Finite element approximation of the first eigenvalue of a nonlinear problem for some special domains

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October 26, 2000

Abstract

In this paper we present a method for the numerical approximation of the smallest eigenvalue of a nonlinear eigenvalue problem using the finite element method. Numerical results are presented for some special domains when the domain is bounded by a square, a "circle", a "semicircle", or a quadrant of a "circle". We compare the exact solutions with the approximate solutions when the exact solutions are known. We show a connection among the first eigenvalues related to different domains.

1 Introduction

Consider the eigenvalue problem

$$\begin{aligned} -Q_p &= \lambda |u|^{p-1} u \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a convex domain in $\mathbf{R}^2$ and Q_p is the nonlinear operator defined by

$$Q_p = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-1}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{p-1}\frac{\partial u}{\partial y}\right) \quad \text{for} \quad 0 < p < \infty.$$

It is known (see [2]) that (1) has a sequence of weak solutions $(\lambda_k(p), u_k(p))$ in $\mathbf{R}\times W_0^{1,p+1}(\Omega)$, where $0 < \lambda_k(p) < \lambda_{k+1}(p)$, $k = 1, 2, \ldots$ .The first eigenvalue can be defined by the variational principle

$$\lambda_1(p) = \inf_{u \in W_0^{1,p+1}(\Omega)} \frac{\int\limits_\Omega \left(|u_x|^{p+1} + |u_y|^{p+1}\right) dx}{\int\limits_\Omega |u|^{p+1}\, dx} =$$

$$= \inf_{\substack{u \in W_0^{1,p+1}(\Omega) \\ \|u\|_{p+1} = 1}} \int\limits_\Omega \left(|u_x|^{p+1} + |u_y|^{p+1}\right) dx. \tag{2}$$

The weak formulation of (1) is

$$\int\limits_\Omega (|u_x|^{p-1} u_x v_x + |u_y|^{p-1} u_y v_{y)} dx = \lambda_1(p) \int\limits_\Omega |u|^{p-1} uv dx \tag{3}$$

for any $v \in W_0^{1,p+1}(\Omega)$. The unique solution $(\lambda_1(p), u_1(p))$ of (3) which saisfies $\|u\|_{p+1} = 1$ is called the first eigenpair of (3). It is known that $u_1(p)$ is positive [1]. From (3) we have got

$$\lambda_1(p) = \int_\Omega \left(|(u_1(p))_x|^{p+1} + |(u_1(p))_y|^{p+1} \right) dx.$$

The knowledge of the first eigenvalue is of considerable importance for the designer of safe and efficient structures. Such structures may appear in microphones, bridges, ships, or space vehicles.

The quantity $\lambda_1(p)$ arises in a variety of situations where it is of interest to know its value more precisely. For example $\lambda_1(p)$ is the principal frequency of Ω. We conceive Ω as the equilibrium position of a stretched membrane fixed along the boundary $\partial\Omega$ of Ω. The frequency of the gravest proper tone of this membrane is $\sqrt[p+1]{\lambda_1(p)}$. In the linear case, $\lambda_1(1)$ is the classical principal eigenvalue of the Poisson equation $\Delta u + \lambda u = 0$. The quantity $\lambda_1(p)$ depends on shape and size of the domain Ω. This principal frequency of vibration has been calculated for various domains. These are: circle, square, quadrant of circle, sector of a circle 60^o, rectangle, equilateral triangle, semicircle [9], [10].

Lord Rayleigh stated that of all clamped membranes with a given area A, the circle has the minimal principal frequency [10]. This property can be expressed by the inequality

$$\lambda_1(1) \geq \frac{\pi j_0^2}{A} \tag{4}$$

with equality only for the circle and where j_0 is the first positive zero of the Bessel function of the first kind $J_0(x)$. It was a conjecture, however Rayleigh supported it not only by numerical evidence, but also by computing the principal frequency of almost circular membranes. G. Faber [5] and E. Krahn [7] found independently the same proof of Rayleigh's conjecture. Another proof was given by G. Polya and G. Szego [9] by using the Steiner symmetrization.

E. Krahn [8] showed that the Rayleigh inequality (4) can be extended to $\mathbf{R}^N$ in the form

$$\lambda_1(1) \geq \left[\frac{V}{A_N} \right]^{\frac{2}{N}} j^2_{\frac{N-2}{2}},$$

where V is the volume of the unit ball in $\mathbf{R}^N$, A_N is the volume of $\Omega \in \mathbf{R}^N$ and $j^2_{\frac{N-2}{2}}$ is the first zero of the Bessel function $J_{\frac{N-2}{2}}$.

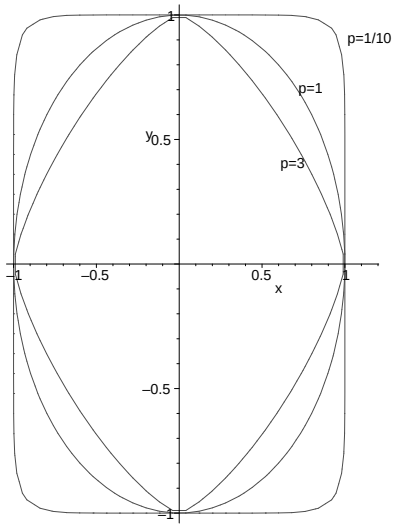
In [1] a lower bound was given for the first eigenvalue of the nonlinear elliptic eigenvalue problem (1) :

$$\lambda_1(p) \geq \left[\frac{P j_0^2}{A} \right]^{\frac{p+1}{2}}, \qquad P = 2\frac{p}{p+1} B\left(\frac{p}{p+1}, \frac{p}{p+1} \right), \tag{5}$$

where A is the area of $\Omega \in \mathbf{R}^2$, and j_0 is the first positive zero of the generalized nonlinear Bessel function $J_0(x)$. In (5) there is equality if and only if $\partial\Omega$ is the curve defined by the equation

$$|x|^{\frac{1}{p}+1} + |y|^{\frac{1}{p}+1} = r^{\frac{1}{p}+1}, \tag{6}$$

with suitable $r \in \mathbf{R}^+$ (Figure.1 with $r = 1$)



This curve is called isoperimetrix which plays the same role in case of nonlinear problem (1) as the circle in case of the Poisson equation. Therefore later we recall this curve as "circle". In the case $p = 1$ the inequality (5) is equivalent to (4) and $P = \pi$. Moreover we obtained that for the simply connected convex domain $\Omega \in \mathbf{R}^2$ the inequality

$$\lambda_1(p) \geq \left[\frac{A+\sigma}{(p+1)\,\varrho A}\right]^{p+1}$$

holds, where ϱ is the radius of the greatest inscribed isoperimetrix of Ω, and σ is the area of the isoperimetrix of radius ϱ [3].

The smallest first eigenvalue $\lambda_1(p)$ is evaluated for the so called "circle" in case of nonlinear problem (1). For the linear case ($p = 1$) the connection among the first eigenvalues regarding to some special domains is given by Rayleigh [10].

In this paper we examine the first eigenvalues for different domains. The exact solution is known only for a special domain (rectangular domain).It is not known that the problem has classical solution on other domains except rectangle. On such a domain we can obtain approximate solutions.We shall present a method for numerical approximation of $\lambda_1(p)$ based on the finite element approximation to $u_1(p)$. Some numerical results will be presented for different special domains when Ω is bounded by a square, a "circle", a "semicircle", a quadrant of a "circle". We compare the exact solutions with the approximate solutions when the exact solutions are known. We show that the membranes of the same area the same connection is valid for the first eigenvalues of the nonlinear problem as of the linear problem ($p = 1$).

2 Preliminary results

For the Dirichlet eigenvalue problem (1) we can find classical solutions when D is bounded by the rectangle

$$D = \{(x, y) :\ 0 \leq x \leq a,\ 0 \leq y \leq b\}\,.$$

The eigenvalues and eigenfunctions are

$$\lambda_{k,l} = p\widetilde{\pi}^{p+1}\left(\frac{k^{p+1}}{a^{p+1}} + \frac{l^{p+1}}{b^{p+1}}\right), \tag{7}$$

$$u_{k,l} = A_{k,l} S_p\left(\frac{k\widetilde{\pi}}{a}x\right) S_p\left(\frac{l\widetilde{\pi}}{b}y\right), \qquad k,l = 1,2,..., \tag{8}$$

where

$$\widetilde{\pi} = \frac{2\frac{\pi}{p+1}}{\sin\frac{\pi}{p+1}},$$

$A_{k,l} = const.$, and the function S_p is the solution of the differential equation

$$S_p" \left|S_p'\right|^{p-1} + |S_p|^{p-1} S_p = 0$$

under conditions

$$S_p(0) = 0, \quad S_p(\widetilde{\pi}) = 0.$$

The function S_p is the generalized sine function which plays the same role in case of nonlinear problem (1) as the sine function in case of Poisson equation. For $p = 1$

$$S_1(x) = \sin x, \quad \widetilde{\pi} = \pi.$$

If the domain D is bounded by a unit square, as a corollary of the above, we get the smallest eigenvalue $\lambda_1(p)$ and the corresponding eigenfunction $u_1(p)$ for the Dirichlet eigenvalue problem of (1) if we put $k = l = 1$ in the expressions of $\lambda_{k,l}$ and $u_{k,l}$:

$$\lambda_1(p) = 2p\widetilde{\pi}^{p+1}, \tag{9}$$

$$u_1(p) = A_{1,1} S_p(\widetilde{\pi}x) S_p(\widetilde{\pi}y).$$

The exact value of the first eigenvalue for the linear eigenvalue problem ($p = 1$) was given by Rayleigh [10]:

$$\lambda_1(1) = 2\pi^2 \approx 19.7392.$$

3 Finite element formulation of the problem

Let Ω be a given convex open subset of $\mathbf{R}^2$ and let $W^{1,p+1}(\Omega)$ denotes the space of all functions which together with their derivatives u_x, u_y belong to $L^{p+1}(\Omega)$. We define the norm in $W^{1,p+1}(\Omega)$ by

$$\begin{aligned} \|u\|_{W^{1,p+1}(\Omega)} &= \left(\int_\Omega \left(|u|^{p+1} + |u_x|^{p+1} + |u_y|^{p+1}\right) dx\right)^{\frac{1}{p+1}} \\ \text{for all} \quad u &\in W^{1,p+1}(\Omega). \end{aligned}$$

As usually, the symbol $W_0^{1,p+1}(\Omega)$ stands for the subspace of $W^{1,p+1}(\Omega)$ obtained by closing the set of all $\mathbf{C}^\infty$-functions with compact support in Ω. On the Sobolev space $W_0^{1,p+1}(\Omega)$ another norm can be defined by

$$\|u\|_{1,p+1} = \left(\int_\Omega \left(|u_x|^{p+1} + |u_y|^{p+1}\right) dx\right)^{\frac{1}{p+1}}$$

which is equivalent to the norm $\|u\|_{W^{1,p+1}(\Omega)}$. The usual norm in $L^{p+1}(\Omega)$ is denoted by $\|u\|_{p+1} = \left(\int\limits_{\Omega} |u|^{p+1}\, dx\right)^{\frac{1}{p+1}}$.

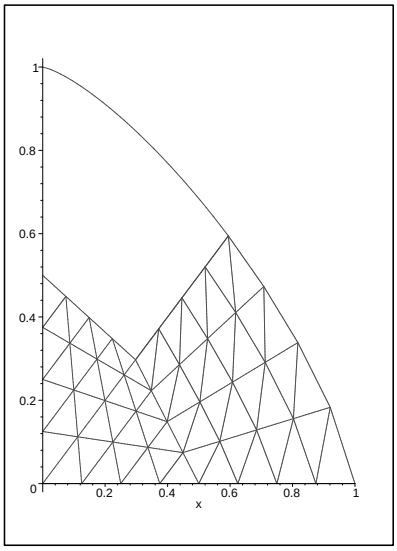
Now we form the the finite element approximation of the problem (1) and (2). Let $V_h(\Omega)$ be the space of continuous linear functions based on regular triangulation of Ω and the subspace of $W_0^{1,p+1}(\Omega)$ as in [4]. The weak formulation of the problem (1) was formulated in (3). The corresponding minimization problem consists of finding the eigenpair $(\lambda_{1h}(p),\, u_h)$ such that

$$\lambda_{1h}(p) = \inf_{v_h \in V_h(\Omega)} \frac{\int\limits_{\Omega} \left(|v_{hx}|^{p+1} + |v_{hy}|^{p+1}\right) dx}{\int\limits_{\Omega} |v_h|^{p+1}\, dx}, \tag{10}$$

and

$$\begin{aligned} \int\limits_{\Omega} (|u_{hx}|^{p-1}\, u_{hx} v_{hx} + |u_{hy}|^{p-1}\, u_{hy} v_{hy}) dx &= \lambda \int\limits_{\Omega} |u_h|^{p-1}\, u_h v_h dx \\ \text{for any} \quad v_h &\in V_h(\Omega). \end{aligned} \tag{11}$$

The domain triangulation depends on the shape of the domain. For square equal mesh sizes is used. For other domain the triangulation has to be generated for each value of p and for each shape separately (Figure 2. the triangulation for the quadrant of the so called circle is given when $p = 3$).



4 Error estimation

Let the Rayleigh-Ritz projection of $u \in W_0^{1,p+1}(\Omega)$ is denoted by $Pu \in V_h(\Omega)$ and Pu is the unique finite element solution of

$$\begin{aligned} &\int\limits_{\Omega} (|Pu_x|^{p-1}\, Pu_x v_{hx} + |Pu_y|^{p-1}\, Pu_y v_{hy}) dx = \\ &\quad = \int\limits_{\Omega} (|u_x|^{p-1}\, u_x v_{hx} + |u_y|^{p-1}\, u_y v_{hy}) dx \end{aligned} \tag{12}$$

for given $u \in W_0^{1,p+1}(\Omega)$ and for any $v_h \in V_h(\Omega)$. From Theorem 5.3.2 [4] follows that

$$\lim_{h\to 0} \|u - Pu\|_{p+1} = 0. \tag{13}$$

Lemma 1 *Let* (λ_1, u_1) *be the first eigenpair of (1) and let*

$$\varepsilon_{1h} = \left| \int_\Omega |u_1|^{p+1}\, dx - \int_\Omega |Pu_1|^{p+1}\, dx \right|.$$

Then

$$\lambda_{1h}(p) \le \frac{\lambda_1}{1-\varepsilon_1^h}$$

provided $\varepsilon_{1h} < 1$.

Proof. We suppose that $Pu_1 \neq 0$. If $Pu_1 = 0$, then $\varepsilon_{1h} = \left| \int_\Omega |u_1|^{p+1}\, dx - \int_\Omega |Pu_1|^{p+1}\, dx \right| = \int_\Omega |Pu_1|^{p+1}\, dx = 1$ and condition $\varepsilon_{1h} < 1$ is not satisfied. Applying (10) we get

$$\lambda_{1h}(p) = \inf_{v_h \in V_h(\Omega)} \frac{\int_\Omega \left(|v_{hx}|^{p+1} + |v_{hy}|^{p+1} \right) dx}{\int_\Omega |v_h|^{p+1}\, dx} \le \frac{\int_\Omega \left(|Pu_{1x}|^{p+1} + |Pu_{1y}|^{p+1} \right) dx}{\int_\Omega |Pu_1|^{p+1}\, dx} \tag{14}$$

for $Pu_1 \in V_h(\Omega)$. Making use of (**??**) with $v_h = Pu_1$, the inequality [6] and the Hölder inequality we obtain

$$\begin{aligned} \int_\Omega \left(|Pu_{1x}|^{p+1} + |Pu_{1y}|^{p+1} \right) dx &= \int_\Omega (|u_{1x}|^{p-1} u_{1x} Pu_{1x} + |u_{1y}|^{p-1} u_{1y} Pu_{1y)} dx \\ &\le \int_\Omega \left[\left(|u_{1x}|^{p+1} + |u_{1y}|^{p+1} \right)^{\frac{p}{p+1}} \left(|Pu_{1x}|^{p+1} + |Pu_{1y}|^{p+1} \right)^{\frac{1}{p+1}} \right] dx \\ &\le \|u_1\|_{1,p+1}^p \, \|Pu_1\|_{1,p+1} \end{aligned}$$

which leads to

$$\|Pu_1\|_{1,p+1}^{p+1} \le \|u_1\|_{1,p+1}^p \, \|Pu_1\|_{1,p+1},$$

$$\|Pu_1\|_{1,p+1} \le \|u_1\|_{1,p+1}^p. \tag{15}$$

Since

$$\begin{aligned} \int_\Omega |Pu_1|^{p+1}\, dx &= \int_\Omega |u_1|^{p+1}\, dx + \int_\Omega (\, |Pu_1|^{p+1} - |u_1|^{p+1}) dx \ge \\ &\ge 1 - \left| \int_\Omega (|Pu_1|^{p+1} - |u_1|^{p+1}) dx \right| \\ &= 1 - \varepsilon_{1h} \end{aligned}$$

we obtain the following estimate from (14) with (15)

$$\lambda_{1h}(p) \leq \frac{\int\limits_{\Omega} \left(|u_{1x}|^{p+1} + |u_{1y}|^{p+1} \right) dx}{\int\limits_{\Omega} |Pu_1|^{p+1} \, dx} \leq \frac{\lambda_1}{1 - \varepsilon_{1h}}$$

which has to be proved. ■

Lemma 2 *There exists a constant* $\alpha > 0$ *depending on* p *and* u*, such that*

$$\varepsilon_{1h} \leq \alpha \, \|u_1\|_{1,p+1}^{p} \, \|u_1 - Pu_1\|_{p+1} . \tag{16}$$

Proof. Applying the inequality

$$\left| x^{p+1} - y^{p+1} \right| \leq (p+1) \, |x - y| \;\; (|x|^{p-1} \, x - |y|^{p-1} \, y) \quad \text{for} \;\; p > 0, \;\; x \geq 0, \;\; y \geq 0.$$

for ε_{1h} we get

$$\begin{aligned} \varepsilon_{1h} &= \left| \int\limits_{\Omega} |u_1|^{p+1} \, dx - \int\limits_{\Omega} |Pu_1|^{p+1} \, dx \right| \\ &\leq (p+1) \, \|u_1 - Pu_1\|_{p+1} \, (\|u_1\|_{p+1}^{p} + \|Pu_1\|_{p+1}^{p}). \end{aligned}$$

By the Poincare inequality [1]

$$\int\limits_{\Omega} |Pu_1|^{p+1} \, dx \leq C_1 \int\limits_{\Omega} \left(|Pu_{1x}|^{p+1} + |Pu_{1y}|^{p+1} \right) dx \;\; \text{with} \;\; C_1 = const.$$

and (15) we have

$$\begin{aligned} \varepsilon_{1h} &\leq (p+1) \, \|u_1 - Pu_1\|_{p+1} \, (\|u_1\|_{p+1}^{p} + C_1^{p} \, \|u_1\|_{p+1}^{p}) \leq \\ &\leq \alpha \, \|u_1\|_{p+1}^{p} \, \|u_1 - Pu_1\|_{p+1} . \end{aligned}$$

■

Thus we can give error estimates for the first eigenvalue

Theorem 3 *Let* (λ_1, u_1) *be the first eigenpair of (1) and let* $(\lambda_{1h}(p), u_{1h})$ *is the counterpart of* (λ_1, u_1) *for (10) and (11) then*

$$\lim_{h \to 0} \lambda_{1h}(p) = \lambda_1. \tag{17}$$

Proof. We can choose h small enough so that $\varepsilon_{1h} < \frac{1}{2}$, then

$$\frac{1}{1 - \varepsilon_{1h}} \leq 1 + 2\varepsilon_{1h}$$

and by Lemma1 we get

$$\lambda_1 \leq \lambda_{1h}(p) \leq \frac{\lambda_1}{1 - \varepsilon_{1h}} \leq \lambda_1(1 + 2\varepsilon_{1h}).$$

Using Lemma 2 we obtain that

$$\begin{aligned} \lambda_1 &\leq \lambda_1(1 + 2\varepsilon_{1h}) \leq \\ &\leq \lambda_1 + 2\lambda_1 \alpha \, \|u_1\|_{p+1}^{p} \, \|u_1 - Pu_1\|_{p+1} \end{aligned}$$

which gives (17) with (13). ■

5 Numerical results

The functions $N_i,\ \ i = 1, 2, ..., m$ will denote a finite element basis of the m dimensional space V_h, thus

$$V_h = \left\{ \sum_{i=1}^{m} x_i N_i \,|(x_1, x_2, ..., x_m) \in \mathbf{R}^m \right\}.$$

The finite element approximation of u_1 is

$$u_h = \sum_{i=1}^{m} x_i N_i = \mathbf{x}^T \ \mathbf{N}^T,$$

$$\nabla u_h = \sum_{i=1}^{m} \nabla (N_i x_i) = \mathbf{x}^T \ \mathbf{B}^T,$$

and from the normalization formula $\int\limits_{\Omega} |u_h|^{p+1}\, dx = 1$ we obtain

$$\int\limits_{\Omega} |\mathbf{N}\ \mathbf{x}|^{p+1}\, dx = 1.$$

For arbitrary $\widetilde{u}_h = \mathbf{N}\ \widetilde{\mathbf{x}} = \mathbf{N}\ (\mathbf{x}a)$, where $a \neq 0$, we have got

$$\int\limits_{\Omega} |\widetilde{u}_h|^{p+1}\, dx = \int\limits_{\Omega} |\mathbf{N}\ \mathbf{x}a|^{p+1}\, dx = \int\limits_{\Omega} |\mathbf{N}\ \mathbf{x}|^{p+1}\, |a|^{p+1}\, dx,$$

therefore

$$|a| = \left[\int\limits_{\Omega} |\widetilde{u}_h|^{p+1}\, dx \right]^{\frac{1}{p+1}}, \quad \text{and} \quad \mathbf{x} = \frac{\widetilde{\mathbf{x}}}{a}.$$

After normalization we obtain

$$\lambda_{1h}(p) = \int\limits_{\Omega} |\mathbf{B}\ \mathbf{x}|^{p+1}\, dx. \tag{18}$$

From $\frac{\partial \lambda_h}{\partial x_i} = 0 \ \ (i = 1, 2, ..., m)$ we get

$$0 = \left[\int\limits_{\Omega} \mathbf{B}^T\, |\mathbf{B}\ \mathbf{x}|^p\, dx \right] \left[\int\limits_{\Omega} |\mathbf{N}\ \mathbf{x}|^{p+1}\, dx \right] - \left[\int\limits_{\Omega} |\mathbf{B}\ \mathbf{x}|^{p+1}\, dx \right] \left[\int\limits_{\Omega} \mathbf{N}^T\ \ |\mathbf{N}\ \mathbf{x}|^p\, dx \right]$$

and using (18) we obtain

$$0 = \int\limits_{\Omega} \mathbf{B}^T\, |\mathbf{B}\ \mathbf{x}|^p\, dx - \lambda_h \int\limits_{\Omega} \mathbf{N}^T\ \ |\mathbf{N}\ \mathbf{x}|^p\, dx. \tag{19}$$

Since (19) is nonlinear we get

$$0 = \int_{\Omega} \mathbf{B}^T \left|\mathbf{B}\ \mathbf{x}_j\right|^p dx - \lambda_{h,j} \int_{\Omega} \mathbf{N}^T\ \left|\mathbf{N}\ \mathbf{x}_j\right|^p dx + p \int_{\Omega} \mathbf{B}^T \left|\mathbf{B}\ \mathbf{x}_j\right|^{p-1} \mathbf{B}\ dx\,(\mathbf{x} - \mathbf{x}_j) \tag{20}$$

by using the Newton's method for given $\mathbf{x}_j$. Applying the following notations

$$\begin{aligned}
\mathbf{H}_j &= p \int_{\Omega} \mathbf{B}^T \left|\mathbf{B}\ \mathbf{x}_j\right|^{p-1} \mathbf{B}\ dx, \\
\mathbf{f}_N &= \int_{\Omega} \mathbf{N}^T\ \left|\mathbf{N}\ \mathbf{x}_j\right|^p dx, \\
\mathbf{f}_B &= \int_{\Omega} \mathbf{B}^T \left|\mathbf{B}\ \mathbf{x}_j\right|^p dx, \\
\Delta\mathbf{x}_j &= (\mathbf{x} - \mathbf{x}_j)\,,
\end{aligned}$$

and we can write (20) as follows

$$\mathbf{H}_j\ \Delta\mathbf{x}_j = \lambda_{1h,j}\mathbf{f}_N - \mathbf{f}_B.$$

In order to give initial values for the eigenfunction and eigenvalue we solve a linear eigenvalue problem ($p = 1$). Then the non-linearity is gradually imposed on the equations, that means, that p is increased or decreased up to the required value.

We briefly describe the algorithm below

1. Start with an initial approximation $\mathbf{x}_1$ $(\mathbf{x}_1 \neq \mathbf{0})$.
2. For given $\mathbf{x}_j$, calculate $\mathbf{H}_j\ \Delta\mathbf{x}_j = \lambda_{1h,j}\mathbf{f}_N - \mathbf{f}_B$, then $\Delta\mathbf{x}_j$.
3. Evaluate $\widetilde{\mathbf{x}}_{j+1} = \mathbf{x}_j + \Delta\mathbf{x}_j$.
4. Evaluate $\mathbf{x}_{j+1} = \dfrac{\widetilde{\mathbf{x}}_{j+1}}{\left(\int_{\Omega} |\mathbf{N}\ \mathbf{x}_{j+1}|^{p+1} dx\right)^{\frac{1}{p+1}}}$.
5. Calculate $\lambda_{1h,j+1} = \int_{\Omega} \left|\mathbf{B}\ \mathbf{x}_{j+1}\right|^{p+1} dx$.
6. Terminate when $\frac{|\lambda_{1h,j+1} - \lambda_{1h,j}|}{\lambda_{1h,j+1}}$ is smaller then a predetermined tolerance.

An experimental computer code written in FORTRAN source language was used for the solution of the approximate eigenvalues. The eigenvalues are calculated by h-version of the finite element method.

In the second section we showed solutions of the nonlinear eigenvalue problem (1) for the unit square. Here the exact values of the first eigenvalues λ_1 are known as a function of p in (9). Using the finite element method we calculated the approximate values of λ_1 for various values of p for different shape of domain Ω in twodimension. These domains are the unit square, the "unit circle" defined in (6), and the "semicircle", and the quadrant of the "circle" with radius 1. The exact values for λ_1 are known from [10] for the linear case $(p = 1)$. In the tables below we give the finite element approximations of the first eigenvalue for different number of nodal points n, when the mesh size is different. The first computation was made for the unit square. The values are reported in Table 1.

$p\backslash n$	9	16	64	256
0.5	14.1856	11.1788	10.7632	10.6683
0.75	18.5345	15.4269	14.8608	14.7285
1	24.0000	20.7733	19.9941	19.8027
2	64.0000	59.7528	57.5335	56.8420
3	164.5714	154.1444	148.9242	146.9674
5	1063.3840	890.3235	862.0190	850.2630

Table 1.

In case $p = 1$ the known value is $\lambda_1(1) = 2\pi^2 \approx 19.7392$. Our approximation differs from the known value by 0.32 %. In case of square domain we can evaluate the relative error for any value of p since we have the formula (9). The relative error is less then 1 % when $p \in [0.25, 10]$. Similar formula is not known for the other domains. Thus we can compare the approximate solution with the exact solution only when $p = 1$. The approximate values are reported in Table 2 for the unit "circle".

$p\backslash n$	67	227	835	3203
0.5	4.1774	4.0573	4.0277	4.0203
0.75	5.0962	4.9366	4.8978	4.8882
1	6.0646	5.8522	5.8004	5.7875
2	10.5134	10.0104	9.8780	9.8435
3	15.9061	15.0216	14.7739	14.7095
5	29.4856	27.6169	27.0521	26.8875

Table 2.

In the linear case the exact solution is 5.7831 [10]. Here the difference between the exact and finite element solution is 0.08 %. For the "semicircle" The calculated values are presented in Table 3.

$p\backslash n$	33	113	417	1601
0.5	8.2958	7.8851	7.7885	7.7645
0.75	11.5277	10.9784	10.8399	10.8066
1	15.7835	14.9546	14.7500	14.6989
2	50.7108	46.2403	44.9594	44.6189
3	150.3274	130.6471	124.0847	122.1544
5	1168.4845	921.3901	822.6043	788.0588

Table 3.

In the linear case the exact solution is given by 14.6842 [10]. The difference is here 0.1 %. The numerical results for λ_1 when the domain is bounded by the quadrant of the "circle" are given in Table 4.

$p\backslash n$	19	61	217	817
0.5	13.2191	12.1671	11.9362	11.8787
0.75	19.7591	18.3118	17.9715	17.8864
1	29.3712	27.1025	26.5555	26.4198
2	144.0884	120.3870	115.2347	113.9156
3	735.6182	504.7015	462.9082	451.2033
5	18610.8737	8298.0772	6753.8355	6269.8435

Table 4.

The known value is 26.3682 for $p = 1$, and the difference is 0.2 %.

The following table gives the relative frequency in certain cases for the gravest tone of membranes under similar mechanical conditions and of equal area (Table 5.). We can conclude that for any value p from [0.5, 5] the "circular" membrane has the gravest mode similarly as in the linear case [10].

p	"circle"	square	quadrant of "circle"	"semicircle"
0.5	4.7525	4.8368	4.8929	5.2117
0.75	4.5075	4.6429	4.7297	5.0154
1	4.2629	4.4429	4.5552	4.8039
2	3.5461	3.8389	4.0105	4.1498
3	3.1221	3.4767	3.6738	3.7477
5	2.6546	3.0741	3.2931	3.2961

Table 5.

Acknowledgement 4 *This work was supported by the National Research Foundation OTKA T026138. The author is grateful to the referee for his valuable comments and helpful*

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