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THE SCHRÖDER EQUATION AND ASYMPTOTIC PROPERTIES OF LINEAR DELAY DIFFERENTIAL EQUATIONS

Jan Čermák

Abstract. We study the asymptotic behaviour of solutions of the differential equation

$$\dot{x}(t) = -p(t)[x(t) - kx(t-\tau(t))] + q(t), \qquad t \in I = [t_0, \infty),$$

where $k \neq 0$ is a scalar, p is a positive function and τ is a positive and unbounded delay. Our aim is to show that under some asymptotic bounds on q the behaviour (as $t \to \infty$) of all solutions of this equation can be estimated via a solution of the Schröder equation

$$\varphi(t-\tau(t)) = \lambda\varphi(t), \qquad t \in I$$

with a suitable positive parameter λ.

1. Introduction and preliminaries

This paper discusses the asymptotic properties of solutions of the delay differential equation

$$\dot{x}(t) = -p(t)[x(t) - kx(t-\tau(t))] + q(t), \qquad t \in I = [t_0, \infty), \tag{1.1}$$

where $k \neq 0$ is a real constant, $p, q \in C(I)$, $p > 0$ on I, $\tau \in C^1(I)$, $\tau > 0$, $t - \tau(t) \to \infty$ as $t \to \infty$ and $0 < \zeta \leq \dot{\tau} < 1$ on I for a real scalar ζ. Throughout this paper we assume that these assumptions on τ are fulfilled (the remaining requirements on p and q will be introduced later).

Equation (1.1) with such a delay has been investigated in many particular cases. The well known pantograph equation

$$\dot{x}(t) = a\,x(\lambda t) + b\,x(t) \tag{1.2}$$

may serve as a prototype of (1.1). The name *pantograph* originated from the work of Ockendon & Taylor [18] on the collection of current by the pantograph head of an electric locomotive. Equations of similar forms appear in many applications such as astrophysics, probability theory on algebraic structures, spectral problem of the Schrödinger equations or quantum mechanics (see [1], [19] or [20]). Particularly, the nonhomogeneous pantograph equation

$$\dot{x}(t) = a\,x(\lambda t) + b\,x(t) + q(t) \tag{1.3}$$

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was derived in [7] as the mathematical model for a problem in electric locomotion. Among the works on equations (1.2), (1.3) we can mention especially those by Kato & McLeod [10] and Lim [15].

The qualitative investigations concerning (1.2) and (1.3) were later extended to more general cases, e.g., to the vector case and nonautonomous case, to equations with a variable (unbounded) delay and to neutral equations (see [8], [9], [13], [14] or [16]). For further related results see also [2], [6], and [11].

Particularly, Makay & Terjéki [17] established conditions under which all solutions of

$$\dot{x}(t) = -p(t)[x(t) - x(\lambda t)]$$

are asymptotically logarithmically periodic. The generalization of this result has been done in [5], where the asymptotic behaviour of all solutions of

$$\dot{x}(t) = -p(t)[x(t) - kx(t - \tau(t))] \tag{1.4}$$

was related to the behaviour of the Schröder functional equation

$$\varphi(t - \tau(t)) = \lambda\varphi(t), \qquad t \in I, \tag{1.5}$$

where

$$\lambda = \sup\{1 - \dot{\tau}(t),\, t \in I\}. \tag{1.6}$$

Our aim is to establish asymptotic conditions on the forcing term q under which results of this type remain valid also for equation (1.1).

Note that equation (1.1) with a positive coefficient at $x(t)$ has been studied in [3], where the asymptotics of solutions have been determined by means of the behaviour of a solution of the auxiliary differential equation

$$\dot{x}(t) = p(t)x(t).$$

Nevertheless, we show that certain similarities in the asymptotics of solutions in both sign cases can be observed.

By a solution of equation (1.1) (and other delay equations occuring in this paper) we mean a real valued function $x \in C([\tau(t_0), \infty))$ satisfying the given equation for all $t \in [t_0, \infty)$. Similarly we introduce the notion of a solution for the Schröder equation (1.5).

Using the step method we can easily prove the following existence result.

Proposition 1. *Consider equation (1.5), where λ is given by (1.6). Let $\varphi_0 \in C^1(I_0)$, where $I_0 = [t_0 - \tau(t_0), t_0]$, be a positive function satisfying $\dot{\varphi}_0 > 0$ on I_0 and*

$$(\varphi_0 \circ (\mathrm{id} - \tau))^{(j)}(t_0) = \lambda\varphi_0^{(j)}(t_0), \qquad j = 0, 1.$$

Then there exists a unique positive solution $\varphi \in C^1(I_0 \cup I)$ of (1.5) such that $\varphi(t) \to \infty$ as $t \to \infty$, $\dot{\varphi}$ is positive and bounded on I and $\varphi \equiv \varphi_0$ on I_0.

2. Results and proofs

In this section, we present asymptotic bounds for all solutions x of (1.1). We recall that all the assumptions imposed on τ in Section 1 are fulfilled and suppose that φ is a solution of the corresponding Schröder equation with the properties guaranteed by Proposition 1.

First we state the following auxiliary result.

Proposition 2. *Let φ be a solution of (1.5) from Proposition 1. Let x be a solution of the equation*

$$\dot{x}(t) = -a(t)x(t) + b(t)x(t-\tau(t)) + f(t), \qquad t \in I, \tag{2.1}$$

where $a, b, f \in C(I)$, $a(t) \geq L/(\varphi(t))^{\alpha}$, $0 < |b(t)| \leq Ka(t)$ for all $t \in I$ and suitable reals $L, K > 0$, $\alpha < 1$. If $f(t) = O((\varphi(t))^{\beta})$ as $t \to \infty$ for a real β, then

$$x(t) = O((\varphi(t))^{\gamma}) \qquad \text{as } t \to \infty, \qquad \gamma > \max{(\alpha + \beta, \log K / \log \lambda^{-1})}.$$

Proof. This result has been proved in particular cases $a(t) \equiv a = \text{const} > 0$ ([4, Lemma 3.3]) and $f \equiv 0$ ([5, Theorem 3.1]). The generalization of both proofs to our case is quite simple and is therefore omitted. □

Proposition 3. *Let φ be a solution of (1.5) from Proposition 1. Let x be a solution of (1.1), where $k \neq 0$, $p, q \in C^1(I)$, $p(t) \geq L/(\varphi(t))^{\alpha}$, $\dot{p}(t) \leq Mp(t)/(\varphi(t))^{\theta}$ for suitable reals $L, M > 0$ and $\alpha < \theta \leq 1$. If $q(t) = O((\varphi(t))^{\beta})$ as $t \to \infty$, $\dot{q}(t) = O((\varphi(t))^{\beta - \tilde{\theta}})$ as $t \to \infty$ for suitable reals β and $\alpha < \tilde{\theta} < \theta$, then*

$$x(t) = O((\varphi(t))^{\delta}) \qquad \text{as } t \to \infty \tag{2.2}$$

and

$$\dot{x}(t) = O((\varphi(t))^{\delta - \tilde{\theta}}) \qquad \text{as } t \to \infty, \tag{2.3}$$

where $\delta > \max{(\alpha + \beta, \log |k| / \log \lambda^{-1})}$.

Proof. The asymptotic relation (2.2) follows immediately from Proposition 2. We prove the validity of (2.3).

If we introduce the function $y = \dot{x}$, then y defines a solution of the delay equation

$$\dot{y}(t) = -[p(t) - \frac{\dot{p}(t)}{p(t)}]y(t) + k(1 - \dot{\tau}(t))p(t)y(t - \tau(t)) + \dot{q}(t) - \frac{\dot{p}(t)}{p(t)}q(t), \qquad t \in I,$$

which is equation (2.1) with coefficients $a(t) = p(t) - \dot{p}(t)/p(t)$, $b(t) = k(1 - \dot{\tau}(t))p(t)$ and $f(t) = \dot{q}(t) - q(t)\dot{p}(t)/p(t)$, $t \in I$. To apply Proposition 2 to this transformed equation we verify its assumptions. First,

$$a(t) = p(t) - \frac{\dot{p}(t)}{p(t)} \geq p(t) - \frac{M}{(\varphi(t))^{\theta}} \geq \frac{L}{(\varphi(t))^{\alpha}} - \frac{M}{(\varphi(t))^{\theta}} \geq \frac{L^*}{(\varphi(t))^{\alpha}}$$

for a suitable $L^* > 0$. Further, let $N = |k|\lambda^{\tilde{\theta}}$. Then

$$0 < |b(t)| = |k|(1 - \dot{\tau}(t))p(t) \leq |k|\lambda p(t) \leq N(p(t) - \frac{\dot{p}(t)}{p(t)}) = Na(t)$$

for all t large enough. Finally,

$$f(t) = \dot{q}(t) - \frac{\dot{p}(t)}{p(t)} q(t) = O((\varphi(t))^{\beta-\tilde{\theta}}) \qquad \text{as } t \to \infty.$$

Since

$$\frac{\log N}{\log \lambda^{-1}} = \frac{\log |k| \lambda^{\tilde{\theta}}}{\log \lambda^{-1}} = \frac{\log |k|}{\log \lambda^{-1}} - \tilde{\theta},$$

a repeated application of Proposition 2 yields

$$y(t) = \dot{x}(t) = O((\varphi(t))^{\delta-\tilde{\theta}}) \qquad \text{as } t \to \infty. \qquad \square$$

Now we formulate the main result which is a refinement of the previous asymptotic estimates.

Theorem. *Let φ be a solution of (1.5) from Proposition 1. Let x be a solution of (1.1) and assume that all the assumptions of Proposition 3 are fulfilled. If we put $\omega := \log |k| / \log \lambda^{-1}$, then*

$$\begin{aligned}
&x(t) = O((\varphi(t))^{\omega}) \qquad \text{as } t \to \infty \qquad \text{if } \alpha + \beta < \omega,\\
&x(t) = O((\varphi(t))^{\omega} \log \varphi(t)) \qquad \text{as } t \to \infty \qquad \text{if } \alpha + \beta = \omega,\\
&x(t) = O((\varphi(t))^{\alpha+\beta}) \qquad \text{as } t \to \infty \qquad \text{if } \alpha + \beta > \omega.
\end{aligned}$$

First we recall the following auxiliary asymptotic result on a special difference equation. This result will be utilized in the proof of Theorem.

Proposition 4 ([15, Lemma 1])**.** *Let z be a solution of the difference equation*

$$z(s) = l z(s-c) + g(s), \qquad s \in J = [s_0, \infty),$$

where $c > 0$ is a real constant, l is a complex constant such that $|l| = \exp\{-\rho c\}$ for a suitable $\rho > 0$ and g is a continuous function fulfilling the property $q(s) = O(\exp\{-\kappa s\})$ as $s \to \infty$ for a suitable $\kappa > 0$. Then

$$\begin{aligned}
&z(s) = O(\exp\{-\rho s\}) \qquad \text{as } s \to \infty \qquad \text{if } \rho < \kappa,\\
&z(s) = O(s \exp\{-\rho s\}) \qquad \text{as } s \to \infty \qquad \text{if } \rho = \kappa,\\
&z(s) = O(\exp\{-\kappa s\}) \qquad \text{as } s \to \infty \qquad \text{if } \rho > \kappa.
\end{aligned}$$

Proof of Theorem. Let $\delta > \max(\alpha + \beta, \omega)$. We set

$$s = \log \varphi(t), \qquad z(s) = (\varphi(t))^{-\delta} x(t)$$

in (1.1) and obtain the differential-difference equation

$$\begin{aligned}
z'(s) = &- [p(h(\exp\{s\}))h'(\exp\{s\}) \exp\{s\} + \delta] z(s)\\
&+ k\lambda^{\delta} p(h(\exp\{s\}))h'(\exp\{s\}) \exp\{s\} z(s-c)\\
&+ \exp\{-\delta s\} q(h(\exp\{s\}))h'(\exp\{s\}) \exp\{s\}, \qquad s \in \varphi(I),
\end{aligned} \tag{2.4}$$

where $c = \log \lambda^{-1}$ and $h \equiv \varphi^{-1}$ on $\varphi(I)$. By Proposition 3, $z(s) = O(1)$ as $s \to \infty$. To estimate z' we write

$$z'(s) = \frac{d}{ds}[x(t)(\varphi(t))^{-\delta}] = O(h'(\exp\{s\})\exp\{(1-\tilde{\theta})s\}) \qquad \text{as } s \to \infty.$$

Then we consider equation (2.4) in the form
(2.5)

$$z(s) = k\lambda^{\delta} z(s-c) + \frac{z'(s) + \delta z(s) - q(h(\exp\{s\}))h'(\exp\{s\})\exp\{s\}\exp\{-\delta s\}}{-p(h(\exp\{s\}))h'(\exp\{s\})\exp\{s\}}.$$

Using the asymptotic properties

$$q(h(\exp\{s\})) = O(\exp\{\beta s\}), \quad \frac{1}{p(h(\exp\{s\}))} = O(\exp\{\alpha s\}) \qquad \text{as } s \to \infty$$

and the above derived estimates on z and z' we can rewrite (2.5) into the asymptotic form

$$z(s) = k\lambda^{\delta} z(s-c) + O(\exp\{-\sigma s\}) \qquad \text{as } s \to \infty,$$

where $\sigma = \min(\tilde{\theta} - \alpha, \delta - \alpha - \beta) > 0$. To prove the required asymptotic bounds we utilize the previous Proposition 4.

First let $\omega > \alpha + \beta$. If $\omega < \beta + \tilde{\theta}$, then $\sigma = \delta - \alpha - \beta$ for a suitable $\delta \in (\omega, \beta + \tilde{\theta})$, $z(s) = O(\exp\{(\omega - \delta)s\})$ as $s \to \infty$, hence $x(t) = O((\varphi(t))^{\omega})$ as $t \to \infty$. If $\omega \geq \beta + \tilde{\theta}$, then $\sigma = \tilde{\theta} - \alpha$ for a suitable $\delta \in (\omega, \omega + \tilde{\theta} - \alpha)$, $z(s) = O(\exp\{(\omega - \delta)s\})$ as $s \to \infty$, hence $x(t) = O((\varphi(t))^{\omega})$ as $t \to \infty$.

Now let $\omega = \alpha + \beta$. If we take any $\delta \in (\omega, \omega + \tilde{\theta} - \alpha)$, then $\sigma = \delta - \alpha - \beta$, $z(s) = O(s\exp\{(\omega - \delta)s\})$ as $s \to \infty$, hence $x(t) = O((\varphi(t))^{\omega}\log\varphi(t))$ as $t \to \infty$.

Finally let $\omega < \alpha + \beta$. If we take a suitable $\delta \in (\alpha + \beta, \tilde{\theta} + \beta)$, then $\sigma = \delta - \alpha - \beta$, $z(s) = O(\exp\{(\alpha + \beta - \delta)s\})$ as $s \to \infty$, hence $x(t) = O((\varphi(t))^{\alpha+\beta})$ as $t \to \infty$. □

Remark. As we have already mentioned in Section 1, equation (1.1) with a positive coefficient at $x(t)$ was the subject of asymptotic investigations in [3]. It is natural to expect that the behaviour of the solutions at infinity described in [3] is quite different from that described in the previous Theorem. Indeed, if p is nondecreasing and q fulfils some asymptotic bounds, then for any solution x of (1.1) (with the positive sign at $x(t)$) there exists a (possibly zero) constant $L \in \mathbb{R}$ such that

$$\exp\left\{-\int_{t_0}^{\infty} p(s)ds\right\} x(t) \to L \qquad \text{as } t \to \infty \tag{2.6}$$

(see [3, Lemma 1]). Of course, the exponential behaviour (if $L \neq 0$) of solutions cannot be estimated via terms occuring in the previous Theorem. However, if relation (2.6) holds with $L = 0$, then using our notation we have $x(t) = O((\varphi(t)^{\omega})$ as $t \to \infty$ for any such solution (see [3, Lemma 2]).

3. Examples

In this section, we illustrate the presented asymptotic estimates in some particular cases of equation (1.1). First note that equation (1.1) involves the nonhomogeneous linear delay equation with constant coefficients

$$\dot{x}(t) = ax(t - \tau(t)) + bx(t) + q(t), \qquad t \in I, \tag{3.1}$$

where $a \neq 0$, $b < 0$ are real scalars. Applying the previous Theorem to equation (3.1) (i.e., with $p \equiv -b$ and $k = a/(-b)$) we obtain the estimates derived in [4, Theorem 2.3]. Hence, we have generalized these estimates to the nonautonomous equation (1.1) (moreover, some assumptions considered in [4] have been weakened).

Another particular case of (1.1) is the corresponding homogeneous equation, i.e. equation of the form (1.4) investigated in [5]. In the cited paper the asymptotics of all solutions of (1.4) are related to the behaviour of the function $(\varphi(t))^\omega$ occuring in the previous Theorem. Here we have posed conditions on q and its derivative under which a similar asymptotic formula holds for the solutions of the nonhomogeneous equation (1.1).

Now we pay our attention to equation (1.1) with a specified form of a lag. To apply the conclusions of Theorem we should be able to solve the corresponding Schröder equation. As an illustration we consider equation (1.1) with a linearly transformed argument (i.e., $t - \tau(t) = \lambda t + \mu$) and equation (1.1) with a power argument (i.e., $t - \tau(t) = t^\xi$). In both cases we can find the required solution of (1.5), (1.6) as follows:

$$\begin{aligned} \varphi(\lambda t + \mu) = \lambda\varphi(t) \quad &\rightarrow \quad \varphi(t) = t - \mu/(1-\lambda), \\ \varphi(t^\xi) = \xi\varphi(t) \quad &\rightarrow \quad \varphi(t) = \log t. \end{aligned}$$

In the general case (if we cannot find the explicit form of a solution of the Schröder equation) we can either use some asymptotic treatment on this equation (see [12]) or replace this functional equation by the corresponding functional inequality.

Corollary 1. *Let x be a solution of*

$$\dot{x}(t) = -p(t)[x(t) - kx(\lambda t + \mu)] + q(t), \qquad t \in I = [\mu/(1-\lambda), \infty), \tag{3.2}$$

where $k \neq 0$, $0 < \lambda < 1$, $\mu \in \mathbb{R}$, $p, q \in C^1(I)$, $p(t) \geq L/t^\alpha$, $\dot{p}(t) \leq Mp(t)/t^\theta$ for suitable reals $L, M > 0$ and $\alpha < \theta \leq 1$. If $q(t) = O(t^\beta)$ as $t \to \infty$, $\dot{q}(t) = O(t^{\beta-\tilde{\theta}})$ as $t \to \infty$ for suitable reals β, $\alpha < \tilde{\theta} < \theta$ and if $\omega := \log|k|/\log\lambda^{-1}$, then

$$\begin{aligned} x(t) &= O(t^\omega) && \text{as } t \to \infty && \text{if } \alpha + \beta < \omega, \\ x(t) &= O(t^\omega \log t) && \text{as } t \to \infty && \text{if } \alpha + \beta = \omega, \\ x(t) &= O(t^{\alpha+\beta}) && \text{as } t \to \infty && \text{if } \alpha + \beta > \omega. \end{aligned} \tag{3.3}$$

Proof. The assertion follows from Theorem applied to equation (3.2). □

Example 1. Consider the equation

$$\dot{x}(t) = -t^{-\alpha}[x(t) - kx(\lambda t + \mu)] + t^\beta, \qquad t \in I = [\mu/(1-\lambda), \infty), \tag{3.4}$$

where $k \neq 0$, $0 < \lambda < 1$, $\alpha < 1$ and β, μ are reals. If $\omega := \log|k|/\log\lambda^{-1}$, then estimates (3.3) hold for any solution x of (3.4).

Quite similarly as Corollary 1 we can derive

Corollary 2. *Let x be a solution of*

$$\dot{x}(t) = -p(t)[x(t) - kx(t^{\xi})] + q(t), \qquad t \in I = [1, \infty), \tag{3.5}$$

where $k \neq 0$, $0 < \xi < 1$, $p, q \in C^1(I)$, $p(t) \geq L/(\log t)^{\alpha}$, $\dot{p}(t) \leq Mp(t)/(\log t)^{\theta}$ for suitable reals $L, M > 0$ and $\alpha < \theta \leq 1$. If $q(t) = O((\log t)^{\beta})$ as $t \to \infty$, $\dot{q}(t) = O((\log t)^{\beta - \tilde{\theta}})$ as $t \to \infty$ for reals β, $\alpha < \tilde{\theta} < \theta$ and if $\omega := \log|k| / \log \xi^{-1}$, then

$$\begin{aligned} &x(t) = O((\log t)^{\omega}) \qquad \text{as } t \to \infty \qquad \text{if } \alpha + \beta < \omega, \\ &x(t) = O((\log t)^{\omega} \log\log t) \qquad \text{as } t \to \infty \qquad \text{if } \alpha + \beta = \omega, \\ &x(t) = O((\log t)^{\alpha+\beta}) \qquad \text{as } t \to \infty \qquad \text{if } \alpha + \beta > \omega. \end{aligned} \tag{3.6}$$

Example 2. Consider the equation

$$\dot{x}(t) = -(\log t)^{-\alpha}[x(t) - kx(t^{\xi})] + (\log t)^{\beta}, \qquad t \in I = [1, \infty), \tag{3.7}$$

where $k \neq 0$, $0 < \xi < 1$, $\alpha < 1$ and β are reals. If $\omega := \log|k| / \log \xi^{-1}$, then estimates (3.6) hold for any solution x of (3.7).

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Technical University of Brno, Brno, Czech Republic
E-mail address: cermakh@um.fme.vutbr.cz

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