof. Multiplying (1.4) by r^{N-1} and integrating on (R,r) gives:

$$r^{N-1}u' = R^{N-1}b - \int_R^r t^{N-1}Kf(u)\,dt. \tag{2.7}$$

Now if $u(r,b)$ is uniformly bounded from above on $[R,2R]$ for all sufficiently large $b>0$ then since f is continuous there exists $C_1>0$ such that $f(u(r,b)) \le C_1$ on $[R,2R]$ for all sufficiently large $b>0$. Recalling (H5), that $\alpha > N > 2$, and estimating in (2.7) we see that:

$$r^{N-1}u' \ge R^{N-1}b - \frac{C_1 d_2 r^{N-\alpha}}{N-\alpha} \quad \text{on } [R,2R]. \tag{2.8}$$

Dividing (2.8) by r^{N-1}, integrating on $[R,2R]$, and recalling $u(R,b)=0$ gives:

$$u(2R,b) \ge \frac{bR[1-(2)^{2-N}]}{N-2} - \frac{C_1 d_2 R^{2-\alpha}(1-2^{2-\alpha})}{(\alpha-2)(N-\alpha)} \to \infty \quad \text{as } b \to \infty.$$

Hence we obtain a contradiction since we assumed that $u(r,b)$ was uniformly bounded from above on $[R,2R]$. This completes the proof of the lemma. □

Lemma 2.3. *Let u satisfy (1.4)–(1.5) and suppose (H1)–(H5) hold. Then $u(r,b)$ has a local maximum on (R,∞) if $b>0$ is sufficiently large.*

Proof. We begin by making the following change of variables:

$$u(r,b) = w(r^{2-N},b). \tag{2.9}$$

Then it is straightforward to show using (1.4)–(1.5):

$$w''(t,b) + h(t)f(w(t,b)) = 0 \quad \text{for } 0<t<R^{2-N}, \tag{2.10}$$

$$w(R^{2-N},b) = 0, \qquad w'(R^{2-N},b) = -\frac{bR^{N-1}}{N-2} < 0 \tag{2.11}$$

where:

$$h(t) = t^{\frac{2(N-1)}{2-N}} K(t^{\frac{1}{2-N}}). \tag{2.12}$$

Since $T(r) = r^{2(N-1)}K(r)$ is increasing by (H5) we see that $h(t) = T(t^{\frac{1}{2-N}})$ is decreasing since $N>2$. Thus:

$$h'(t) < 0 \text{ on } (0,R^{2-N}] \text{ and by (H5) } h(t) \sim \frac{1}{t^q} \text{ for small positive } t \text{ where } q = \tfrac{2(N-1)-\alpha}{N-2}. \tag{2.13}$$

We note since $N < \alpha < 2(N-1)$ it follows that $0<q<1$ and thus $h(t)$ is integrable on $(0,R^{2-N}]$.

Suppose now that $u(r,b)$ does not have a local maximum on $[R,\infty)$ for sufficiently large b. Then $u'(r,b)>0$ for $r \ge R$ and so we see that $\max_{[R,2R]} u(r,b) = u(2R,b) = \min_{[2R,\infty)} u(r,b)$. From this and Lemma 2.2 it follows that $\min_{[2R,\infty)} u(r,b) \to \infty$ as $b\to\infty$ hence from (2.9) we see that:

$$\min_{(0,(2R)^{2-N}]} w(t,b) \to \infty \quad \text{as } b \to \infty. \tag{2.14}$$

In addition, $u'(r,b) > 0$ on $[R,\infty)$ so from (2.9) we see $w'(t,b) < 0$ on $(0, R^{2-N}]$. Next we define:

$$C(b) = \frac{1}{2} \min_{(0,(2R)^{2-N}]} h(t) \frac{f(w(t,b))}{w(t,b)}. \tag{2.15}$$

It follows from (2.14) and (H2) that $\min_{(0,(2R)^{2-N}]} \frac{f(w(t,b))}{w(t,b)} \to \infty$ as $b \to \infty$. In addition, since $h'(t) < 0$ on $(0, R^{2-N}]$ then we see:

$$C(b) \geq \frac{1}{2} h((2R)^{2-N}) \min_{(0,(2R)^{2-N}]} \frac{f(w(t,b))}{w(t,b)} \to \infty \text{ as } b \to \infty. \tag{2.16}$$

Now we let $y(t)$ be the solution of:

$$y'' + C(b)y = 0 \tag{2.17}$$

such that:

$$y((2R)^{2-N}) = w((2R)^{2-N}, b) > 0 \text{ and } y'((2R)^{2-N}) = w'((2R)^{2-N}, b) < 0. \tag{2.18}$$

Multiplying (2.17) by w, multiplying (2.10) by y, and subtracting gives:

$$(yw' - wy')' + \left(h(t)\frac{f(w)}{w} - C(b) \right) wy = 0. \tag{2.19}$$

Now it is well-known that the general nontrivial solution of equation (2.17) is $y(t) = c_1 \sin\left(\sqrt{C(b)}(t - c_2)\right)$ for some constants $c_1 \neq 0$ and c_2. Thus any interval of length $\frac{\pi}{\sqrt{C(b)}}$ contains a zero of $y(t)$. Since $C(b) \to \infty$ as $b \to \infty$ (by (2.16)) it follows that if b is sufficiently large then $y(t)$ has a zero on $(\frac{1}{2}(2R)^{2-N}, (2R)^{2-N})$. In particular, since $y((2R)^{2-N}) = w((2R)^{2-N}, b) > 0$ and $y'((2R)^{2-N}) = w'((2R)^{2-N}, b) < 0$ it follows that there is an m_b with $\frac{1}{2}(2R)^{2-N} < m_b < (2R)^{2-N}$ such that $y(t)$ has a local maximum at m_b, $y'(t) < 0$ on $(m_b, (2R)^{2-N}]$, and $y(t) > 0$ on $(m_b, (2R)^{2-N})$.

We claim now that $w(t,b)$ has a local maximum on $(\frac{1}{2}(2R)^{2-N}, (2R)^{2-N})$. So suppose by way of contradiction that this is not the case. Then $w'(t,b) < 0$ on $(\frac{1}{2}(2R)^{2-N}, (2R)^{2-N})$ and since $w((2R)^{2-N}, b) > 0$ then $w(t,b) > 0$ on $(\frac{1}{2}(2R)^{2-N}, (2R)^{2-N})$. Next integrating (2.19) on $(m_b, (2R)^{2-N})$ and using (2.18) gives:

$$-y(m_b)w'(m_b, b) + \int_{m_b}^{(2R)^{2-N}} \left(h(t)\frac{f(w)}{w} - C(b) \right) wy \, dt = 0. \tag{2.20}$$

By definition of $C(b)$ in (2.15) it follows that $h(t)\frac{f(w)}{w} - C(b) > 0$ on $(m_b, (2R)^{2-N})$. Also since $y > 0$ and $w > 0$ on $(m_b, (2R)^{2-N})$, we see that the integral in (2.20) is positive. In addition, $y(m_b) > 0$ thus we see from (2.20) that $w'(m_b, b) > 0$ but this contradicts our assumption that $w'(t,b) < 0$ on $(\frac{1}{2}(2R)^{2-N}, (2R)^{2-N})$. Thus $w(t,b)$ has a local maximum, Q_b, such that $Q_b \in (\frac{1}{2}(2R)^{2-N}, (2R)^{2-N})$ with $w'(t,b) < 0$ on $(Q_b, (2R)^{2-N})$ and consequently by (2.9) it follows that $u(r,b)$ has a local maximum at $M_b = Q_b^{\frac{1}{2-N}} \in (R, \infty)$ and $u'(r,b) > 0$ on $[R, M_b)$ if $b > 0$ is sufficiently large. This completes the proof. □

Lemma 2.4. *Let u satisfy (1.4)–(1.5) and suppose (H1)–(H5) hold. Then $\lim_{b\to\infty} u(M_b, b) = \infty$ and $\lim_{b\to\infty} M_b = R$.*

Proof. Integrating (2.10) and using (2.11) on (Q_b, R^{2-N}) gives:

$$\frac{bR^{N-1}}{N-2} + \int_{Q_b}^{R^{2-N}} h(t) f(w(t,b))\, dt = 0. \tag{2.21}$$

If the $u(M_b, b)$ are uniformly bounded by some constant C_2 for all sufficiently large b then the same is true for $w(Q_b, b)$ and therefore $f(w(t,b))$ is uniformly bounded on $(Q_b, R^{2-N}) \subset (0, R^{2-N})$. Now recall from (2.13) that h is integrable on $(0, R^{2-N})$. Thus the integral term in (2.21) is uniformly bounded whereas $\frac{bR^{N-1}}{N-2} \to \infty$ as $b \to \infty$ which contradicts (2.21). Thus we see that $u(M_b, b) \to \infty$ as $b \to \infty$. This completes the first part of the proof.

Next a straightforward computation using (2.10) shows:

$$\left(\frac{1}{2}\frac{w'^2}{h(t)} + F(w)\right)' = -\frac{w'^2 h'}{h^2} \geq 0 \text{ since } h'(t) < 0 \text{ on } (0, R^{2-N}]. \tag{2.22}$$

Therefore we have:

$$\frac{1}{2}\frac{w'^2(t,b)}{h(t)} + F(w(t,b)) \geq F(w(Q_b, b)) \quad \text{for } Q_b \leq t \leq R^{2-N}. \tag{2.23}$$

After rewriting (2.23), recalling that $w' < 0$ on (Q_b, R^{2-N}), and integrating on (Q_b, R^{2-N}) we obtain:

$$\begin{aligned}\int_0^{w(Q_b,b)} \frac{dt}{\sqrt{2}\sqrt{F(w(Q_b,b)) - F(t)}} &= \int_{Q_b}^{R^{2-N}} \frac{|w'(t,b)|\, dt}{\sqrt{2}\sqrt{F(w(Q_b,b)) - F(w(t,b))}} \\ &\geq \int_{Q_b}^{R^{2-N}} \sqrt{h(t)}\, dt.\end{aligned} \tag{2.24}$$

Now we will show $\int_0^{w(Q_b,b)} \frac{dt}{\sqrt{2}\sqrt{F(w(Q_b,b)) - F(t)}} \to 0$ as $b \to \infty$. Proceeding as we did in [8] it follows from (H2) that $f(x) \geq \frac{1}{2}x^p$ for large x and thus for x sufficiently large we have $\min_{[\frac{1}{2}x, x]} f \geq \frac{1}{2^{p+1}} x^p$. Therefore since $p > 1$ we see that:

$$\lim_{x \to \infty} \frac{x}{\min_{[\frac{1}{2}x, x]} f} = 0. \tag{2.25}$$

In particular, since we saw $u(M_b, b) \to \infty$ as $b \to \infty$ from the first part of this proof it follows from (2.9) that $w(Q_b, b) \to \infty$ as $b \to \infty$ and:

$$\frac{w(Q_b, b)}{S_b} \to 0 \quad \text{as } b \to \infty \tag{2.26}$$

where:

$$S_b = \min_{[\frac{1}{2}w(Q_b,b), w(Q_b,b)]} f. \tag{2.27}$$

We now divide the domain of the integral on the left-hand side of (2.24) into $(0, w(Q_b, b)/2))$ and $(w(Q_b, b)/2, w(Q_b, b))$ and then show that each of these integrals goes to 0 as $b \to \infty$. First let $w(Q_b, b)/2 \leq t \leq w(Q_b, b)$. By (2.27) and the mean value theorem there exists a C_3 with $w(Q_b, b)/2 \leq C_3 \leq w(Q_b, b)$ such that:

$$F(w(Q_b, b)) - F(t) = f(C_3)(w(Q_b, b) - t) \geq S_b\, (w(Q_b, b) - t)\,. \tag{2.28}$$

Hence by (2.26) and (2.28):

$$\int_{w(Q_b,b)/2}^{w(Q_b,b)} \frac{dt}{\sqrt{2}\sqrt{F(w(Q_b,b)) - F(t)}}$$
$$\leq \int_{w(Q_b,b)/2}^{w(Q_b,b)} \frac{dt}{\sqrt{2S_b}\sqrt{w(Q_b,b) - t}} = \sqrt{\frac{w(Q_b,b)}{S_b}} \to 0 \quad \text{as } b \to \infty. \tag{2.29}$$

Next when $0 \leq t \leq w(Q_b,b)/2$ and b is sufficiently large we have $F(t) \leq F(w(Q_b,b)/2)$. By (2.27) and the mean value theorem there exists a C_4 with $w(Q_b,b)/2 \leq C_4 \leq w(Q_b,b)$ such that:

$$\begin{aligned} F(w(Q_b,b)) - F(t) &\geq F(w(Q_b,b)) - F(w(Q_b,b)/2) = f(C_4)w(Q_b,b)/2 \\ &\geq S_b w(Q_b,b)/2. \end{aligned} \tag{2.30}$$

Thus by (2.26) and (2.30):

$$\int_0^{w(Q_b,b)/2} \frac{dt}{\sqrt{2}\sqrt{F(w(Q_b,b)) - F(t)}} \leq \frac{w(Q_b,b)/2}{\sqrt{2}\sqrt{F(w(Q_b,b)) - F(w(Q_b,b)/2)}}$$
$$\leq \frac{1}{2}\sqrt{\frac{w(Q_b,b)}{S_b}} \to 0 \quad \text{as } b \to \infty. \tag{2.31}$$

Combining (2.29)–(2.31) we see that the left-hand side of (2.24) goes to 0 as $b \to \infty$. Thus the right-hand side of (2.24) must also go to zero and thus $Q_b \to R^{2-N}$ as $b \to \infty$. Since $Q_b = M_b^{2-N}$ (as we saw in Lemma 2.3 this implies $M_b \to R$ as $b \to \infty$. This completes the proof. □

Lemma 2.5. *Let u satisfy (1.4)–(1.5) and suppose (H1)–(H5) hold. If $b > 0$ is sufficiently large then $u(r,b)$ has an arbitrarily large number of zeros for $r > R$.*

Proof. Let:

$$v_\lambda(r,b) = \lambda^{-\frac{2}{p-1}} u(M_b + \frac{r}{\lambda}, b)$$

where:

$$\lambda^{\frac{2}{p-1}} = u(M_b, b)$$

and M_b is the local maximum that we have shown to exist by Lemma 2.4. Then:

$$v_\lambda'' + \frac{N-1}{\lambda M_b + r} v_\lambda' + \lambda^{\frac{-2p}{p-1}} K\left(M_b + \frac{r}{\lambda}\right) f(\lambda^{\frac{2}{p-1}} v_\lambda) = 0,$$

$$v_\lambda(0) = 1, \qquad v_\lambda'(0) = 0.$$

From Lemma 2.4 we see that as $b \to \infty$ then $\lambda^{\frac{2}{p-1}} = u(M_b, b) \to \infty$.

Now we let:

$$E_\lambda = \frac{1}{2} \frac{v_\lambda'^2}{K(M_b + \frac{r}{\lambda})} + \frac{F(\lambda^{\frac{2}{p-1}} v_\lambda)}{\lambda^{\frac{2(p+1)}{p-1}}}. \tag{2.32}$$

It is straightforward to show that:

$$E_\lambda' = \left(\frac{1}{2} \frac{v_\lambda'^2}{K(M_b + \frac{r}{\lambda})} + \frac{F(\lambda^{\frac{2}{p-1}} v_\lambda)}{\lambda^{\frac{2(p+1)}{p-1}}} \right)' \leq 0.$$

Denoting $G(u) = \int_0^u g(u)$ then from (H2)–(H3) we see $F(u) = \frac{1}{p+1}|u|^{p+1} + G(u)$ where $\frac{G(u)}{|u|^{p+1}} \to 0$ as $|u| \to \infty$. Then for $r > 0$:

$$\frac{1}{2}\frac{v_\lambda'^2}{K(M_b + \frac{r}{\lambda})} + \frac{1}{p+1}|v_\lambda|^{p+1} + \frac{G(\lambda^{\frac{2}{p-1}}v_\lambda)}{\lambda^{\frac{2(p+1)}{p-1}}} = \frac{1}{2}\frac{v_\lambda'^2}{K(M_b + \frac{r}{\lambda})} + \frac{F(\lambda^{\frac{2}{p-1}}v_\lambda)}{\lambda^{\frac{2(p+1)}{p-1}}} \tag{2.33}$$

$$= E_\lambda(r) \leq E_\lambda(0) = \frac{F(\lambda^{\frac{2}{p-1}})}{\lambda^{\frac{2(p+1)}{p-1}}} \leq \frac{1}{p+1} + \frac{G(\lambda^{\frac{2}{p-1}})}{\lambda^{\frac{2(p+1)}{p-1}}}. \tag{2.34}$$

Since $\frac{G(u)}{|u|^{p+1}} \to 0$ as $|u| \to \infty$ it follows that the right-hand side of (2.34) is bounded for large λ and also since $\frac{G(u)}{|u|^{p+1}} \to 0$ as $|u| \to \infty$ it follows that there is a constant G_0 such that $|G(u)| \leq \frac{1}{2(p+1)}|u|^{p+1} + G_0$ for all u. Therefore it follows from (2.33)–(2.34) that v_λ and v_λ' are uniformly bounded and so by the Arzelà–Ascoli theorem there is a subsequence (again labeled v_λ) such that $v_\lambda \to v$ uniformly on compact subsets of $[0, \infty)$ where v satisfies:

$$v'' + K(R)|v|^{p-1}v = 0$$

$$v(0) = 1, \qquad v'(0) = 0.$$

Now it is straightforward to show that v has an infinite number of zeros on $[0, \infty)$ and thus given n then v_λ has at least n zeros for large enough λ so that u has at least n zeros for large enough b. This completes the proof. □

Lemma 2.6. *Solutions of* (2.10)–(2.11) *with* (H1)–(H5) *depend continuously on the parameter b.*

Proof. Let $a_1, a_2 \in \mathbb{R}$ and suppose $a_1 \leq a \leq a_2$. It is straightforward to show that if $w'' + h(t)f(w) = 0$ on $(0, R_0)$ with $w(R_0) = 0$ and $w'(R_0) = a$ where $R_0 > 0$ then:

$$w(t) = a(R_0 - t) - \int_t^{R_0}\int_s^{R_0} h(x)f(w(x))\,dx\,ds. \tag{2.35}$$

It follows from (2.22) that:

$$F(w(t)) \leq \frac{1}{2}\frac{w'^2(t)}{h(t)} + F(w(t)) \leq \frac{1}{2}\frac{a^2}{h(R_0)} \quad \text{on } (t, R_0).$$

Since $F(w) \to \infty$ as $|w| \to \infty$ by (H2)–(H3) we see that there is a constant C_5 such that $|w(t)| \leq C_5$ for all $t \in [0, R_0]$ and for all a where $a_1 \leq a \leq a_2$. Therefore there is a constant C_6 such that $|f(w(t))| \leq C_6$ for all $t \in [0, R_0]$ and for all a where $a_1 \leq a \leq a_2$. Also since $h(t) \sim \frac{1}{t^q}$ with $0 < q < 1$ (by (2.12)) there is a $C_7 > 0$ such that:

$$\int_s^{R_0} h(x)\,dx \leq C_7 \quad \text{for } 0 \leq s \leq R_0.$$

Thus it follows from (2.35) and since h is decreasing that:

$$|w(t)| \leq |a|R_0 + \int_t^{R_0}\int_s^{R_0} h(x)|f(w(x))|\,dx\,ds \leq |a|R_0 + \int_t^{R_0} h(s)\,ds \int_t^{R_0} |f(w(x))|\,dx$$
$$\leq |a|R_0 + \int_t^{R_0} C_6C_7 \leq |a|R_0 + C_6C_7R_0 \leq (|a_1| + |a_2| + C_6C_7)\,R_0 \quad \text{on } [0, R_0].$$

Thus for $B = (|a_1| + |a_2| + C_6C_7)\,R_0$ we see that $|w(t)| \leq B$ on $[0, R_0]$ for all a with $a_1 \leq a \leq a_2$.

So now suppose w_1 and w_2 are solutions of (2.10) with $w_1(R_0) = w_2(R_0) = 0$, $w_1'(R_0) = a_1$, and $w_2'(R_0) = a_2$. Then from (2.35):

$$w_1(t) - w_2(t) = (a_1 - a_2)(R_0 - t) - \int_t^{R_0} \int_s^{R_0} h(x)[f(w_1) - f(w_2)]\, dx\, ds \quad \text{for } 0 < t < R_0.$$

Since f is locally Lipschitz it follows that on $[0, B]$ there exists a $D > 0$ such that $|f(w_1) - f(w_2)| \leq D|w_1 - w_2|$ for all $w_i \in [0, B]$. Then since $h' < 0$:

$$\begin{aligned} |w_1(t) - w_2(t)| &\leq |(a_1 - a_2)(R_0 - t)| + D \int_t^{R_0} \int_s^{R_0} h(x)|w_1(x) - w_2(x)|\, dx\, ds \\ &\leq |(a_1 - a_2)(R_0 - t)| + D \int_t^{R_0} h(s)\, ds \int_t^{R_0} |w_1(x) - w_2(x)|\, dx. \end{aligned}$$

Then for $C_{10} = C_7 D$ we obtain:

$$|w_1(t) - w_2(t)| \leq |a_1 - a_2| R_0 + C_{10} \int_t^{R_0} |w_1(x) - w_2(x)|\, dx \quad \text{on } [0, R_0].$$

Then from the usual Gronwall inequality [4] we obtain:

$$|w_1(t) - w_2(t)| \leq |a_1 - a_2| R_0 e^{C_{10} R_0} \quad \text{on } [0, R_0].$$

Thus we obtain continuous dependence on $[0, R_0]$. Thus if a_1 is sufficiently close to a_2 then w_1 is close to w_2 on all on $[0, R_0]$. □

Lemma 2.7. *Suppose* (H1)–(H5) *hold. If $u(r, b_n)$ is a solution of* (1.4)–(1.5) *that has n zeros on (R, ∞) and $\lim_{r\to\infty} u(r, b_n) = 0$ then if b is sufficiently close to b_n then $u(r, b)$ has at most $n + 1$ zeros on (R, ∞).*

Proof. We do the proof in the case $n = 0$. The proof for the other cases is similar. Suppose $u(r, b_0) \to 0$ as $r \to \infty$ and $u(r, b_0)$ is a positive solution of (1.4)–(1.5). Suppose now that b is close to b_0 and $u(r, b)$ has a first zero, $z_b > R$. We want to show that there is not a second zero $z_{2,b} > z_b$. So suppose there is. Then there is a local minimum, m_b, such that $z_b < m_b < z_{2,b}$ such that $u' \leq 0$ on (z_b, m_b) and since $E' \leq 0$ then $F(u(m_b, b)) = E(m_b) \geq E(z_{2,b}) \geq 0$ so that $u(m_b, b) \leq -\gamma$. Then there is a p_b and q_b with $z_b < p_b < q_b < m_b < z_{2,b}$ such that $u(p_b, b) = -\frac{3\beta+\gamma}{4}$ and $u(q_b, b) = -\frac{\beta+\gamma}{2}$. Returning to (2.4), integrating on $[p_b, q_b]$ where $u' < 0$ and recalling that F is even gives:

$$\begin{aligned} \int_{\frac{3\beta+\gamma}{4}}^{\frac{\beta+\gamma}{2}} \frac{dt}{\sqrt{\frac{b^2}{K(R)} - 2F(t)}} &= \int_{p_b}^{q_b} \frac{-u'(r, b)\, dr}{\sqrt{\frac{b^2}{K(R)} - 2F(u(r, b))}} \leq \int_{p_b}^{q_b} \sqrt{d_2} r^{-\frac{\alpha}{2}} \\ &= \frac{\sqrt{d_2}\left(p_b^{1-\frac{\alpha}{2}} - q_b^{1-\frac{\alpha}{2}}\right)}{\frac{\alpha}{2} - 1}. \end{aligned} \tag{2.36}$$

Now as $b \to b_0^+$ then $z_b \to \infty$ (otherwise a subsequence of z_b would converge to some z and $u(z, b_0) = 0$ but we know that $u(r, b_0) > 0$) and thus $p_b \to \infty$ and $q_b \to \infty$. Therefore the right-hand side of (2.36) goes to 0 as $b \to b_0^+$ since $\alpha > 2$ but the left-hand side goes to the positive constant

$$\int_{\frac{3\beta+\gamma}{4}}^{\frac{\beta+\gamma}{2}} \frac{dt}{\sqrt{\frac{b_0^2}{K(R)} - 2F(t)}} > 0.$$

Thus we obtain a contradiction so no such $z_{2,b}$ exists. This completes the proof. □

3 Proof of Theorem 1.1

By Lemma 2.1 we see that $\{b > 0 \mid u(r,b) > 0 \text{ for all } r > R\}$ is nonempty and by Lemma 2.5 this set is bounded from above so we define:

$$0 < b_0 = \sup\{b > 0 \mid u(r,b) > 0 \text{ for all } r > R\}.$$

It follows that $u(r,b_0) > 0$ for $r > R$ because if there were a smallest $z > R$ such that $u(z,b_0) = 0$ then it follows by uniqueness of solutions of initial value problems that $u'(z,b_0) < 0$ and so $u(r,b_0) < 0$ for r slightly larger than z. Then by continuous dependence of solutions on initial conditions, it follows that $u(r,b)$ would get negative for r near z and for slightly smaller $b < b_0$ contradicting the definition of b_0. Thus $u(r,b_0) > 0$ on (R,∞).

Next we claim $E(r,b_0) \geq 0$ for $r \geq R$. If not then there is an $r_0 > R$ such that $E(r_0,b_0) < 0$. Then by continuous dependence on initial conditions it follows that $E(r_0,b) < 0$ for b slightly larger than b_0. In addition for $b > b_0$ then $u(r,b)$ must have a zero so there exists z_b such that $u(z_b,b) = 0$. It follows that $E(z_b,b) \geq 0$. Since E is nonincreasing we have $E(r_0,b) < 0 \leq E(z_b,b)$ so it then follows that $z_b < r_0$. Thus a subsequence of the z_b converges to some z as $b \to b_0$ and since $u(r,b) \to u(r,b_0)$ uniformly on the compact set $[R, r_0+1]$ it follows that $u(z,b_0) = 0$. However, we proved earlier that $u(r,b_0) > 0$ and so we obtain a contradiction. Thus it must be that $E(r,b_0) \geq 0$ for all $r \geq R$.

Next we show that $u(r,b_0)$ has a local maximum. So we suppose not. Then $u(r,b_0)$ is increasing for $r \geq R$. Since $F(u(r,b)) \leq \frac{1}{2}\frac{b^2}{K(R)}$ it follows that $u(r,b)$ is bounded so then there is an L such that $u(r,b_0) \to L$ as $r \to \infty$. Now for $b > b_0$ we see that $u(r,b)$ must have a zero, z_b, and hence a local maximum, M_b, with $R < M_b < z_b$. Since $E' \leq 0$ we have:

$$0 \leq E(z_b,b) \leq \frac{1}{2}\frac{u'^2(r,b)}{K(r)} + F(u(r,b)) = E(r) \leq E(M_b,b) = F(u(M_b,b)) \text{ for } M_b \leq r \leq z_b. \tag{3.1}$$

Thus $u(M_b,b) \geq \gamma$ and now rewriting (3.1), using (H5), and integrating on (M_b,z_b) we get:

$$\int_0^{\gamma} \frac{dt}{\sqrt{2}\sqrt{F(u(M_b,b)) - F(t)}}$$
$$\leq \int_0^{u(M_b,b)} \frac{dt}{\sqrt{2}\sqrt{F(u(M_b,b)) - F(t)}} = \int_{M_b}^{z_b} \frac{|u'(r,b)|\,dr}{\sqrt{2}\sqrt{F(u(M_b,b)) - F(u(r,b))}} \tag{3.2}$$
$$\leq \int_{M_b}^{z_b} \sqrt{K(r)}\,dr \leq \int_{M_b}^{z_b} \sqrt{d_2} r^{-\frac{\alpha}{2}}\,dr = \sqrt{d_2}\left(\frac{z_b^{1-\frac{\alpha}{2}} - M_b^{1-\frac{\alpha}{2}}}{\frac{\alpha}{2} - 1}\right). \tag{3.3}$$

Now if $M_b \to \infty$ then since $M_b < z_b$ then also $z_b \to \infty$ and since $\alpha > 2$ the right-hand side of (3.3) goes to 0 as $b \to \infty$.

On the left-hand side we know that the $u(M_b,b)$ are bounded for b near b_0 because $F(u(M_b,b)) \leq \frac{1}{2}\frac{b^2}{K(R)} \leq \frac{1}{2}\frac{(b_0+1)^2}{K(R)} = C_{12}$ for all b near b_0. Also from (H3) it follows that there is an $F_0 > 0$ such that $F(u) \geq -F_0$ for all u. Thus $F(u(M_b,b)) - F(t) \leq C_{12} + F_0$. This implies the left-hand side (3.2) is bounded from below by a positive constant contradicting that the right-hand side of (3.3) goes to 0. Thus it must be that the M_b are uniformly bounded. Hence a subsequence of them converges to some M_{b_0} as $b \to b_0$ and since $u(r,b) \to u(r,b_0)$ uniformly on $[R, M_{b_0}+1]$ it follows that $u(r,b_0)$ has a local maximum at M_{b_0}.

Next since $E(r,b_0) \geq 0$ it follows that $u(r,b_0)$ cannot have a positive local minimum $m_{b_0} > M_{b_0}$ for at such an m_{b_0} we would have $F(u(m_{b_0},b_0)) = E(m_{b_0},b_0) \geq 0$ implying that $u(m_{b_0},b_0) \geq \gamma$. On the other hand, since m_{b_0} is a local minimum then $u'(m_{b_0},b_0) = 0$ and

$u''(m_{b_0}, b_0) \geq 0$. Thus $f(u(m_{b_0}, b_0)) \leq 0$ which implies $0 < u(m_{b_0}, b_0) \leq \beta$ which contradicts that $u(m_{b_0}, b_0) \geq \gamma$. Thus $u'(r, b_0) \leq 0$ for $r > M_{b_0}$ and so there exists an $L \geq 0$ such that $\lim_{r\to\infty} u(r, b_0) = L \geq 0$.

From Lemma 2.6 it follows that $w(t, b) \to w(t, b_0)$ uniformly on $[0, R^{2-N}]$. In addition, for $b > b_0$ then $w(t, b)$ has a zero, $Z_b \in [0, R^{2-N}]$. Thus the Z_b are bounded and so a subsequence of them converges with $Z_b \to Z \geq 0$ as $b \to b_0$. In fact $Z = 0$. If not a subsequence converges to a $Z > 0$ and $0 = w(Z_b, b) \to w(Z, b_0)$ by Lemma 2.6 but we showed $w(t, b_0) > 0$ on $(0, R^{2-N})$ earlier in the proof. Thus $Z = 0$ and therefore we see by Lemma 2.6 that $0 = w(Z_b, b) \to w(0, b_0)$ hence $w(0, b_0) = 0$. Since w is continuous then:

$$\lim_{t\to 0^+} w(t, b_0) = 0.$$

Hence it follows from (2.9) that:

$$\lim_{r\to\infty} u(r, b_0) = 0.$$

Thus we have a positive solution of (1.4)–(1.5) such that $\lim_{r\to\infty} u(r, b_0) = 0$.

Next by Lemma 2.7 it follows that

$$\{b > 0 \mid u(r, b) \text{ has exactly one zero for } r > R\}$$

is nonempty and by Lemma 2.5 this set is bounded above. So we let:

$$b_1 = \{b > 0 \mid u(r, b) \text{ has exactly one zero for } r > R\}.$$

Then as we did above it is possible to show $u(r, b_1)$ is a solution of (1.4)–(1.5) which has exactly one zero for $r > R$ and:

$$\lim_{r\to\infty} u(r, b_1) = 0.$$

Similarly for any nonnegative integer n there is a $b_n > b_{n-1}$ such that $u(r, b_n)$ is a solution which has exactly n zeros for $r > R$ and:

$$\lim_{r\to\infty} u(r, b_n) = 0.$$

This completes the proof of Theorem 1.1. □

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