Regularity, nonoscillation and asymptotic behaviour of solutions of second order linear equations

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Abstract

The role of the regularly varying functions in the sense of Karamata in the qualitative theory of the equations in question is presented.

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1. Introduction.

We present here a survey of some results on regularity, nonoscillation and asymptotic behaviour of solutions of the equation

$$y'' + f(x)y = 0 \tag{1.1),}$$

where f is continuous on a half-axis $[a, \infty)$ for some $a > 0$, and need not to be of constant sign.

These results are then easily transfered to the equation

$$z'' + g(x)z' + h(x)z = 0 \tag{1.2}$$

by the usual transformation $y = \exp\{1/2 \int g(x)dx\}z$.

If especially in equation (1.1) one has $f(x) < 0$, for convenience it is written as

$$y'' - f(x)y = 0 \tag{1.1$'$}$$

with $f(x) > 0$ and some additional results are given.

Besides the regularity this is an ancient topic; (non) oscillation criteria begun with Kneser in 1893 (for positive $f(x)$). The asymptotics (meaning as usual to find a function φ such that $y(x)/\varphi(x) \to 1$, as $x \to \infty$, denoted by $y(x) \sim \varphi(x)$) goes back to Liouville's and Green's work of 1837 (for equation (1.1')).

There is an immense number of papers dealing with these questions both in mathematics and in theoretical physics since (1.1) is an one-dimensional (or radially symmetric) Schrödinger equation with oscillating potential if $f(x)$ changes sign (as it is the case here). The later fact is probably the predominant reason for the contemporary interest in the subject [17].

In spite of such a long lasting and abundant activity, our approach is quite new since it makes use of the notion of regularly (or especially - slowly) varying functions as introduced by Jovan Karamata in 1930 [12].

Since this class of functions is not a common tool in the theory of differential equations, in order to help the reader we give here the basic definitions and a fundamental property of the class of regularly varying functions.

Definition 1 [12]. *A positive continuous (measurable) function r defined on $[a, \infty)$, $a > 0$ is called regularly varying (at infinity), if*

$$\lim_{x\to\infty} r(tx)/r(x) = t^{\rho} \qquad \textit{for every } t > 0,$$

where ρ is called the index of regular variation.

In the case $\rho = 0$, one has

Definition 2 [12]. *A positive continuous (measurable) function L defined on $[a, \infty)$, $a > 0$ is called slowly varying (at infinity) if*

$$\lim_{x\to\infty} L(tx)/L(x) = 1 \qquad \textit{for every}\, t > 0.$$

Definitions 1 and 2 imply that any regularly varying function $r(x)$ has the form

$$r(x) = x^{\rho} L(x).$$

Hence the class of regularly varying functions represents a generalization of powers of x in which the slowly varying function $L(x)$ plays the essential role.

All positive functions tending to positive constants, the function

$$L(x) = \prod_{\nu=1}^{n} (ln_{\nu} x)^{\xi_{\nu}}$$

where ξ_{ν} are real and $ln_{\nu}(\cdot)$ denotes the ν-th iteration of logarithm, the function $\frac{1}{x}\int\limits_a^x \frac{dt}{lnt}$, the function $\exp\{lnlnx\}$ are examples of slowly varying functions.

It is worthwhile to mention that a slowly varying function might oscillate (in the sense of real functions), even infinitely as shown by the example

$$L(x) = \exp\{(lnx)^{1/3} \cos(lnx)^{1/3}\}$$

where

$$\lim_{x\to\infty} \inf L(x) = 0, \quad \lim_{x\to\infty} \sup L(x) = \infty.$$

Definition 3 [2]. *A positive continuous (measurable) function g defined on $[a, \infty)$, $a > 0$ is called rapidly varying at infinity if*

$$\lim_{x\to\infty} g(tx)/g(x) = 0 \textit{ or } \infty \qquad \textit{for every}\, t > 1.$$

Iterated exponential functions or the function $y(x) = \exp(\ell n^2 x)$ may serve as examples of rapidly varying functions.

We shall refer to all three mentioned classes as *Karamata functions.*

From the extensively developed theory of Karamata functions we mention only one of the most useful results both in the theory and in the applications of these functions:

Proposition 1 [12]. *(Representation theorem). The function L is slowly varying if and only if it can be written in the form*

$$L(x) = c(x)\exp\{\int_a^x \frac{\varepsilon(t)}{t}dt\}$$

for some $a > 0$, where $c(x)$ is continuous (measurable) and, for $x \to \infty$, $c(x) \to$ const (> 0), $\varepsilon(x) \to 0$.

If, in particular $c(x) =$ *const*. then $L(x)$ is called *normalized*, and usually denoted by $L_0(x)$.

The first monograph on Karamata's functions was written by E. Seneta [16]. The state of the art both of the theory and of various applications is presented in the comprehensive treatise of Bingham, Goldie and Teugels [3].

Originally introduced by Karamata for application in Tauberian theory (resulting among other, in what is today known as Hardy-Littlewood-Karamata Tauberian theorem), Karamata functions have been later applied in several branches of analysis: Abelian theorems (asymptotics of series and integrals - Fourier ones in particular), analytic (entire) functions, analytic number theory, etc. The great potential of these functions in the probability theory and its applications is shown in the well known Feller's treatise [4] and a new impetus is given by de Haan in [7]. First application to the theory of ordinary differential equations related to the Thomas Fermi one was done by V.G. Avakumović in 1947 [1].

2. Results.

2.1. Regularity and nonoscillation of solutions.

The existence of regularly varying solutions of equation (1.1) is established by the following two theorems.

Theorem 1 [11]. *Let $-\infty < c < 1/4$, and let $\alpha_1 < \alpha_2$ be the roots of the equation*

$$\alpha^2 - \alpha + c = 0.$$

Further, let L_i, $i = 1, 2$, denote two normalized slowly varying functions.

Then there exist two linearly independent regularly varying solutions of the form

$$y_i(x) = x^{\alpha_i} L_i(x), \quad i = 1, 2, \tag{2.1}$$

where $L_2(x) \sim \{(1 - 2\alpha_1)L_1(x)\}^{-1}$ *as* $x \to \infty$, *if and only if*

$$\phi(x) := x \int_x^\infty f(t)dt - c \to 0, \quad \text{as } x \to \infty. \tag{2.2}$$

Notice that for $c = 0$, one has $\alpha_1 = 0$, $\alpha_2 = 1$ so that the first solution is of the form $y_1(x) = L_1(x)$, hence slowly varying and the second one is of the form $y_2(x) = xL_2(x)$ hence regularly varying of index one. We may mention that the proof of the necessity part of Theorem 1 also implies the convergence of the occuring integral.

Theorem 2 [11]. *Let* $c = 1/4$ *and let the integral* $\int^\infty \frac{|\phi(t)|}{t}dt$ *converge. Put*

$$\psi(x) := \int_x^\infty \frac{|\phi(t)|}{t}dt$$

and assume

$$\int^\infty \frac{\psi(t)}{t}dt < \infty.$$

Further let L_i, $i = 1, 2$ *denote two normalized slowly varying functions. Then there exist two linearly independent solutions of equation (1.1) of the form*

$$y_1(x) = x^{1/2} L_1(x) \quad \text{and} \quad y_2(x) = x^{1/2} lnx L_2(x),$$

where, L_i, $i = 1, 2$ *tend to a constant as* $x \to \infty$ *and* $L_2(x) \sim L_1^{-1}(x)$, *if and only if condition (2.2) holds.*

Here, unfortunately, additional conditions are introduced which restrict slowly varying functions $L_i(x)$ to the special ones tending to constants as $x \to \infty$.

If $c > 1/4$ the solutions are oscillatory which, for $f(x) > 0$, follows from the following well known (non) oscillation criterion of Hille.

Proposition 2 [18]. *If*

$$\lim_{x\to\infty} \sup x \int_x^\infty f(t)dt < \frac{1}{4},$$

then equation (1.1) is nonoscillatory; if

$$\lim_{x\to\infty} \inf x \int_x^\infty f(t)dt > \frac{1}{4},$$

then (1.1) is oscillatory.

It is worthy to note that the same integral occurs both in our and in the above Hille's condition. Our condition is more restrictive requiring the existence of the limit of the intervening function, but as a compensation, we obtain the regularity and the representation of solutions.

Since by Definitions 1 and 2 regularly (slowly) varying functions are positive, condition (2.2) is a necessary and sufficient condition for nonoscillation.

Although this holds only for the subclass of solutions of (1.1) of the form (2.1) it might be of interest since results giving necessary and sufficient conditions for nonoscillation are quite scarce in the existing literature.

For more general equation (1.2) similar results hold. (The notation $y(x)$ for solutions is kept for convenience.)

Theorem 3 [14]. *Let* $c, c_1 \in \mathbb{R}$ *such that*

$$c_1/2 - c_1^2/4 + c < \frac{1}{4},$$

and let $\alpha_1 < \alpha_2$ *be the roots of the equation* $\alpha^2 - \alpha + \gamma = 0$ *with*

$$\gamma = c_1/2 - c_1^2/4 + c.$$

Further let h *be continuous and* g *continuously differentiable on* $[x_0, \infty)$ *such that*

$$\lim_{x\to\infty} xg(x) = c_1.$$

Let L_i, $i = 1, 2$ *denote some normalized slowly varying functions.*

Then there exists two linearly independent regularly varying solutions y_i of equation (1.2)*, of the form $y_1(x) = x^{\alpha_1 - c_1/2} L_1(x)$ and $y_2(x) = x^{\alpha_2 - c_1/2} L_2(x)$, where for $x \to \infty$,*

$$L_2(x) \sim \{(1 - 2\alpha_1) L_1(x)\}^{-1} \exp\left(\int_a^x \frac{\varepsilon(s)}{s} dt \right)$$

with some $\varepsilon(s) \to 0$, as $s \to 0$, if and only if

$$x \int_x^\infty h(t)dt \to c, \text{ as } x \to \infty.$$

If in equation (1.1) $f(x)$ is ultimately negative then all (positive) solutions are convex which offers a possibility to improve Theorem 1 and to prove some additional results. Namely, for the solutions of equation (1.1') Theorem 1 holds verbatim for all $c \geq 0$ except that slowly varying functions in (2.1) may be completely general (not only normalized ones). In addition, there holds

Theorem 4 [13]. *Let y_1 be any decreasing solution of (1.1') and y_2 any linearly independent one. Then y_1 and y_2 are rapidly varying if and only if*

$$\lim_{x \to \infty} x \int_x^{\lambda x} f(t)dt = \infty \tag{2.3}$$

for every $\lambda > 1$.

One would prefer to have the same upper limit in integrals in (2.2) and (2.3). This is indeed possible as it is shown in [6]. If fact, the upper limit of the integral in (2.2) can be replaced by λx (but condition (2.2) is easier to verify and thus kept as it stands). The converse is not true since if in (2.3) one replaces λx by ∞, the integral might even diverge. Hence there holds the following

Corollary 1 *All solutions of equation (1.1') are Karamata functions if and only if there exists the finite or infinite limit*

$$\mu := \lim_{x \to \infty} x \int_x^{\lambda x} f(t)dt \tag{2.4}$$

for every $\lambda > 1$.

Since, for the decreasing solution $y_1(x)$ of (1.1') integral $\int^{\infty} y_1^{-2}(t)dt$ diverges, $y_1(x)$ is, by definition, a *principal solution* and there holds the following *trichotomy result*:

Corollary 2 [13]. *All principal solutions of (1.1') are slowly or regularly or rapidly varying functions according as* $\mu = 0$, $\mu \in (0, \infty)$, $\mu = \infty$ *in (2.4).*

2.2. Asymptotic behaviour of solutions.

To determine the asymptotic behaviour for $x \to \infty$, of solutions under consideration of equations (1.1) and (1.1'), due to representation (2.1) and since $L_2(x) \sim \{(1 - 2\alpha_1)L_1(x)\}^{-1}$, one has to do it for $L_1(x)$ only.

We consider separately two cases: $c = 0$ and $c \neq 0$, $-\infty < c < 1/4$. The reason is that in the former case one obtains a more general result than in the later.

For $c = 0$, Theorem 1 guarantees the existence of two linearly independent solutions of the form $y_1(x) = L_1(x)$ and $y_2(x) = xL_2(x)$ for which there holds

Theorem 5 *Let* $F(x) = \int_x^{\infty} f(t)dt$,

$$Z_0(x) = 0,\ Z_n(x) = -\int_x^{\infty} (F(t) - Z_{n-1}(t))dt \text{ for } n = 1, 2, \dots .$$

Suppose that condition (2.2) holds for $c = 0$ *and let* $c(x)$ *be a positive decreasing continuous function such that* $|\phi(x)| \leq c(x)$.

If for some positive integer n

$$\int^{\infty} (c^{n+1}(t)/t)dt < \infty, \tag{2.5}$$

then two linearly independent solutions of (1.1), $y_1(x) = L_1(x)$, $y_2(x) = xL_2(x)$ *where* $L_i(x)$ *are normalized slowly varying functions with* $L_2(x) \sim 1/L_1(x)$, *possess the following asymptotic representation for* $x \to \infty$,

$$y_1(x) \sim \exp\left\{\int_a^x (F(t) - Z_{n-1}(t))dt\right\},$$

$$y_2(x) \sim x \exp\left\{-\int_a^x (F(t) - Z_{n-1}(t))dt\right\}$$

and

$$xy_1'(x)/y_1(x) = o(1), \quad y_2'(x) \sim y(x)/x.$$

Observe that for equation (1.1') the same result holds if F is replaced by $-F$. Recall that here the functions L_i are not normalized in general.

For $n = 1$, Theorem 1 reduces to a result of Hartman and Wintner [9] (which holds also for complex-valued f).

For $-\infty < c < 1/4$, $c \neq 0$, again Theorem 1 gives the existence of two solutions of the form (2.1) for which there holds

Theorem 6 [5]. *Define*

$$g(x) = 2(\alpha_1 + \phi(x))/x, \quad \rho(x) = \exp(\int_1^x g(t)dt),$$

where ϕ is defined in (2.2).

Suppose that condition (2.2) holds for $c \in (-\infty, 1/4)$, $c \neq 0$. If

$$\int^\infty \frac{\phi^2(t)}{t} dt < \infty,$$

then for two linearly independent solutions y_1, y_2 of (1.1) there hold for $x \to \infty$,

$$y_1(x) \sim x^{\alpha_1} \exp\{\int_a^x (\frac{\phi(t)}{t} + 2\alpha_1 \int_t^\infty \frac{\rho(\tau)}{\rho(t)} \frac{\phi(\tau)}{\tau^2} d\tau)dt\},$$

$$y_2(x) \sim (1 - 2\alpha_1)^{-1} x^{\alpha_2} L_1^{-1}(x)$$

and

$$y_1'(x) \sim \alpha_1 y_1(x)/x, \quad y_2'(x) \sim \alpha_2 y_2(x)/x.$$

If integral $\int^\infty \frac{\phi(t)}{t} dt$ converges, Theorem 6 is reduced to a result of Mařik and Ráb, [15], where $L_i(x)$ tend to constants.

By introducing some additional hypotheses one can largely simplify the above asymptotic formula. This is done in

Theorem 7 [5]. *Suppose that condition (2.2) holds for* $c \in (-\infty, 1/4)$, $c \neq 0$. *Denote by* ϕ_0 *a slowly varying function satisfying*

$$\int^{\infty} \frac{\phi_0^2(t)}{t} dt < \infty.$$

If $\phi(x) = O(\phi_0(x))$ *for* $x \to \infty$, *then the two solutions* $y_i(x)$, $i = 1, 2$, *possess for* $x \to \infty$, *the asymptotics*

$$y_1(x) \sim x^{\alpha_1} \exp\{\frac{1}{1-2\alpha_1} \int_a^x \frac{\phi(t)}{t} dt\},$$

$$y_2(x) \sim (1-2\alpha_1)^{-1} x^{\alpha_2} L_1^{-1}(x)$$

and

$$y_1'(x) \sim \alpha_1 y_1(x)/x, \quad y_2'(x) \sim \alpha_2 y_2(x)/x.$$

Observe that for $c = 1/4$, Theorem 2 also gives the asymptotics of solution since the occuring slowly varying functions tend to constants.

To emphasize a possible significance of the results on asymptotics presented here, we recall the G.H. Hardy class of logarithmico-exponential functions defined on some half-axis (a, ∞) by a finite combination of symbols $+$, $-$, $\times$, $:$, $\sqrt[n]{}$, ln, exp, acting on the real variable x and on real constants. More generally one defines Hardy field as a set of germs of real-valued functions defined on (a, ∞) that is closed under differentiation and that form a field under usual addition and multiplication of germs.

The logarithmico-exponential class of functions, or more generally Hardy fields, have been considered as the natural domain of asymptotic analysis where all rules hold without qualifying conditions.

In [8] Hardy expressed his opinion that the law of increase of all functions which appeared in the analysis can be stated in logarithmico-exponential terms. This is even true for the very complicated arithmetic functions occuring in the number theory so that no "genuinely new modes of increase" appear.

But all Hardy functions (or any element of Hardy fields) together with the derivatives are ultimately monotonic, of constant sign and $\lim_{x\to\infty} f(x) \leq \infty$ exists.

But as one can see in Theorem 5, solutions of equation (1.1) may behave as slowly varying functions which may oscillate even infinitely as it is shown by the example following Definition 2. Therefore such solutions may exibit a "genuinely new mode of increase".

3. Examples.

Example 1.

$$y'' + \frac{a + b\sin x}{x^\alpha ln^\beta x} y = 0$$

$\alpha \geq 1$, β, a and b are real numbers. Obviously $f(x)$ will be of arbitrary sign if $|a| < b$.

We shall treat two cases $a \neq 0$, $a = 0$ while b is arbitrary real.

1. **Case $\mathbf{a \neq 0}$.**

If $\alpha > 2$, then for any b, β condition (2.2) with $c = 0$ is fulfilled and by Theorem 1, there exist two solutions of the form $y_1(x) = L_1(x)$, $y_2(x) = xL_2(x)$. Also we choose for some $M > 0$,

$$c(x) = Mx^{2-\alpha} ln^{-\beta} x$$

so that condition (2.5) is fulfilled with $n = 1$ and an application of Theorem 5 gives $y_1(x) \to const.$, as $x \to \infty$.

If $\alpha = 2$ and $\beta > 0$, one can take

$$c(x) = M ln^{-\beta} x$$

so that condition (2.5) gives

$$\int^{\infty} (c^{(n+1)}(t)/t) dt \leq M \int^{\infty} (t^{-1} ln^{-(n+1)\beta}) dt.$$

Consequently, for $\beta > 1/2$ one can take in the above integral $n = 1$ to obtain for $x \to \infty$:

$$\begin{array}{lll} \text{For} & \beta > 1, & y_1(x) \to const., \\ \text{for} & \beta = 1, & y_1(x) \sim (lnx)^\alpha, \\ \text{for} & 1/2 < \beta < 1, & y_1(x) \sim \exp(\frac{a}{1-\beta} ln^{1-\beta} x). \end{array}$$

For $1/3 < \beta \leq 1/2$, one takes $n = 2$ and obtains:

$$\text{For} \quad \beta = 1/2, \qquad y_1(x) \sim \exp(2a(lnx)^{1/2})(lnx)^{a^2},$$

$$\text{for} \quad 1/3 < \beta < 1/2, \quad y_1(x) \sim \exp(\tfrac{a}{1-\beta} ln^{1-\beta} x) \exp(\tfrac{a^2}{1-2\beta} ln^{1-2\beta} x),$$

etc; the asymptotics can be obtained for any positive β. All these solutions are slowly varying and the behaviour of the second solution follows from $y_2(x) \sim x/y_1(x)$.

If $\alpha = 2$ and $\beta = 0$, by Theorem 1, for $a \in (-\infty, 1/4)$, there exist two linearly independent solutions of the form

$$y_1(x) = x^{(1-\sqrt{1-4a})/2} L_1(x), \; y_2(x) = x^{(1+\sqrt{1-4a})/2} L_2(x), \tag{2.6}$$

where $L_2(x) \sim (\sqrt{1-4a} L_1(x))^{-1}$. Theorem 6 gives that functions $L_1(x)$ (and so $L_2(x)$) tend to some constants and (2.6) give the asymptotic behaviour.

2. **Case $\mathbf{a = 0}$.**

If $\alpha > 1$ condition (2.2) with $c = 0$ holds for any β, so that $y_1(x)$ is slowly varying by Theorem 1. By Theorem 5, with $n = 1$, y_1 tends to a nonzero constant as $x \to \infty$.

If $\alpha = 1$ and $\beta > 1/2$ the situation is the same.

For $\beta = 1/2$, take $n = 2$ and then $y_1(x) \sim (lnx)^{b^2/2}$, for $1/3 < \beta < 1/2$, take again $n = 2$ and get

$$y_1(x) \sim \exp(\frac{b^2}{2(1-2\beta)}(lnx)^{1-2\beta}),$$

etc.

By diminishing β, the number of factors will increase as in the case $a \neq 0$. We emphasize that such is the case in the extension of the classical Liouville Green approximation as obtained in [10].

Example 2. Take

$$\phi(x) = \frac{(lnx)^{1/5} \sin(lnx)^{1/5} - \cos(lnx)^{1/5}}{5(lnx)^{4/5}}$$

and for $-\infty < c < 1/4$, $c \neq 0$

$$f(x) = cx^{-2} + (\phi(x)/x)'.$$

Since $(\phi(x)/x)' = o(x^{-2})$ and $|\phi(x)| \leq m(lnx)^{-3/5}$, Theorem 7 gives

$$y_1(x) \sim x^{-\alpha_1} \exp\{\frac{1}{1-2\alpha_1}(lnx)^{1/5}\cos(lnx)^{1/5}\}.$$

Note that the slowly varying function multiplying $x^{-\alpha_1}$ oscillates infinitely between 0 and ∞.

Example 3. For the Legendre equation of order one,

$$y'' + \frac{2x}{x^2-1}y' - \frac{2}{x^2-1}y = 0.$$

Theorem 3 gives for $x \to \infty$

$$y_1(x) = x^{-2}L_1(x),\ y_2(x) = xL_2(x),\ L_2(x) \sim (3L_1(x))^{-1}.$$

Indeed, the equation has two linearly independent solutions $y_1(x) = 1 - \frac{x}{2}ln(\frac{x+1}{1-x})$, $y_2(x) = x$.

Example 4.

$$y'' + (\frac{x^{-2}}{4} + e^{-x})y = 0.$$

Here

$$\lim_{x\to\infty} x\int_x^\infty f(t)dt = \frac{1}{4}.$$

This is the borderline case so that Theorem 2 gives $y_1(x) \sim x^{1/2}$, $y_2(x) \sim x^{1/2}lnx$, which are obviously nonoscillatory. Nonoscillation (but not the behaviour) can be also obtained by Potter's criterion [18].

Remark. (added in proof) Recently (April 2000) the book of the author *Regular variation and differential equations*, Springer-Verlag, Berlin, Heidelberg, New York, Lecture Notes in Mathematics 1726 is published. Besides the results presented here, in the book there are some additional ones on the subject, and the complete proofs of all the statements as well.

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