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ON THE NON–EXPONENTIAL DECAY TO EQUILIBRIUM OF SOLUTIONS OF NONLINEAR SCALAR VOLTERRA INTEGRO–DIFFERENTIAL EQUATIONS

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Abstract. We study the rate of decay of solutions of the scalar nonlinear Volterra equation

$$x'(t) = -f(x(t)) + \int_0^t k(t-s)g(x(s))\,ds, \quad x(0) = x_0$$

which satisfy $x(t) \to 0$ as $t \to \infty$. We suppose that $xg(x) > 0$ for all $x \neq 0$, and that f and g are continuous, continuously differentiable in some interval $(-\delta_1, \delta_1)$ and $f(0) = 0$, $g(0) = 0$. Also, k is a continuous, positive, and integrable function, which is assumed to be subexponential in the sense that $k(t-s)/k(t) \to 1$ as $t \to \infty$ uniformly for s in compact intervals. The principal result of the paper asserts that $x(t)$ cannot converge to 0 as $t \to \infty$ faster than $k(t)$.

1. Introduction

In this note, we consider the initial-value problem

$$x'(t) = -f(x(t)) + \int_0^t k(t-s)g(x(s))\,ds, \quad t > 0, \tag{1}$$

$$x(0) = x_0. \tag{2}$$

Here f and g are continuous, f and g are C^1 on $(-\delta_1, \delta_1)$ for some $\delta_1 > 0$, $xg(x) > 0$ for all $x \neq 0$, and $f(0) = 0$, $g(0) = 0$. Thus $x(t) \equiv 0$ is a solution of (1), called the zero solution. If $x(t) \to 0$ as $t \to \infty$, we investigate here the rate at which this solution decays. We expect to infer information about the stability properties of the zero solution of (1) from that of the linear equation

$$x'(t) = -ax(t) + g'(0)\int_0^t k(t-s)x(s)\,ds, \quad t > 0, \tag{3}$$

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where $a = f'(0)$. According to Brauer [4], if the zero solution of (3) is uniformly asymptotically stable, then so is that of (1).

A number of authors have investigated necessary conditions for the exponential stability of solutions of (3). Some of these studies have been motivated by a question of Lakshmikantham and Corduneanu, posed in [9], which asked: if all solutions of (1) satisfy $x(t) \to 0$ as $t \to \infty$, must the convergence be exponentially fast. The question was natural in view of the fact that the asymptotic stability of the zero solution of equations with bounded delay implies the exponential asymptotic stability of the zero solution. However Murakami showed in [14] that exponential asymptotic stability does not follow automatically from the property of (uniform) asymptotic stability of the zero solution. His result requires that $k \in L^1(0,\infty) \cap C[0,\infty)$ and is of one sign, and that the zero solution of (3) is uniformly asymptotically stable. The result concludes that the zero solution of (3) is exponentially asymptotically stable if and only if k is exponentially integrable. We term a function $k \in L^1(0,\infty)$ exponentially integrable if

$$\int_0^\infty |k(s)|e^{\gamma s}\, ds < \infty \quad \text{for some } \gamma > 0. \tag{4}$$

It is therefore natural to ask at what rate does $x(t) \to 0$ as $t \to \infty$ if k is not exponentially integrable.

In [1], Appleby and Reynolds obtained a lower bound on the rate of decay of solutions of (3). Therein, it is assumed that k is a positive, continuously differentiable and integrable function satisfying $k'(t)/k(t) \to 0$ as $t \to \infty$. This last condition prevents k from being exponentially integrable. Using elementary analysis, it is shown that if $x(t) \to 0$ as $t \to \infty$, then there is a positive lower bound for $\liminf_{t\to\infty} x(t)/k(t)$.

In this paper, we pose the following question: given that the uniform asymptotic stability of solutions of (3) transfers to those of (1) which start sufficiently close to the zero equilibrium, do the asymptotically stable solutions of the nonlinear equation (1) inherit the non–exponential decay rate of solutions of the linear equation (3). We ask this for problems whose kernel contain those studied in [1] as a subclass. We prove that if $x(t) \to 0$ as $t \to \infty$, then $\liminf_{t\to\infty} x(t)/k(t)$ has a positive lower bound. In the case of $a > 0$ and $g'(0) = 0$, one might expect that asymptotically stable solutions of (1) approach zero exponentially fast, as the linear equation (3) corresponding to (1) is $x'(t) = -ax(t)$. However, a positive lower bound on $\liminf_{t\to\infty} x(t)/k(t)$ is still obtained.

2. Technical Discussion and Results

In this paper, we consider the integro–differential equation (1). The following hypotheses are imposed on the functions f, g and k occurring in it.

(H_1) $f:\mathbb{R}\to\mathbb{R}$ is continuous, and is C^1 on an interval $(-\delta_1,\delta_1)$ with $\delta_1>0$. Also $f(0)=0$. We use the notation $a=f'(0)$.

(H_2) $g:\mathbb{R}\to\mathbb{R}$ is a continuous function with $g(0)=0$ and $xg(x)>0$ for all $x\neq 0$. Also g is C^1 on $(-\delta_1,\delta_1)$, and $g'(0)$ is either 0 or 1.

(H_3) $k:[0,\infty)\to\mathbb{R}$ is a nontrivial, continuous, integrable function with $k(t)\geq 0$ for all $t\geq 0$.

Since $xg(x)>0$ for $x\neq 0$, $g'(0)\geq 0$. If $g'(0)>0$, g and k can be redefined in such a way as to ensure that $g'(0)=1$. There is therefore no loss of generality in restricting $g'(0)$ to be either 0 or 1.

The paper is concerned with the following question: if a solution of (1) satisfies $x(t)\to 0$ as $t\to\infty$, what is its rate of decay to zero. In particular, we are interested in non–exponential rates of decay. By Murakami's analysis, the linear equation (3) exhibits such non–exponential decay if

$$\int_0^\infty k(s)e^{\varepsilon s}\,ds=\infty \quad \text{for all } \varepsilon>0. \tag{5}$$

This constraint is trivially satisfied, for instance, if

$$\lim_{t\to\infty} k(t)e^{\varepsilon t}=\infty \quad \text{for all } \varepsilon>0. \tag{6}$$

We state three further hypotheses for k.

(H_4) $k(t)>0$ for all $t\geq 0$, and for all $T>0$,

$$\lim_{t\to\infty}\sup_{0\leq s\leq T}\left|\frac{k(t-s)}{k(t)}-1\right|=0.$$

(H_5) k is C^1 on $(0,\infty)$, $k(t)>0$ for all $t\geq 0$ and $k'(t)/k(t)\to 0$ as $t\to\infty$.

(H_6) k satisfies $\int_t^\infty k(s)\,ds>0$ for all sufficiently large $t\geq 0$.

We remark that (H_4) implies (6) (cf., e.g., Chistyakov [8, Theorem 2] or [3, Lemma 2.2]), and that (H_4) follows from (H_5). The non–exponential convergence of solutions of the linearised equation (3) was investigated in [1] under the conditions (H_3) and (H_5) on k, and $g'(0)\neq 0$. The present work extends this result to the nonlinear case, under the weaker hypotheses (H_3) and (H_4) imposed on k. Indeed, we show that the results in [1] can be recovered under these hypotheses as a special case.

The main result of the paper, motivated by [1, Theorem 2], is now stated.

Theorem 1. *Suppose that* (H_1)-(H_3) *hold. Let x be a solution of (1) satisfying*

$$\lim_{t\to\infty} x(t) = 0. \tag{7}$$

Then $a \geq 0$. Further if (H_4) *is true, this solution obeys*

$$\liminf_{t\to\infty} \frac{|x(t)|}{k(t)} \geq \frac{1}{a}\int_0^\infty |g(x(s))|\, ds > 0, \tag{8}$$

where the righthand side of (8) is interpreted as ∞ if either $g \circ x$ is not in $L^1(\mathbb{R}^+)$ or $a = 0$. Moreover, for each $\varepsilon > 0$

$$\lim_{t\to\infty} |x(t)| e^{\varepsilon t} = 0. \tag{9}$$

To prove (9), we observe that it is a direct consequence of (8) and (6). But, as mentioned above, (6) follows from (H_4). We prove (8) in Section 3.

In the absence of the hypothesis (H_4), it is still possible to prove a result about the decay rate of the solution of (1), by a slight reworking and extension of a result of Burton [6, Theorem 1.3.7]. His result pertains to a linear equation of the form (3), under the assumption that k satisfies (H_3) and (H_6).

Theorem 2. *Suppose that* (H_1)-(H_3) *and* (H_6) *hold. Suppose that x is a solution of* (1) *and* (2) *satisfying*

$$\lim_{t\to\infty} x(t) = 0, \quad x \in L^1(0,\infty). \tag{10}$$

Then x obeys

$$\liminf_{t\to\infty} \frac{\int_t^\infty |x(s)|\, ds}{\int_t^\infty k(s)\, ds} \geq \frac{1}{a}\int_0^\infty |g(x(s))|\, ds > 0. \tag{11}$$

Furthermore if $k(t) > 0$ for all t sufficiently large, then

$$\limsup_{t\to\infty} \frac{|x(t)|}{k(t)} \geq \frac{1}{a}\int_0^\infty |g(x(s))|\, ds > 0. \tag{12}$$

The proof of Theorem 2 is given in Section 3.

It is interesting to compare the weaker result (12) with (8), which is proven under an stronger hypothesis (H_4). (H_4) provides additional pointwise asymptotic control on k, enabling a stronger asymptotic result to be obtained on x.

The hypotheses of Theorems 1 and 2 involve not only assumptions on the data for (1), but also properties of the solution. We establish in Theorems 3 and 4 some necessary and sufficient conditions, solely in terms of the data, for (7) and (10) to hold. In Brauer [4] it shown that uniform asymptotic stability of the zero solution of the linearised equation implies the uniform asymptotic stability of the zero solution of the nonlinear equation (1). However we cannot conclude from this that solutions of (1) with initial values near 0 are integrable, though the equivalence of uniform asymptotic stability and integrability of such solutions has been shown for a class of linear Volterra equations by Miller [13] and Grossman and Miller [11]. Thus we use a separate argument using Lyapunov functions to demonstrate that solutions are integrable.

We make a further hypothesis on the existence, uniqueness and continuation of solutions of the initial-value problem consisting of (1) and (2). For existence, uniqueness and continuation results for nonlinear Volterra equations we refer to standard texts such as Burton [5], Burton [6] and Gripenberg, Londen and Staffans [10].

(H_7) There is $0 < \delta_2 \leq \delta_1$ such that (1) has a solution in $C[0,\infty) \cap C^1(0,\infty)$ satisfying (2) for each $|x_0| < \delta_2$; moreover this solution is unique.

Theorem 3. *Suppose that* (H_1)-(H_3) *and* (H_7) *hold with* $g'(0) = 0$. *Then* $a > 0$ *implies that there is a* $0 < \delta_3 \leq \delta_2$ *such* (10) *holds for every solution of* (1) *and* (2) *with* $|x_0| < \delta_3$. *Conversely, if there is a solution of* (1) *and* (2) *for which* (10) *holds, then* $a \geq 0$.

Theorem 4. *Suppose that* (H_1)-(H_3) *and* (H_7) *hold with* $g'(0) = 1$. *If*

$$a > \int_0^\infty k(s)\,ds, \tag{13}$$

then there is a $0 < \delta_4 \leq \delta_2$ *such* (10) *holds for every solution of* (1) *and* (2) *with* $|x_0| < \delta_4$. *Conversely if there is a number* $\delta_4 > 0$ *such that* (10) *holds for every solution of* (1) *and* (2) *with* $|x_0| < \delta_4$, *then*

$$a \geq \int_0^\infty k(s)\,ds. \tag{14}$$

These results are proven in Section 3. The proofs extend the analysis of Burton and Mahfoud [7, Theorem 1], which is concerned with a linear equation, to the nonlinear problem (1).

Using Theorems 3 and 4, we immediately have the following corollary to Theorems 1 and 2.

Corollary 5. *Suppose that* (H_1)-(H_3) *and* (H_7) *hold. Also suppose that either* $g'(0) = 0$ *and* $a > 0$, *or* $g'(0) = 1$ *and* (13) *is true. Then the solution* x *of* (1) *and* (2) *satisfies* (10), *and obeys (11) and (12). Moreover, the quantity on the righthand sides of (11) and (12) is finite.*

Suppose in addition that k *satisfies* (H_4). *Then the solution of (1) and* (2) *obeys (8) and (9), and the quantity on the righthand sides of (8) and (9) is finite.*

The above results can be applied to the linear Volterra equation

$$x'(t) = -ax(t) + \int_0^t k(t-s)x(s)\,ds, \quad t > 0, \tag{15}$$

by choosing $f(x) = ax$ and $g(x) = x$. If k satisfies (H$_3$) with $k(t) > 0$ for $t \geq 0$, the stability properties of the zero solution of (15) are well understood. Brauer [4] showed that the zero solution could not be stable if $a < \int_0^\infty k(s)\,ds$, thereby precluding its asymptotic stability. A slight modification to the argument in Kordonis and Philos [12] shows that the zero solution is stable if $a = \int_0^\infty k(s)\,ds$. Indeed, some further analysis reveals that the solution is asymptotically stable in this case if and only if $\int_0^\infty sk(s)\,ds = \infty$. These results sharpen those of Burton [5, Theorem 5.2.3]. In the case $a > \int_0^\infty k(s)\,ds$, the zero solution is asymptotically stable; moreover every solution tends to zero. In fact, the solution is integrable if and only if $a > \int_0^\infty k(s)\,ds$. The results for $a > \int_0^\infty k(s)\,ds$ are proved in Burton and Mahfoud [7, Theorem 1]. Consequently, if $x(t) \to 0$ as $t \to \infty$ then $a \geq \int_0^\infty k(s)\,ds$. If the solution to (15) is integrable, we must have $a > \int_0^\infty k(s)\,ds$ and

$$\int_0^\infty x(s)\,ds = \frac{x_0}{a - \int_0^\infty k(s)\,ds},$$

and $x(t) \to 0$ as $t \to \infty$. Hence we obtain the following corollary to Theorem 2.

Corollary 6. *Suppose that* (H$_1$)-(H$_3$) *and* (H$_6$) *hold. If x is a nontrivial integrable solution of (15), then*

$$\liminf_{t\to\infty} \frac{\int_t^\infty |x(s)|\,ds}{\int_t^\infty k(s)\,ds} \geq \frac{|x_0|}{a(a - \int_0^\infty k(s)\,ds)},$$

the expression on the righthand side of this inequality being positive and finite.

This is Theorem 1 in [1], and can be inferred from Burton [6, Theorem 1.3.7]. We may also obtain a corollary to Theorem 1 for the linear equation (15).

Corollary 7. *Suppose that* (H$_1$)-(H$_4$) *hold. If the solution of* (15) *and* (2) *satisfies* (7), *then*

$$\liminf_{t\to\infty} \frac{|x(t)|}{k(t)} \geq \frac{|x_0|}{a(a - \int_0^\infty k(s)\,ds)}, \tag{16}$$

where the quantity on the righthand side is strictly positive, and interpreted to be ∞ if $a = \int_0^\infty k(s)\,ds$.

This furnishes an improvement to [1, Theorem 2], which requires the stronger hypothesis (H_6) in place of (H_4).

It is shown in Appleby and Reynolds [2, Theorem 6.2] that the solution of (15) satisfies

$$\lim_{t\to\infty}\frac{x(t)}{k(t)} = \frac{x(0)}{\left(a-\int_0^\infty k(s)\,ds\right)^2}, \tag{17}$$

if $a > \int_0^\infty k(s)\,ds$, and in addition to ($H_3$) and ($H_4$), k satisfies

$$\lim_{t\to\infty}\frac{\int_0^t k(t-s)k(s)\,ds}{k(t)} = 2\int_0^\infty k(s)\,ds. \tag{18}$$

Thus the exact value of the lefthand side of (16) is known in that case.

We can use Theorem 2 to obtain an elementary proof that the exponential integrability of k is necessary if there exists a non–trivial solution of (1) which converges to zero exponentially fast. A version of this result for linear Volterra integro–differential equations, where k is of one sign, is proven in [14, Theorem 2] using Laplace transform techniques.

Before we embark on this proof, we wish to re-examine one of the hypotheses required of k in Theorem 2. If k is a continuous, non–negative and integrable function defined on $\mathbb{R}^+$ which is not compactly supported, then $\int_t^\infty k(s)\,ds > 0$ for all t sufficiently large, and (H_6) holds. Hence, if (1) is genuinely an equation with unbounded delay, then (H_6) is true.

Corollary 8. *Suppose that* (H_1)-(H_3) *hold, and that k does not have compact support. Suppose that there is a non–trivial solution of (1) satisfing $|x(t)| \le ce^{-\alpha t}$ for some $c > 0$ and $\alpha > 0$. Then there is $0 < \gamma < \alpha$ such that*

$$\int_0^\infty k(s)e^{\gamma s}\,ds < \infty.$$

3. Proofs

As a preliminary step, we first develop a representation for solutions of (1). Define

$$\tilde{f}(x) = \begin{cases} f(x)/x, & x \neq 0, \\ a, & x = 0. \end{cases} \tag{19}$$

Due to (H_1), $\tilde{f}$ is continuous, and hence $t \mapsto \tilde{f}(x(t))$ is continuous. Introduce the function

$$\varphi(t) = \exp\left(-\int_0^t \tilde{f}(x(s))\,ds\right). \tag{20}$$

Then $\varphi(t) > 0$ for all $t \geq 0$. It also satisfies $\varphi'(t) = -\tilde{f}(x(t))\varphi(t)$. Defining $y(t) = x(t)/\varphi(t)$, we obtain

$$y'(t) = \varphi(t)^{-1} \int_0^t k(t-s)g(x(s))\,ds.$$

Therefore

$$x(t) = \varphi(t)x_0 + \varphi(t) \int_0^t \varphi(s)^{-1} \int_0^s k(s-u)g(x(u))\,du\,ds. \tag{21}$$

Next we observe that solutions cannot change sign.

Lemma 9. *Suppose that* (H_1)-(H_3) *hold. Let x be the solution of (1) and* (2) *with $x_0 \neq 0$. Then $x(t)/x_0 > 0$ for all $t \geq 0$.*

Proof. Suppose that $x_0 > 0$. Let $[0, t_0)$ be the maximal interval for which $x(t) > 0$ for all $0 \leq t < t_0$. Clearly $t_0 > 0$. If t_0 is finite, consider

$$r(t) = \int_0^t k(t-s)g(x(s))\,ds.$$

Then $r(t) \geq 0$ for all $t \in [0, t_0]$. Since $\varphi(t) > 0$ for all $t \geq 0$ and $x(0) > 0$, (21) implies that

$$0 = x(t_0) = \varphi(t_0)x(0) + \varphi(t_0) \int_0^{t_0} \varphi(s)^{-1} r(s)\,ds \geq \varphi(t_0)x(0) > 0,$$

a contradiction. Hence $x(t) > 0$ for all $t \geq 0$. The proof in the case $x_0 < 0$ is identical. □

We continue our proof with an asymptotic estimate.

Lemma 10. *Let p be in $C[0,\infty)$ with $p(t) \geq 0$ for all $t \geq 0$. If k satisfies* (H_3) *and* (H_4)*, then*

$$\liminf_{t\to\infty} \frac{\int_0^t k(t-s)p(s)\,ds}{k(t)} \geq \int_0^\infty p(s)\,ds,$$

where we interpret the righthand side as ∞ if $p \notin L^1(0,\infty)$.

Proof. The proof when $p \notin L^1(0,\infty)$ is similar to that when $p \in L^1(\mathbb{R}^+)$, and so the former case is omitted. Choose $T > 0$ and let $t > T$. Then

$$\frac{\int_0^t k(t-s)p(s)\,ds}{k(t)} \geq \int_0^T \left(\frac{k(t-s)}{k(t)} - 1\right) p(s)\,ds + \int_0^T p(s)\,ds. \tag{22}$$

However

$$\left|\int_0^T \left(\frac{k(t-s)}{k(t)} - 1\right) p(s)\,ds\right| \leq \sup_{0\leq s\leq T} \left|\frac{k(t-s)}{k(t)} - 1\right| \int_0^T p(s)\,ds$$

has zero limit as $t \to \infty$, due to (H_4). By taking the limit inferior as $t \to \infty$ on both sides of the relation (22), and then the limit as $T \to \infty$, the result follows. $\square$

Proof of Theorem 1. Hereinafter, it is assumed that $x_0 > 0$; the proof for $x_0 < 0$ is identical and thus omitted. A consequence of $x_0 > 0$ is that $x(t) > 0$ for all $t \geq 0$.

We can infer from the positivity of x and (21) that $x(t) \geq \varphi(t)x_0$. Since $x(t) \to 0$ as $t \to \infty$, $\varphi(t) \to 0$ as $t \to \infty$. But $x(t) \to 0$ as $t \to \infty$ and (19) imply

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \tilde{f}(x(s))\, ds = a.$$

Since $t^{-1} \log \varphi(t) \to -a$ as $t \to \infty$, it follows that $a \geq 0$.

We now establish the result, assuming that the positive function $g \circ x$ is in $L^1(0, \infty)$. The argument is sufficiently similar in the case of $g \circ x$ not being $L^1(0, \infty)$ to warrant its exclusion. Let $0 < \varepsilon < 1$ and $T > 0$. It is a consequence of (21) that

$$\frac{x(t)}{k(t)} > \int_{t-T}^t \frac{k(s)}{k(t)} \varphi(t)\varphi(s)^{-1} \frac{1}{k(s)} \int_0^s k(s-u) g(x(u))\, du\, ds. \tag{23}$$

for all $t > T$. By Lemma 10 there is $T_1(\varepsilon) > 0$ such that

$$\frac{1}{k(t)} \int_0^t k(t-s) g(x(s))\, ds > (1-\varepsilon) \int_0^\infty g(x(s))\, ds, \tag{24}$$

whenever $t > T_1(\varepsilon)$. By (20), and the fact that $x(t) \to 0$ as $t \to \infty$, there exists $T_2(\varepsilon) > 0$ such that $t \geq s > T_2(\varepsilon)$ implies

$$\varphi(t)\varphi(s)^{-1} \geq e^{-a(1+\varepsilon)(t-s)}. \tag{25}$$

Here we make use of the fact that $\tilde{f}(x(t)) \to a$ as $t \to \infty$, and also that $a \geq 0$. Now let $T(\varepsilon) = \max(T_1(\varepsilon), T_2(\varepsilon))$, and take $t > T(\varepsilon) + T$. Using (24), (25), the inequality (23) becomes

$$\frac{x(t)}{k(t)} \geq \int_{t-T}^t \frac{k(s)}{k(t)} e^{-a(1+\varepsilon)(t-s)}\, ds \cdot (1-\varepsilon) \int_0^\infty g(x(s))\, ds.$$

But

$$\int_{t-T}^t \frac{k(s)}{k(t)} e^{-a(1+\varepsilon)(t-s)}\, ds \\ \geq \int_0^T e^{-a(1+\varepsilon)u}\, du + \int_0^T \left(\frac{k(t-u)}{k(t)} - 1 \right) e^{-a(1+\varepsilon)u}\, du.$$

Arguing as in Lemma 10 gives

$$\liminf_{t\to\infty} \frac{x(t)}{k(t)} \geq (1-\varepsilon) \int_0^T e^{-a(1+\varepsilon)u}\, du \int_0^\infty g(x(s))\, ds. \tag{26}$$

If $a = 0$, letting $T \to \infty$ on both sides of (26) yields the result, with the righthand side of (8) interpreted as infinity. If $a > 0$, letting $T \to \infty$ both sides of (26) gives

$$\liminf_{t\to\infty} \frac{x(t)}{k(t)} \geq (1-\varepsilon)\frac{1}{a(1+\varepsilon)} \int_0^\infty g(x(s))\,ds.$$

We now obtain (8) by taking the limit as $\varepsilon \downarrow 0$. $\square$

Proof of Theorem 2. We assume again, without loss of generality, that $x(0) > 0$. Then $x(t) > 0$ for all $t \geq 0$, and therefore $g(x(t)) > 0$. Let $\varepsilon > 0$. Since $x(t) \to 0$ as $t \to \infty$, by (H_1) and (H_2) there is a $T_1(\varepsilon) > 0$ such that

$$|f(x(t))| \leq (|f'(0)| + \varepsilon)|x(t)|, \quad |g(x(t))| \leq (|g'(0)| + \varepsilon)|x(t)|,$$

for all $t > T_1(\varepsilon)$. Because $x \in L^1(0,\infty)$, it follows that $f \circ x$ and $g \circ x$ are also in $L^1(0,\infty)$. Moreover as $k \in L^1(0,\infty)$, the convolution $k * (g \circ x)$ is in $L^1(0,\infty)$. Since $x(t) \to 0$ as $t \to \infty$, the integration of (1) over $[t,\infty)$ yields

$$-x(t) = -\int_t^\infty f(x(s))\,ds + \int_t^\infty \int_0^s k(s-u)g(x(u))\,du\,ds.$$

Availing of the non–negativity of k and proceeding as in [6, Theorem 1.3.7], we get

$$\int_t^\infty f(x(s))\,ds \geq \int_t^\infty k(s)\,ds \int_0^t g(x(u))\,du. \tag{27}$$

By Theorem 1, $a + \varepsilon > 0$. As $\tilde{f}(x(t)) \to a$ as $t \to \infty$, there exists $T_2(\varepsilon) > 0$ such that

$$f(x(t)) < (a+\varepsilon)x(t),$$

for $t > T_2(\varepsilon)$. Hence for $t > T_2(\varepsilon)$,

$$(a+\varepsilon)\int_t^\infty x(s)\,ds \geq \int_t^\infty f(x(s))\,ds \geq \int_t^\infty k(s)\,ds \int_0^t g(x(u))\,du.$$

¿From (H_6) and the fact that $(g \circ x)$ is in $L^1(0,\infty)$, we obtain

$$\liminf_{t\to\infty} \frac{\int_t^\infty x(s)\,ds}{\int_t^\infty k(s)\,ds} \geq \frac{1}{a+\varepsilon}\int_0^\infty g(x(u))\,du.$$

Allowing $\varepsilon \downarrow 0$ proves (11).

To see that (12) follows from (11), under the additional hypotheses that $k(t) > 0$ for all $t > T_0 > 0$, we suppose to the contrary that

$$\limsup_{t\to\infty} \frac{x(t)}{k(t)} < \frac{1}{a}\int_0^\infty g(x(s))\,ds.$$

Therefore there exists $0 \leq \beta < 1$ such that

$$\limsup_{t\to\infty} \frac{x(t)}{k(t)} = \beta \frac{1}{a} \int_0^\infty g(x(s))\, ds.$$

Thus for all $0 < \varepsilon < 1 - \beta$, there is a $T(\varepsilon) > T_0$ such that

$$x(t) < \frac{k(t)(\beta + \varepsilon)}{a} \int_0^\infty g(x(u))\, du,$$

for all $t > T(\varepsilon)$. Hence for all $t > T(\varepsilon)$,

$$\frac{\int_t^\infty x(s)\, ds}{\int_t^\infty k(s)\, ds} < \frac{(\beta + \varepsilon)}{a} \int_0^\infty g(x(u))\, du.$$

Taking the limit inferior of each side as $t \to \infty$, we deduce from (11) that

$$\frac{1}{a} \int_0^\infty g(x(u))\, du \leq (\beta + \varepsilon) \frac{1}{a} \int_0^\infty g(x(u))\, du.$$

Letting $\varepsilon \downarrow 0$ yields $\beta \geq 1$, which is a contradiction. □

Proof of Theorem 3. We first recall that it was established in Theorem 1 that (7) implies $a \geq 0$. Thus the converse has already been proved.

It is now proved that $a > 0$ implies (10) holds for every solution of (1) and (2) with $|x_0|$ small enough. Again we confine attention to the case $x_0 > 0$. Then by Lemma 9, $x(t) > 0$ for all $t \geq 0$. Since $a > 0$,

$$2\varepsilon = \frac{a}{1 + \int_0^\infty k(s)\, ds} > 0. \tag{28}$$

By assumption, there exists $0 < \delta_3 \leq \delta_2$ such that $|x| \leq \delta_3$ implies

$$|f(x) - ax| \leq \varepsilon |x|, \quad |g(x)| \leq \varepsilon |x|. \tag{29}$$

Assume that $0 < x_0 < \delta_3$, and let $[0, t_1)$ be the maximal interval with the property that $0 < x(t) < \delta_3$ for all $0 \leq t < t_1$. Clearly $t_1 > 0$ is either finite and positive, or $[0, t_1) = [0, \infty)$. We show that it cannot be finite. By construction $x'(t_1) \geq 0$. However by (28) and (29),

$$\begin{aligned} x'(t_1) &= -f(x(t_1)) + \int_0^{t_1} k(t_1 - s) g(x(s))\, ds \\ &\leq -(a - \varepsilon) x(t_1) + \varepsilon \int_0^{t_1} k(t_1 - s) x(s)\, ds \\ &= \delta_3 \left(-(a - \varepsilon) + \varepsilon \int_0^{t_1} k(s)\, ds \right) < 0, \end{aligned}$$

a contradiction. Thus $0 < x(t) < \delta_3$ for all $t > 0$ if $0 < x_0 < \delta_3$.

We now prove the asymptotic stability and integrability of the solutions with $0 < x_0 < \delta_3$. We introduce the non–negative function

$$V(t) = x(t) + \int_0^t \int_t^\infty k(\tau - s)g(x(s))\,d\tau\,ds. \tag{30}$$

This is based on a Lyapunov functional used in Theorem 1 of Burton and Mahfoud [7]. Differentiation and use of (29) leads to

$$\frac{dV}{dt}(t) = -f(x(t)) + g(x(t))\int_0^\infty k(s)\,ds \le -\beta x(t), \tag{31}$$

where

$$\beta = a - \varepsilon(1 + \int_0^\infty k(s)\,ds) > 0.$$

As $V(t) > 0$ for all $t \ge 0$, integration of (31) over $[0, t]$ gives

$$\beta \int_0^t x(s)\,ds \le V(0) = x(0).$$

Hence $x \in L^1(\mathbb{R}^+)$. Since $0 < x(t) < \delta_3$, it follows from (29) that

$$|g(x(t))| \le \varepsilon|x(t)|, \quad |f(x(t))| \le (|a| + \varepsilon)|x(t)|,$$

and so $f \circ x$ and $g \circ x$ are in $L^1(0,\infty)$. Consequently $k * (g \circ x)$ is in $L^1(0,\infty)$. Integrating (1), we see that $\lim_{t\to\infty} x(t)$ exists. But $x \in L^1(0,\infty)$ implies $x(t) \to 0$ as $t \to \infty$. □

Proof of Theorem 4. We prove the converse only, as the first assertion can be established by following the model of proof in Theorem 3 above. Indeed, the proof uses the same function V defined in (30).

To prove the converse we again suppose that $x_0 > 0$, so that $x(t) > 0$ for all $t \ge 0$. Indeed we assume that $0 < x_0 < \delta_4$ so that implies $x(t) \to 0$ as $t \to \infty$. Note also that Theorem 1 implies that $a \ge 0$. Since $k(t) \not\equiv 0$ and is continuous, $\int_0^\infty k(s)\,ds > 0$.

Suppose $a < \int_0^\infty k(s)\,ds$. We prove this is false by contradiction. To this end, fix $T > 0$ so that $\int_0^T k(s)\,ds > a$. Let $0 < \varepsilon < 1$ satisfy

$$0 < \varepsilon < \frac{\int_0^T k(s)\,ds - a}{\int_0^T k(s)\,ds + 1}.$$

There exists $\delta_5 > 0$ such that

$$0 < f(x) \le (a + \varepsilon)x, \quad g(x) \ge (1 - \varepsilon)x,$$

whenever $0 < x \le \delta_5$. There is $t_1 \ge 0$ such that $0 < x(t) < \delta_5$ for all $t > t_1$. Since $\min_{t\in[0,\tau]} x(t) > 0$ for all finite $\tau > 0$, there also exists $0 < \delta_6 < \delta_5$ and $t_2 > t_1 + T$ for which $t_2 = \min\{t > 0 : x(t) = \delta_6\}$. The minimality of t_2 implies $x'(t_2) \leq 0$. This construction now implies the following sequence of inequalities:

$$
\begin{aligned}
x'(t_2) &= -f(x(t_2)) + \int_0^{t_2} k(t_2 - s) g(x(s))\, ds \\
&\geq -(a+\varepsilon) x(t_2) + \int_{t_1}^{t_2} k(t_2 - s) g(x(s))\, ds \\
&\geq -(a+\varepsilon) x(t_2) + (1-\varepsilon) \int_{t_1}^{t_2} k(t_2 - s) x(s)\, ds \\
&\geq \left(-(a+\varepsilon) + (1-\varepsilon) \int_{t_1}^{t_2} k(t_2 - s)\, ds \right) \delta_6 \\
&\geq \left(-(a+\varepsilon) + (1-\varepsilon) \int_0^{T} k(s)\, ds \right) \delta_6 \\
&> 0,
\end{aligned}
$$

which is a contradiction. Therefore $a \geq \int_0^\infty k(s)\, ds$, as required. ☐

Proof of Corollary 8. The assumption on x implies $x(t) \to 0$ as $t \to \infty$, and $x \in L^1(0, \infty)$. All the hypotheses of Theorem 2 hold, and hence there is a $c_1 > 0$ such that

$$
\frac{\int_t^\infty |x(s)|\, ds}{\int_t^\infty k(s)\, ds} \geq c_1, \quad t \geq 0.
$$

Thus if $c_2 = c/(\alpha c_1) > 0$, we have

$$
\int_t^\infty k(s)\, ds \leq c_2 e^{-\alpha t}, \quad t \geq 0.
$$

Let $0 < \varepsilon < \alpha$ and fix $\gamma = \alpha - \varepsilon$. Then

$$
\int_0^\infty e^{\gamma t} \int_t^\infty k(s)\, ds\, dt \leq \frac{c_2}{\varepsilon}.
$$

The non–negativity of the integrand justifies reversing the order of integration, yielding

$$
\int_0^\infty e^{\gamma t} \int_t^\infty k(s)\, ds\, dt = \int_0^\infty k(s) \frac{1}{\gamma} (e^{\gamma s} - 1)\, ds.
$$

Hence

$$
\frac{1}{\gamma} \int_0^\infty e^{\gamma s} k(s)\, ds \leq \frac{c_2}{\varepsilon} + \frac{1}{\gamma} \int_0^\infty k(s)\, ds,
$$

and the result is true. The ideas here have been partially inspired by arguments in the proof of Theorem 1 of [7]. ☐

References

[1] J. A. D. Appleby and D. W. Reynolds, *On the non–exponential convergence of asymptotically stable solutions of linear scalar Volterra integro–differential equations*, J. Integral Equations Appl. **14** (2002), no. 2, 109–118.

[2] ______, *Subexponential solutions of linear Volterra integro-differential equations and transient renewal equations*, Proc. Roy. Soc. Edinburgh. Sect. A **132A** (2002), 521–543.

[3] ______, *Subexponential solutions of linear integro– differential equations*, Submitted (2004).

[4] F. Brauer, *Asymptotic stability of a class of integro-differential equations*, J. Differential Equations **28** (1978), 180–188.

[5] T. A. Burton, *Volterra integral and differential equations*, Mathematics in Science and Engineering, vol. 167, Academic Press, 1983.

[6] ______, *Stability and periodic solutions of functional differential equations*, Mathematics in Science and Engineering, vol. 178, Academic Press, Orlando, Florida, 1985.

[7] T. A. Burton and W. E. Mahfoud, *Stability criterion for Volterra equations*, Trans. Amer. Math. Soc. **279** (1983), 143–174.

[8] V. P. Chistyakov, *A theorem on sums of independent positive random variables and its application to branching random processes*, Theory Probab. Appl. **9** (1964), 640–648.

[9] C. Corduneanu and V. Lakshmikantham, *Equations with unbounded delay: a survey*, Nonlinear Anal. **4** (1980), 831–877.

[10] G. Gripenberg, S. O. Londen, and O. Staffans, *Volterra integral and functional equations*, Encylopedia of Mathematics and its Applications, Cambridge University Press, 1990.

[11] S. I. Grossman and R. K. Miller, *Nonlinear Volterra integrodifferential equations*, J. Differential Equations **13** (1973), 551–566.

[12] I.-G. E. Kordonis and Ch. G. Philos, *The behavior of solutions of linear integro-differential equations with unbounded delay*, Comput. Math. Appl. **38** (1999), no. 2, 45–50.

[13] R. K. Miller, *Asymptotic stability properties of linear Volterra integrodifferential equations*, J. Differential Equations **10** (1971), 485–506.

[14] S. Murakami, *Exponential asymptotic stability of scalar linear Volterra equations*, Differential Integral Equations **4** (1991), no. 3, 519–525.

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