

THE ABSTRACT RENEWAL EQUATION

ŠTEFAN SCHWABIK

Praha

Abstract. The abstract Perron-Stieltjes integral defined in the Kurzweil-Henstock sense is used for introducing Stieltjes convolutions. The corresponding facts on integration are given in [6], [7] and [8].

The approach is used for obtaining the basic existence result for the abstract renewal equation which was studied e. g. by Diekmann, Gyllenberg and Thieme in [1] and [2].

For a given Banach space X let $L(X)$ be the Banach space of all bounded linear operators $A : X \rightarrow X$ with the uniform operator topology.

For $B : L(X) \times X \rightarrow X$ given by $B(A, x) = Ax \in X$ for $A \in L(X)$ and $x \in X$, we obtain the bilinear triple $(L(X), X, X)$ because we have

$$\|B(A, x)\|_X \leq \|A\|_{L(X)} \|x\|_X$$

for the bilinear form B . Similarly, if we define the bilinear form $B^* : L(X) \times L(X) \rightarrow L(X)$ by $B^*(A, C) = AC \in L(X)$ for $A, C \in L(X)$ where AC is the composition of the linear operators A and C we get the bilinear triple $(L(X), L(X), L(X))$ because

$$\|B^*(A, C)\|_{L(X)} \leq \|AC\|_{L(X)} \leq \|A\|_{L(X)} \|C\|_{L(X)}.$$

Assume that the interval $[0, b] \subset \mathbb{R}$ is bounded.

Given $A : [0, b] \rightarrow L(X)$, the function A is of *bounded variation on* $[0, b]$ if

$$\text{var}_{[0, b]}(A) = \sup \left\{ \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} \right\} < \infty,$$

where the supremum is taken over all finite partitions

$$D : 0 = \alpha_0 < \alpha_1 < \dots < \alpha_{k-1} < \alpha_k = b$$

of the interval $[0, b]$. The set of all functions $A : [0, b] \rightarrow L(X)$ with $\text{var}_{[0, b]}(A) < \infty$ will be denoted by $BV([0, b]; L(X))$.

For $A : [0, b] \rightarrow L(X)$ and a partition D of the interval $[0, b]$ define

$$V_0^b(A, D) = \sup \left\{ \left\| \sum_{j=1}^k [A(\alpha_j) - A(\alpha_{j-1})] x_j \right\|_X \right\},$$

The paper is in final form and no version of it will be submitted for publication elsewhere.

This work was supported by the grant 201/97/0218 of the Grant Agency of the Czech Republic.

where the supremum is taken over all possible choices of $x_j \in X, j = 1, \dots, k$ with $\|x_j\|_X \leq 1$.

Let us set

$$s \text{ var}_{[0,b]}(A) = \sup V_0^b(A, D)$$

where the supremum is taken over all finite partitions D of the interval $[0, b]$.

An operator valued function $A : [0, b] \rightarrow L(X)$ with $s \text{ var}_{[0,b]}(A) < \infty$ is called a *function of bounded semi-variation on $[0, b]$* (cf. [4]).

We denote by $BSV([0, b]; L(X))$ the set of all functions $A : [0, b] \rightarrow L(X)$ with

$$s \text{ var}_{[0,b]}(A) < \infty.$$

Assume that $\eta \geq 0$ is given and define

$$\text{var}_{[0,b]}^{(\eta)}(A) = \sup \left\{ \sum_{j=1}^k e^{-\eta\alpha_{j-1}} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} \right\}$$

where the supremum is taken over all finite partitions D of the interval $[0, b]$.

Similarly define

$$V_0^b(\eta, A, D) = \sup \left\{ \left\| \sum_{j=1}^k [A(\alpha_j) - A(\alpha_{j-1})] x_j e^{-\eta\alpha_{j-1}} \right\|_X \right\},$$

where the supremum is taken over all possible choices of $x_j \in X, j = 1, \dots, k$ with $\|x_j\|_X \leq 1$ and set

$$s \text{ var}_{[0,b]}^{(\eta)}(A) = \sup V_0^b(\eta, A, D)$$

where the supremum is taken over all finite partitions D of the interval $[0, b]$.

Since for every $j = 1, \dots, k$ we have

$$e^{-\eta b} \leq e^{-\eta\alpha_{j-1}} \leq 1$$

we get

$$(1) \quad e^{-\eta b} \text{ var}_{[0,b]}(A) \leq \text{var}_{[0,b]}^{(\eta)}(A) \leq \text{var}_{[0,b]}(A)$$

and also

$$e^{-\eta b} V_0^b(A, D) \leq V_0^b(\eta, A, D) \leq V_0^b(A, D).$$

The last inequalities lead immediately to

$$(2) \quad e^{-\eta b} s \text{ var}_{[0,b]}(A) \leq s \text{ var}_{[0,b]}^{(\eta)}(A) \leq s \text{ var}_{[0,b]}(A).$$

Let us mention that

$$\text{var}_{[0,b]}^{(0)}(A) = \text{var}_{[0,b]}(A) \text{ and } s \text{ var}_{[0,b]}^{(0)}(A) = s \text{ var}_{[0,b]}(A).$$

It is well known that $BV([0, b]; L(X))$ with the norm

$$\|A\|_{BV} = \|A(0)\|_{L(X)} + \text{var}_{[0,b]}(A)$$

is a Banach space and in [8] it was shown that with the norm

$$\|A\|_{SV} = \|A(0)\|_{L(X)} + s \text{ var}_{[0,b]}(A)$$

the space $BSV([0, b]; L(X))$ is also a Banach space.

Taking into account the inequalities (1) and (2) we get the following statement.

1. Proposition. For every $\eta \geq 0$ the space $BV([0, b]; L(X))$ with the norm

$$\|A\|_{BV,\eta} = \|A(0)\|_{L(X)} + \text{var}_{[0,b]}^{(\eta)}(A)$$

is a Banach space and the space $BSV([0, b]; L(X))$ with the norm

$$\|A\|_{SV,\eta} = \|A(0)\|_{L(X)} + s \text{var}_{[0,b]}^{(\eta)}(A)$$

is also a Banach space.

The norms $\|A\|_{BV,\eta}$ and $\|A\|_{BV}$ are equivalent on $BV([0, b]; L(X))$ and the norms $\|A\|_{SV,\eta}$ and $\|A\|_{SV}$ are equivalent on $BSV([0, b]; L(X))$.

Given $x : [0, b] \rightarrow X$, the function x is called *regulated on* $[0, b]$ if it has one-sided limits at every point of $[0, b]$, i.e. if for every $s \in [0, b)$ there is a value $x(s+) \in X$ such that

$$\lim_{t \rightarrow s+} \|x(t) - x(s+)\|_X = 0$$

and if for every $s \in (0, b]$ there is a value $x(s-) \in X$ such that

$$\lim_{t \rightarrow s-} \|x(t) - x(s-)\|_X = 0.$$

The set of all regulated functions $x : [0, b] \rightarrow X$ will be denoted by $G([0, b]; X)$.

The space $G([0, b]; X)$ endowed with the norm

$$\|x\|_{G([0,b];X)} = \sup_{t \in [0,b]} \|x(t)\|_X, \quad x \in G([0, b]; X)$$

is known to be a Banach space (see [4, Theorem 3.6]).

It is clear that the space $C([0, b]; X)$ of continuous functions $x : [0, b] \rightarrow X$ is a closed subspace of $G([0, b]; X)$, i.e.

$$C([0, b]; X) \subset G([0, b]; X).$$

We are using the concept of abstract Perron-Stieltjes integral based on the Kurzweil-Henstock definition presented via integral sums (for more detail see e.g. [5], [6], [7]).

A finite system of points

$$\{\alpha_0, \tau_1, \alpha_1, \tau_2, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

such that

$$0 = \alpha_0 < \alpha_1 < \dots < \alpha_{k-1} < \alpha_k = b$$

and

$$\tau_j \in [\alpha_{j-1}, \alpha_j] \quad \text{for } j = 1, \dots, k$$

is called a *P-partition* of the interval $[0, b]$.

Any positive function $\delta : [0, b] \rightarrow (0, \infty)$ is called a *gauge on* $[0, b]$.

For a given gauge δ on $[0, b]$ a *P-partition* $\{\alpha_0, \tau_1, \alpha_1, \tau_2, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$ of $[0, b]$ is called *δ -fine* if

$$[\alpha_{j-1}, \alpha_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)) \quad \text{for } j = 1, \dots, k.$$

Definition. Assume that functions $A, C : [0, b] \rightarrow L(X)$ and $x : [0, b] \rightarrow X$ are given.

We say that the Stieltjes integral $\int_0^b d[A(s)]x(s)$ exists if there is an element $J \in X$ such that for every $\varepsilon > 0$ there is a gauge δ on $[0, b]$ such that for

$$S(dA, x, D) = \sum_{j=1}^k [A(\alpha_j) - A(\alpha_{j-1})]x(\tau_j)$$

we have

$$\|S(dA, x, D) - J\|_X < \varepsilon$$

provided D is a δ -fine P -partition of $[0, b]$. We denote $J = \int_0^b d[A(s)]x(s)$.

Analogously we say that the Stieltjes integral $\int_0^b d[A(s)]C(s)$ exists if there is an element $J \in L(X)$ such that for every $\varepsilon > 0$ there is a gauge δ on $[0, b]$ such that for

$$S(dA, C, D) = \sum_{j=1}^k [A(\alpha_j) - A(\alpha_{j-1})]C(\tau_j)$$

we have

$$\|S(dA, C, D) - J\|_{L(X)} < \varepsilon$$

provided D is a δ -fine P -partition of $[0, b]$.

Similarly we can define the Stieltjes integral $\int_0^b A(s)d[C(s)]$ using Stieltjes integral sums of the form

$$S(A, dC, D) = \sum_{j=1}^k A(\tau_j)[C(\alpha_j) - C(\alpha_{j-1})].$$

Assume that $U, V : [0, \infty) \rightarrow L(X)$ and $x : [0, \infty) \rightarrow X$ are given and define the convolutions

$$(U * x)(t) = \int_0^t d[U(s)]x(t-s)$$

and

$$(U * V)(t) = \int_0^t d[U(s)]V(t-s)$$

for $t \in [0, \infty)$.

Let us denote by $BSV_{loc}([0, \infty), L(X))$ the set of all $U : [0, \infty) \rightarrow L(X)$ for which $U \in BSV([0, b], L(X))$ for every $b > 0$.

In [8] it was shown that if $U, V \in G([0, \infty), L(X)) \cap (\mathcal{B})BV_{loc}([0, \infty), L(X))$ and $x \in G([0, \infty), X)$ then the convolutions $(U * x)(t)$ and $(U * V)(t)$ are well defined for every $t \in [0, \infty)$ when the abstract Perron-Stieltjes integral is used.

It was also shown in [8] that

$$(3) \quad \|(U * V)(t)\|_{L(X)} \leq \|U\|_{SV} \cdot \|V\|_{SV}$$

holds for every $t \geq 0$.

2. Lemma. Assume that

$$U \in G([0, \infty), L(X)) \cap BSV_{loc}([0, \infty), L(X)), \quad f \in G([0, \infty), X)$$

and that $\eta \geq 0$ is given.

Then the integral $\int_0^b d[U(s)]e^{-\eta s} f(s) \in X$ exists for every $b > 0$ and

$$(4) \quad \left\| \int_0^b d[U(s)]e^{-\eta s} f(s) \right\|_X \leq s \operatorname{var}_{[0,b]}^{(\eta)}(U) \cdot \sup_{s \in [0,b]} \|f(s)\|_X$$

holds.

Proof. The existence of the integral $\int_0^b d[U(s)]e^{-\eta s} f(s)$ is clear because the function $e^{-\eta s} f(s)$ is regulated on $[0, \infty)$ (c.f. [6, Proposition 15]).

Assume that $b > 0$ is fixed. By the existence of the integral, for any $\varepsilon > 0$ there is a gauge δ on $[0, b]$ such that for every δ -fine P -partition

$$D = \{0 = \alpha_0, \tau_1, \alpha_1, \tau_2, \dots, \alpha_{k-1}, \tau_k, \alpha_k = b\}$$

of $[0, b]$ the inequality

$$\left\| \int_0^b d[U(s)]e^{-\eta s} f(s) - \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta \tau_j} f(\tau_j) \right\|_X < \varepsilon$$

holds. Hence

$$(5) \quad \left\| \int_0^b d[U(s)]e^{-\eta s} f(s) \right\|_X < \varepsilon + \left\| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta \tau_j} f(\tau_j) \right\|_X.$$

Let us choose a fixed δ -fine P -partition D of $[0, b]$ for which $\alpha_{j-1} < \tau_j$ for every $j = 1, \dots, k$. Then

$$\begin{aligned} & \left\| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta \tau_j} f(\tau_j) \right\|_X = \\ & = \left\| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta \alpha_{j-1}} e^{-\eta(\tau_j - \alpha_{j-1})} f(\tau_j) \right\|_X = \\ & = \left\| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta \alpha_{j-1}} \frac{e^{-\eta(\tau_j - \alpha_{j-1})} f(\tau_j)}{\|f(\tau_j)\|_X} \|f(\tau_j)\|_X \right\|_X \end{aligned}$$

and we have

$$\left\| \frac{e^{-\eta(\tau_j - \alpha_{j-1})} f(\tau_j)}{\|f(\tau_j)\|_X} \right\|_X \leq 1$$

for $j = 1, \dots, k$.

Hence

$$\begin{aligned} & \left\| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta \alpha_{j-1}} \frac{e^{-\eta(\tau_j - \alpha_{j-1})} f(\tau_j)}{\|f(\tau_j)\|_X} \|f(\tau_j)\|_X \right\|_X \leq \\ & \leq \sup_{j=1, \dots, k} \|f(\tau_j)\|_X \cdot \left\| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta \alpha_{j-1}} \frac{e^{-\eta(\tau_j - \alpha_{j-1})} f(\tau_j)}{\|f(\tau_j)\|_X} \right\|_X \leq \\ & \leq \sup_{s \in [0,b]} \|f(s)\|_X \cdot s \operatorname{var}_{[0,b]}^{(\eta)}(U) \end{aligned}$$

and this together with (5) gives the result.

3. Proposition. Assume that $U, V \in G([0, \infty), L(X)) \cap BSV_{loc}([0, \infty), L(X))$ and that $U(0) = V(0) = 0$.

Then the convolution

$$(U * V)(t) = \int_0^t d[U(s)]V(t-s) \in L(X)$$

is well defined for every $t \in [0, \infty)$, and for every $b > 0$, $\eta \geq 0$ the inequality

$$(6) \quad s \operatorname{var}_{[0,b]}^{(\eta)}(U * V) \leq s \operatorname{var}_{[0,b]}^{(\eta)}(U) \cdot s \operatorname{var}_{[0,b]}^{(\eta)}(V)$$

holds.

Proof.

Define

$$\tilde{V}(\sigma) = V(\sigma) \quad \text{for } \sigma \geq 0$$

and

$$\tilde{V}(\sigma) = 0 \quad \text{for } \sigma < 0.$$

Assume that $b \geq 0$ and let $0 = \alpha_0 < \alpha_1 < \dots < \alpha_k = b$ be an arbitrary partition of $[0, b]$.

Using the definition of \tilde{V} we have for every $\alpha \in [0, b]$ the equality

$$\int_0^\alpha d[U(s)]V(\alpha-s) = \int_0^\alpha d[U(s)]\tilde{V}(\alpha-s)$$

and therefore we obtain for any choice of $x_j \in X$, $\|x_j\|_X \leq 1$, $j = 1, \dots, k$ the equalities

$$\begin{aligned} & \left\| \sum_{j=1}^k [(U * V)(\alpha_j) - (U * V)(\alpha_{j-1})] x_j e^{-\eta \alpha_{j-1}} \right\|_X = \\ & = \left\| \sum_{j=1}^k \left[\int_0^{\alpha_j} d[U(s)]V(\alpha_j-s) - \int_0^{\alpha_{j-1}} d[U(s)]V(\alpha_{j-1}-s) \right] x_j e^{-\eta \alpha_{j-1}} \right\|_X = \\ & = \left\| \sum_{j=1}^k \int_0^b d[U(s)] [\tilde{V}(\alpha_j-s) - \tilde{V}(\alpha_{j-1}-s)] x_j e^{-\eta \alpha_{j-1}} \right\|_X = \\ (7) \quad & = \left\| \int_0^b d[U(s)] e^{-\eta s} \sum_{j=1}^k [\tilde{V}(\alpha_j-s) - \tilde{V}(\alpha_{j-1}-s)] x_j e^{-\eta(\alpha_{j-1}-s)} \right\|_X. \end{aligned}$$

The function

$$s \mapsto \sum_{j=1}^k [\tilde{V}(\alpha_j-s) - \tilde{V}(\alpha_{j-1}-s)] x_j e^{-\eta(\alpha_{j-1}-s)} \in X$$

is evidently regulated on $[0, b]$ because $V \in G([0, b], L(X))$ and therefore by Lemma 2 we obtain

$$\begin{aligned} & \left\| \int_0^b d[U(s)] e^{-\eta s} \sum_{j=1}^k [\tilde{V}(\alpha_j - s) - \tilde{V}(\alpha_{j-1} - s)] x_j e^{-\eta(\alpha_{j-1} - s)} \right\|_X \leq \\ & \leq s \operatorname{var}_{[0, b]}^{(\eta)}(U) \cdot \sup_{s \in [0, b]} \left\| \sum_{j=1}^k [\tilde{V}(\alpha_j - s) - \tilde{V}(\alpha_{j-1} - s)] x_j e^{-\eta(\alpha_{j-1} - s)} \right\|_X. \end{aligned}$$

On the other hand, for every $s \in [0, b]$ we have

$$\left\| \sum_{j=1}^k [\tilde{V}(\alpha_j - s) - \tilde{V}(\alpha_{j-1} - s)] x_j e^{-\eta(\alpha_{j-1} - s)} \right\|_X \leq s \operatorname{var}_{[0, b]}^{(\eta)}(V)$$

and this gives

$$\begin{aligned} & \left\| \sum_{j=1}^k [(U * V)(\alpha_j) - (U * V)(\alpha_{j-1})] x_j e^{-\eta \alpha_{j-1}} \right\|_X \leq \\ & \leq s \operatorname{var}_{[0, b]}^{(\eta)}(U) \cdot s \operatorname{var}_{[0, b]}^{(\eta)}(V) \end{aligned}$$

and by the definition also

$$s \operatorname{var}_{[0, b]}^{(\eta)}(U * V) \leq s \operatorname{var}_{[0, b]}^{(\eta)}(U) \cdot s \operatorname{var}_{[0, b]}^{(\eta)}(V).$$

This inequality yields by (2) also that

$$s \operatorname{var}_{[0, b]}(U * V) < \infty,$$

i.e. that

$$(8) \quad U * V \in BSV_{loc}([0, \infty), L(X))$$

because $b \geq 0$ can be taken arbitrarily.

Analogously it can be proved that the following statement holds.

4. Proposition. *Assume that $U, V \in BV_{loc}([0, \infty), L(X))$ and that $U(0) = V(0) = 0$.* ■

Then the convolution

$$(U * V)(t) = \int_0^t d[U(s)] V(t - s) \in L(X)$$

is well defined for every $t \in [0, \infty)$ and for every $b > 0, \eta \geq 0$ the inequality

$$(9) \quad \operatorname{var}_{[0, b]}^{(\eta)}(U * V) \leq \operatorname{var}_{[0, b]}^{(\eta)}(U) \cdot \operatorname{var}_{[0, b]}^{(\eta)}(V)$$

holds.

In [8] the following result has been proved.

5. Proposition. For every $b > 0$ the set of all $U : [0, b] \rightarrow L(X)$ with $U \in C([0, b], L(X)) \cap BSV([0, b], L(X))$ and $U(0) = 0$ is a Banach algebra with the Stieltjes convolution $U * V$ as multiplication and $s \text{var}_{[0, b]}(U)$ as the norm.

See [8 ,Theorem 15].

6. Remark. Unfortunately a statement of the form:

For every $b > 0$ the set of all $U : [0, b] \rightarrow L(X)$ with $U \in BV([0, b], L(X))$ and $U(0) = 0$ is a Banach algebra with the Stieltjes convolution

$$(U * V)(t) = \int_0^t d[U(s)]V(t - s)$$

as multiplication and $\text{var}_{[0, b]}(U)$ as the norm.

does not hold because in this case the multiplication given by the convolution is not associative.

It was also shown [8 ,Proposition 12 and 13] that the following two statements hold.

7. Proposition. If $U, V \in BV_{loc}([0, \infty), L(X))$ and $U(0) = V(0) = 0$ then $U * V \in BV_{loc}([0, \infty), L(X))$.

8. Proposition. If $U, V \in C([0, \infty), L(X)) \cap BSV_{loc}([0, \infty), L(X))$ and $U(0) = V(0) = 0$ then $U * V \in C([0, \infty), L(X)) \cap BSV_{loc}([0, \infty), L(X))$.

9. Lemma. Assume that $A \in BSV([0, b], L(X))$ for some $b > 0$. Then for every $\eta \geq 0$ and $c \in (0, b]$ we have

$$(10) \quad s \text{var}_{[0, b]}^{(\eta)}(A) \leq s \text{var}_{[0, c]}^{(\eta)}(A) + e^{-\eta c} s \text{var}_{[c, b]}^{(\eta)}(A).$$

Proof. Assume that D is a partition of $[0, b]$ given by the points

$$0 = \alpha_0 < \alpha_1 < \dots < \alpha_k = b$$

and that $x_j \in X$ with $\|x_j\|_X \leq 1$ for $j = 1, \dots, k$. Then there is an index $l = 1, \dots, k$ such that $c \in (\alpha_{l-1}, \alpha_l]$ and

$$\begin{aligned} & \sum_{j=1}^k [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} = \\ & = \sum_{j=1}^{l-1} [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} + [A(\alpha_l) - A(\alpha_{l-1})]x_l e^{-\eta\alpha_{l-1}} + \\ & \quad + \sum_{j=l+1}^k [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}}. \end{aligned}$$

Taking into account that

$$[A(\alpha_l) - A(\alpha_{l-1})]x_l e^{-\eta\alpha_{l-1}} =$$

$$= [A(\alpha_l) - A(c)]x_l e^{-\eta\alpha_{l-1}} + [A(c) - A(\alpha_{l-1})]x_l e^{-\eta\alpha_{l-1}}$$

we obtain

$$\begin{aligned} & \left\| \sum_{j=1}^k [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} \right\|_X = \\ & = \left\| \sum_{j=1}^{l-1} [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} + [A(c) - A(\alpha_{l-1})]x_l e^{-\eta\alpha_{l-1}} + \right. \\ & \left. + [A(\alpha_l) - A(c)]x_l e^{-\eta\alpha_{l-1}} + \sum_{j=l+1}^k [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} \right\|_X \leq \\ & \leq \left\| \sum_{j=1}^{l-1} [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} + [A(c) - A(\alpha_{l-1})]x_l e^{-\eta\alpha_{l-1}} \right\|_X + \\ & \quad + \left\| [A(\alpha_l) - A(c)]x_l e^{-\eta\alpha_{l-1}} + \sum_{j=l+1}^k [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} \right\|_X. \end{aligned}$$

For the first term on the right hand side of this inequality we have evidently

$$\begin{aligned} & \left\| \sum_{j=1}^{l-1} [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} + [A(c) - A(\alpha_{l-1})]x_l e^{-\eta\alpha_{l-1}} \right\|_X \leq \\ & \leq s \operatorname{var}_{[0,c]}^{(\eta)}(A) \end{aligned}$$

and for the second

$$\begin{aligned} & \left\| [A(\alpha_l) - A(c)]x_l e^{-\eta\alpha_{l-1}} + \sum_{j=l+1}^k [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} \right\|_X = \\ & = \left\| [A(\alpha_l) - A(c)]x_l e^{-\eta\alpha_{l-1}} + e^{-\eta c} \sum_{j=l+1}^k [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta(\alpha_{j-1}-c)} \right\|_X \leq \\ & \leq e^{-\eta c} V_c^b(\eta, A, D_+) \leq e^{-\eta c} \operatorname{var}_{[c,b]}^{(\eta)}(A) \end{aligned}$$

(D_+ is the partition of $[c, b]$ given by the points $c \leq \alpha_l < \dots < \alpha_k = b$). Hence

$$\begin{aligned} & \left\| \sum_{j=1}^k [A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} \right\|_X \leq \\ & \leq s \operatorname{var}_{[0,c]}^{(\eta)}(A) + e^{-\eta c} \operatorname{var}_{[c,b]}^{(\eta)}(A) \end{aligned}$$

and the lemma is proved.

Similarly it can be shown that the following statement is valid.

10. Lemma. Assume that $A \in BV([0, b], L(X))$ for some $b > 0$. Then for every $\eta \geq 0$ and $c \in (0, b]$ we have

$$(11) \quad \text{var}_{[0, b]}^{(\eta)}(A) \leq \text{var}_{[0, c]}^{(\eta)}(A) + e^{-\eta c} \text{var}_{[c, b]}^{(\eta)}(A).$$

11. Proposition. If $A \in C([0, b], L(X)) \cap BSV([0, b], L(X))$, $A(0) = 0$ and if there is a $c \in (0, b]$ such that

$$(12) \quad s \text{var}_{[0, c]}(A) < 1,$$

then there exists a unique $R \in C([0, b], L(X)) \cap BSV([0, b], L(X))$ with $R(0) = 0$ such that

$$(13) \quad R(t) - \int_0^t d[A(s)]R(t-s) = A(t), \quad t \in [0, b]$$

and

$$(14) \quad R(t) - \int_0^t d[R(s)]A(t-s) = A(t), \quad t \in [0, b].$$

Proof. By Lemma 9, (2) and (12) we have

$$\begin{aligned} s \text{var}_{[0, b]}^{(\eta)}(A) &\leq s \text{var}_{[0, c]}^{(\eta)}(A) + e^{-\eta c} s \text{var}_{[c, b]}^{(\eta)}(A) \leq \\ &\leq s \text{var}_{[0, c]}(A) + e^{-\eta c} s \text{var}_{[c, b]}(A) \end{aligned}$$

and this yields that taking $\eta > 0$ sufficiently large we get

$$(15) \quad s \text{var}_{[0, b]}^{(\eta)}(A) < 1.$$

Let us now define $A_0(t) = A(t)$ and $A_{n+1}(t) = (A * A_n)(t)$, $t \in [0, b]$ and put

$$(16) \quad R(t) = \sum_{n=0}^{\infty} A_n(t).$$

By (6) from Proposition 3 we get the inequalities

$$s \text{var}_{[0, b]}^{(\eta)}(A_n) \leq (s \text{var}_{[0, b]}^{(\eta)}(A))^n, \quad n \in \mathbb{N}.$$

Since (15) holds, this implies the convergence of the series (16) in $BSV([0, b], L(X))$ and by Proposition 8 also the continuity of its sum $R(t)$, i. e. $R \in C([0, b], L(X)) \cap BSV([0, b], L(X))$ and clearly also $R(0) = 0$.

By the definitions we have

$$\left(\left(\sum_{n=0}^N A_n \right) * A \right)(t) = \left(A * \left(\sum_{n=0}^N A_n \right) \right)(t) = \sum_{n=1}^{N+1} A_n(t) = \sum_{n=0}^{N+1} A_n(t) - A(t)$$

for every $N \in \mathbb{N}$ and passing to the limit for $N \rightarrow \infty$ we obtain (13) and (14).

Concerning the uniqueness let us assume that

$$Q \in C([0, b], L(X)) \cap BSV([0, b], L(X))$$

also satisfies (13) and (14). Then

$$Q - A * Q = A \quad \text{and} \quad R - R * A = A.$$

Using the associativity of convolution products we get

$$\begin{aligned} R &= A + R * A = A + R * (Q - A * Q) = A + R * Q - R * A * Q = \\ &= A + (R - R * A) * Q = A + A * Q = Q \end{aligned}$$

and the unicity is proved.

12. Corollary. Assume that $A : [0, \infty) \rightarrow L(X)$, $A(0) = 0$. If

$$A \in C([0, \infty), L(X)) \cap BSV_{loc}([0, \infty), L(X))$$

and if there is a $c \in (0, b]$ such that

$$s \operatorname{var}_{[0, c]}(A) < 1$$

then there exists a unique $R : [0, \infty) \rightarrow L(X)$,

$$R \in C([0, \infty), L(X)) \cap BSV_{loc}([0, \infty), L(X))$$

with $R(0) = 0$ such that for every $b > 0$ (13) and (14) hold.

$R \in C([0, \infty), L(X)) \cap BSV_{loc}([0, \infty), L(X))$ given in Corollary 12 is called the *resolvent* of $A \in C([0, \infty), L(X)) \cap BSV_{loc}([0, \infty), L(X))$.

13. Theorem. Assume that $A : [0, \infty) \rightarrow L(X)$, $A(0) = 0$, $A \in C([0, \infty), L(X)) \cap BSV_{loc}([0, \infty), L(X))$ and that there is a $c \in (0, b]$ such that

$$s \operatorname{var}_{[0, c]}(A) < 1.$$

Then for every $F \in G([0, \infty), L(X))$ and $f \in G([0, \infty), X)$ there exist unique solutions $X : [0, \infty) \rightarrow L(X)$ and $x : [0, \infty) \rightarrow X$ for the abstract renewal equations

$$(17) \quad X(t) = F(t) + \int_0^t d[A(s)]X(t-s)$$

and

$$(18) \quad x(t) = f(t) + \int_0^t d[A(s)]x(t-s),$$

respectively, and the relations

$$(19) \quad X(t) = F(t) + \int_0^t d[R(s)]F(t-s),$$

$$(20) \quad x(t) = f(t) + \int_0^t d[R(s)]f(t-s)$$

hold for $t > 0$ where R is the resolvent of A .

Proof. The expression on the right hand side of (19) is well defined and it reads $X(t) = F(t) + (R * F)(t)$.

Hence using (13) we obtain

$$A * X(t) = A * F(t) + (A * (R * F))(t) = ((A + A * R) * F)(t) = (R * F)(t) = X(t) - F(t)$$

and this yields that by (19) a solution of (17) is given.

The analogous result for (18) can be shown similarly.

For renewal equations see also the excellent book [3].

The author expresses his thanks to the referee for pointing out that the statement given in Remark 6 is not valid.

REFERENCES

- [1] O. Diekmann, M. Gyllenberg, H. R. Thieme, *Perturbing semigroups by solving Stieltjes renewal equations*, Differential Integral Equations **6** (1993), 155–181.
- [2] O. Diekmann, M. Gyllenberg, H. R. Thieme, *Perturbing evolutionary systems by step responses on cumulative outputs*, Differential Integral Equations **8** (1995), 1205–1244.
- [3] G. Gripenberg, S.-O. Londen, O. Staffans, *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.
- [4] Ch. S. Hⁿig, *Volterra – Stieltjes Integral Equations*, North – Holland Publ. Comp., Amsterdam, 1975.
- [5] J. Kurzweil, *Nichtabsolut konvergente Integrale*, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1980.
- [6] Š. Schwabik, *Abstract Perron – Stieltjes integral*, Mathematica Bohemica **121** (1996), 425–447.
- [7] Š. Schwabik, *A note on integration by parts for abstract Perron - Stieltjes integrals*, Mathematica Bohemica (to appear).
- [8] Š. Schwabik, *Operator-valued functions of bounded semivariation and convolutions*, Mathematica Bohemica (to appear).

Z itna 25, 115 67 Praha 1, C zech R epublic
E-mail address: schwabik@math.cas.cz