Electronic Journal of Qualitative Theory of Differential Equations
Proc. 7th Coll. QTDE, 2004, No. **11** 1-13;
http://www.math.u-szeged.hu/ejqtde/

Multiple Positive Solutions of a Boundary Value Problem for Ordinary Differential Equations

John R. Graef*[†], Chuanxi Qian[‡], Bo Yang[§]

This paper is dedicated to László Hatvani on the occasion of his sixtieth birthday.

Abstract

The authors consider the three point boundary value problem consisting of the nonlinear differential equation

$$u''''(t) = g(t)f(u), \quad 0 < t < 1, \tag{E}$$

and the boundary conditions

$$u(0) = u'(1) = u''(1) = u''(0) - u''(p) = 0. \tag{B}$$

Sufficient conditions for the existence of multiple positive solutions to the problem (E)–(B) are given.

This paper is in final form and no version of it will be submitted for publication elsewhere.

*Research supported in part by the University of Tennessee at Chattanooga Center of Excellence for Computer Applications.

[†]Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403
[‡]Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762
[§]Department of Mathematics, Kennesaw State University, Kennesaw, GA 30144

1 Introduction

In this paper, we consider the fourth order nonlinear ordinary differential equation

$$u''''(t) = g(t)f(u), \quad 0 < t < 1, \tag{E}$$

together with the boundary conditions

$$u(0) = u'(1) = u''(1) = u''(0) - u''(p) = 0. \tag{B}$$

Throughout the remainder of the paper, we assume that:

(C_1) $f : [0,\infty) \to [0,\infty)$ and $g : [0,1] \to [0,\infty)$ are continuous;

(C_2) $\int_0^1 g(t)dt > 0$;

(C_3) $p \in (0,1)$ is a fixed constant.

If $f(0) = 0$, then the boundary value problem (E)–(B) always has the trivial solution, but here we are only interested in positive solutions, i.e., a solution $x(t)$ such that $x(t) > 0$ on $(0,1)$. Moreover, in this paper, we wish to obtain results that imply the existence of multiple positive solutions.

Due to their important role in both theory and applications, boundary value problems for ordinary differential equations have generated a great deal of interest over the years. They are often used to model various phenomena in physics, biology, chemistry, and engineering. Equation (E), which is sometimes referred to as the beam equation, has been studied in conjunction with a variety of boundary conditions, and we refer the reader to the works of Love [19], Prescott [22], and Timoshenko [25] on elasticity, the monographs by Mansfield [21] and Soedel [24] on deformation of structures, and Dulácska [9] on the effects of soil settlement for various specific applications. For surveys of known results on various types of boundary value problems, we recommend the monographs by Agarwal [1] and Agarwal, O'Regan, and Wong [2]. Recent contributions to the literature on multipoint problems and/or the existence of multiple positive solutions include the papers of Agarwal and Wong [3], Avery et al. [4], Baxley and Haywood [5, 6], Chyan and Henderson [7], Davis et al. [8], Eloe and Henderson [10], Graef and Henderson [11], Graef et al. [12, 13, 14], He and Ge [16], Henderson and Thompson [17], Ma [20], Raffoul [23], Webb [26], and Wong [27].

Graef and Yang [15] and others have considered boundary conditions of the form

$$u(0) = u'(1) = u''(1) = u'''(0) = 0,$$

which can actually be considered as the limiting case of the conditions (B). In fact, $u''(0) - u''(p) = 0$, which is one of the boundary conditions in (B), implies that there exists $q \in (0,p)$ such that $u'''(q) = 0$. As $p \to 0^+$, we have $q \to 0^+$, and the condition

$$u''(0) - u''(p) = 0$$

"tends to" the condition

$$u'''(0) = 0.$$

The following result, known as Krasnosel'skii's Fixed Point Theorem [18], will be the main tool used to prove our existence results.

Theorem K. *Let $\mathcal{X}$ be a Banach space and let $\mathcal{P} \subset \mathcal{X}$ be a cone in $\mathcal{X}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{X}$ with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let*

$$L : \mathcal{P} \cap (\overline{\Omega_2} - \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that either

(i) $||Lu|| \leq ||u||$ *if* $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $||Lu|| \geq ||u||$ *if* $u \in \mathcal{P} \cap \partial\Omega_2$, *or*

(ii) $||Lu|| \geq ||u||$ *if* $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $||Lu|| \leq ||u||$ *if* $u \in \mathcal{P} \cap \partial\Omega_2$.

Then L has a fixed point in $\mathcal{P} \cap (\overline{\Omega_2} - \Omega_1)$.

In the next section, we define the Green's functions for the problem (E)–(B) and prove a lemma that provides estimates for the positive solutions of this boundary value problem. Section 3 contains our existence results for multiple positive solutions.

2 Green's Functions and Estimates for Solutions

The Green's function $G_1 : [0,1] \times [0,1] \to [0,\infty)$ for the boundary value problem

$$y'' = 0, \quad y(0) = y'(1) = 0$$

is given by

$$G_1(t,s) = \begin{cases} t, & t \leq s, \\ s, & s \leq t, \end{cases}$$

while the Green's function $G_2 : [0,1] \times [0,1] \to [0,\infty)$ for the problem

$$y'' = 0, \quad y(0) - y(p) = y(1) = 0$$

is given by

$$G_2(t,s) = \begin{cases} 1-t, & t \geq s \geq p, \\ 1-s, & s \geq t \text{ and } s \geq p, \\ \frac{s}{p}(1-t), & s \leq p \text{ and } t \geq s, \\ \dfrac{s-ps+pt-st}{p}, & t \leq s \leq p. \end{cases}$$

If we define $J : [0,1] \times [0,1] \to [0,\infty)$ by

$$J(t,s) = \int_0^1 G_1(t,v)G_2(v,s)dv,$$

then $J(t,s)$ is the Green's function for the problem (E)–(B). It is not difficult to see that solving the boundary value problem (E)–(B) is equivalent to solving the integral equation

$$u(t) = \lambda \int_0^1 J(t,s)g(s)f(u(s))ds, \ 0 \leq t \leq 1, \tag{I}$$

as well as being equivalent to solving the problem

$$u''(t) = -\lambda \int_0^1 G_2(t,s)g(s)f(u(s))ds, \quad u(0) = u'(1) = 0.$$

We define the functions

$$a(t) = \begin{cases} t, & 0 \leq t \leq p, \\ \frac{1}{1-p}(2t - t^2 - p), & p \leq t \leq 1, \end{cases}$$

and

$$b(t) = t^3 - 3t^2 + 3t.$$

These functions will be used in the following lemma to estimate the positive solutions of the problem (E)–(B). While a proof of this lemma actually appears in [14], we will include a proof here as well for the sake of completeness.

Lemma 1. If $x \in C^4[0,1]$,

$$x(0) = x'(1) = x''(1) = x''(0) - x''(p) = 0,$$

and

$$x''''(t) \geq 0 \quad \text{and} \quad x''''(t) \not\equiv 0 \quad \text{on} \quad (0,1),$$

then

$$x(1) > x(t) > 0 \text{ for } t \in (0,1), \tag{1}$$

$$x'(t) > 0 \text{ on } [0,1), \tag{2}$$

$$x''(t) < 0 \text{ on } [0,1), \tag{3}$$

$$x(t) \geq a(t)x(1) \text{ on } [0,1], \tag{4}$$

and

$$x(t) \leq b(t)x(1) \text{ on } [0,1]. \tag{5}$$

Proof. Since $x''(0) = x''(p)$, there exists $q \in (0,p)$ such that $x'''(q) = 0$. We then have

$$x'''(t) \leq 0 \text{ on } (0,q),$$

$$x'''(t) \geq 0 \text{ on } (q,1),$$

and

$$x'''(t) \not\equiv 0 \text{ on } [0,1].$$

We will first show that $x''(q) < 0$. Since $x''''(t) \geq 0$ on $[0,1]$, $x''(t)$ is concave upwards there. Now $x''(1) = 0$, so it follows that $x''(q) \leq 0$. Thus, we just need to show that $x''(q) \neq 0$.

Suppose $x''(q) = 0$. Then, $x'''(t) \geq 0$ on $(q,1)$ and $x''(1) = 0$ imply $x''(t) \equiv 0$ on $(q,1)$, and so $x''(p) = 0$. Thus, we have that $x''(0) = x''(p) = 0$, and this means that $x''(t) \equiv 0$ on $(0,q)$. Therefore, $x''(t) \equiv 0$ on $[0,1]$, so $x''''(t) \equiv 0$ on $[0,1]$. This contradiction shows that $x''(q) < 0$.

We know that $x''(t)$ is concave upwards, and since $x''(1) = 0$ and $x''(q) < 0$, we have $x''(t) < 0$ on $(q,1)$. Hence, $x''(p) < 0$, which means that $x''(0) < 0$. Since $x''(0) = x''(p) < 0$ and $x''(t)$ is concave up, we have $x''(t) < 0$ on $(0,p)$. Thus, we have proved that $x''(t) < 0$ on $[0,1)$. Since $x'(1) = 0$, we have $x'(t) > 0$ on $[0,1)$, which implies that $0 < x(t) < x(1)$ for $t \in (0,1)$. Therefore, (1)–(3) hold.

With no loss in generality in the remainder of the proof, we may assume that $x(1) = 1$. In order to prove (5), we let

$$y(t) = b(t) - x(t) = t^3 - 3t^2 + 3t - x(t) \quad \text{for} \quad 0 \leq t \leq 1.$$

Then,

$$y'(t) = 3t^2 - 6t + 3 - x'(t),$$

$$y''(t) = 6t - 6 - x''(t),$$

$$y'''(t) = 6 - x'''(t),$$

and

$$y''''(t) = -x''''(t).$$

It follows that

$$y(0) = y(1) = 0,$$

$$y'(1) = 0,$$

$$y''(1) = 0,$$

and

$$y''''(t) \le 0 \quad \text{and} \quad y''''(t) \not\equiv 0 \quad \text{for} \quad t \in (0,1).$$

Now $y(0) = y(1) = 0$, so there exists $r_1 \in (0,1)$ such that $y'(r_1) = 0$. Since $y'(r_1) = y'(1) = 0$, we see that there exists $r_2 \in (r_1, 1)$ such that $y''(r_2) = 0$. The fact that $y''(1) = y''(r_2) = 0$ implies there exists $r_3 \in (r_2, 1)$ such that $y'''(r_3) = 0$. We then have

$$y'''(t) \ge 0 \text{ on } (0, r_3),$$

$$y'''(t) \le 0 \text{ on } (r_3, 1),$$

and

$$y'''(t) \not\equiv 0 \text{ on } (0,1).$$

Because $y''(1) = y''(r_2) = 0$, we have

$$y''(t) \le 0 \text{ on } (0, r_2),$$

$$y''(t) \ge 0 \text{ on } (r_2, 1),$$

and

$$y''(t) \not\equiv 0 \text{ on } (0,1).$$

We then have $y'(r_1) = y'(1) = 0$, so

$$y'(t) \ge 0 \text{ on } (0, r_1),$$

$$y'(t) \le 0 \text{ on } (r_1, 1),$$

and

$$y'(t) \not\equiv 0 \text{ on } (0,1).$$

And finally, $y(0) = y(1) = 0$, so we have

$$y(t) > 0 \text{ for } t \in (0,1).$$

Thus, (5) is proved.

To prove (4), note that $x(0) = 0$, $x(1) = 1$, and $x(t)$ is concave down, so we have that $x(t) \geq t$ for each $t \in [0, 1]$. Thus, $x(t) \geq a(t)$ on $[0, p]$. For $t \in [p, 1]$, we define

$$z(t) = x(t) - a(t) = x(t) - \frac{1}{1-p}(2t - t^2 - p).$$

It suffices to show that $z(t) > 0$ for $t \in (p, 1)$. We have

$$z'(t) = x'(t) - \frac{1}{1-p}(2 - 2t),$$

$$z''(t) = x''(t) + \frac{2}{1-p},$$
$$z'''(t) = x'''(t),$$

and

$$z''''(t) = x''''(t).$$

Hence,

$$z(p) > 0, \quad z(1) = 0,$$
$$z'(1) = 0,$$
$$z''(1) > 0,$$

and

$$z'''(t) \geq 0 \text{ on } (p, 1) \subset (q, 1).$$

There are two possibilities for z':

(i) $z'(t) \leq 0$ for each $t \in [p, 1]$, or

(ii) there exists $r_4 \in (p, 1)$ such that

$$z'(t) \geq 0 \text{ on } (p, r_4),$$

$$z'(t) \leq 0 \text{ on } (r_4, 1).$$

Since $z(p) > 0$ and $z(1) = 0$, in either case we have $z(t) > 0$ for $t \in [p, 1]$ so (4) holds, and this completes the proof of the lemma.

3 Existence of Multiple Positive Solutions

For our Banach space, we take $\mathcal{X} = C[0,1]$ with the norm

$$\|x\| = \max_{t\in[0,1]} |x(t)|, \ x \in \mathcal{X},$$

and we see that

$$\mathcal{P} = \{x \in \mathcal{X} \mid x(1) \geq 0, \ x(t) \text{ is nondecreasing}, \ a(t)x(1) \leq x(t) \leq b(t)x(1) \text{ on } [0,1]\}$$

is a positive cone in $\mathcal{X}$. Moreover, if $x \in \mathcal{X}$, then $\|x\| = x(1)$. Define the operator $T : \mathcal{P} \to \mathcal{X}$ by

$$Tu(t) = \int_0^1 J(t,s)g(s)f(u(s))ds, \ 0 \leq t \leq 1, \text{ for all } u \in \mathcal{P}.$$

By arguments similar to those used in the proof of Lemma 1, it is not difficult to show that $T(\mathcal{P}) \subset \mathcal{P}$. In addition, a standard argument shows that $T : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator. In view of (I), it is easy to see that solving the boundary value problem (E)–(B) is equivalent to finding a fixed point of the operator T in $\mathcal{P}$.

Next, we define the constant

$$K = \int_0^1 J(1,s)g(s)ds,$$

and for each $r \in (0,1)$, we let

$$L(r) = \int_r^1 J(1,s)g(s)ds.$$

The following two lemmas are needed to prove our main results.

Lemma 2. If $c > 0$, $f(z) \leq \frac{c}{K}$ for $z \in [0,c]$, and $x \in \mathcal{P}$ with $\|x\| = c$, then $\|Tx\| \leq c$.

Proof. If $x \in \mathcal{P}$ with $\|x\| = c$, then

$$\begin{aligned} \|Tx\| &= (Tx)(1) \\ &\leq \int_0^1 J(1,s)g(s)f(x(s))ds \\ &\leq \frac{c}{K}\int_0^1 J(1,s)g(s)ds \\ &= c. \end{aligned}$$

The proof of the lemma is now complete.

Lemma 3. If $c > 0$, $r \in (0,1)$, $f(z) \geq \frac{c}{L(r)}$ for $z \in [ca(r), c]$, and $x \in \mathcal{P}$ with $\|x\| = c$, then $\|Tx\| \geq c$.

Proof. If $x \in \mathcal{P}$ with $\|x\| = c$, then, for each $t \in [r, 1]$, we have

$$x(t) \geq a(t)\|x\| \geq a(r)\|x\| = ca(r).$$

Thus,

$$\begin{aligned}\|Tx\| = (Tx)(1) &= \int_0^1 J(1,s)g(s)f(x(s))ds \\ &\geq \int_r^1 J(1,s)g(s)f(x(s))ds \\ &\geq \frac{c}{L(r)} \int_r^1 J(1,s)g(s)ds \\ &= b.\end{aligned}$$

This completes the proof of the lemma.

We are now ready to prove our existence results.

Theorem 1. If there are constants $0 < c_1 < c_2 < c_3 < c_4$ and r_2, $r_3 \in (0,1)$ such that

1. $f(z) \leq \frac{c_i}{K}$ for $z \in [0, c_i]$, $i = 1, 4$, and
2. $f(z) \geq \frac{c_i}{L(r_i)}$ for $z \in [c_i a(r_i), c_i]$, $i = 2, 3$,

then the boundary value problem (E)–(B) has at least two positive solutions.

Proof. Define

$$\Omega_i = \{x \in \mathcal{X} \mid \|x\| < c_i\}, \ i = 1, 2, 3, 4.$$

By Lemmas 2 and 3, we have

$$\|Tu\| \leq \|u\| \text{ for } u \in \mathcal{P} \cap \partial\Omega_i, \ i = 1, 4,$$

$$\|Tu\| \geq \|u\| \text{ for } u \in \mathcal{P} \cap \partial\Omega_i, \ i = 2, 3,$$

and

$$\overline{\Omega_1} \subset \Omega_2 \text{ and } \overline{\Omega_3} \subset \Omega_4.$$

By Theorem K, T has two fixed points, one in $\mathcal{P} \cap (\overline{\Omega_4} - \Omega_3)$ and one in $\mathcal{P} \cap (\overline{\Omega_2} - \Omega_1)$. This completes the proof of the theorem.

In a similar fashion, we can prove the following result.

Theorem 2. If there exist $0 < c_1 < c_2 < c_3 < c_4$ and r_1, $r_4 \in (0,1)$ such that

1. $f(z) \le \frac{c_i}{K}$ for $z \in [0, c_i]$, $i = 2, 3$, and

2. $f(z) \ge \frac{c_i}{L(r_i)}$ for $z \in [c_i a(r_i), c_i]$, $i = 1, 4$,

then the boundary value problem (E)–(B) has at least two positive solutions.

Theorems 1 and 2 are for the existence of two positive solutions. It is possible to prove similar results for three or four such solutions. In fact, for each positive integer n, we can impose conditions on f so that the problem (E)–(B) has at least n positive solutions, or even infinitely many positive solutions. Here is one such result.

Theorem 3. If there are constants $0 < c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < c_7 < c_8 < \cdots$ and r_2, r_3, r_6 ,r_7, r_{10}, $r_{11} \cdots \in (0, 1)$ such that

1. $f(z) \le \frac{c_i}{K}$ for $z \in [0, c_i]$, $i = 1, 4, 5, 8, 9, 12, 13, \cdots\cdots$, and

2. $f(z) \ge \frac{c_i}{L(r_i)}$ for $z \in [c_i a(r_i), c_i]$, $i = 2, 3, 6, 7, 10, 11, \cdots\cdots$,

then the boundary value problem (E)–(B) has infinitely many positive solutions.

In order to illustrate our results, we present the following example.

Example. Consider the boundary value problem

$$u''''(t) = g(t)f(u(t)), \tag{e_1}$$

$$u(0) = u'(1) = u''(1) = u''(0) - u''(\frac{1}{5}) = 0, \tag{b_1}$$

where

$$g(t) = t \quad \text{and} \quad f(u) = 10(1 + u^2).$$

We wish to apply Theorem 2 to show that the problem (e_1)–(b_1) has at least two positive solutions.

Choose $r_1 = r_4 = \frac{1}{2}$; then values of K, $L(r_1)$, $L(r_4)$, $a(r_1)$, and $a(r_4)$ become:

$$K = \frac{11}{225}, \quad L(r_1) = L(r_4) = \frac{29}{960}, \quad a(r_1) = a(r_4) = \frac{11}{16}.$$

With these values, Theorem 2 reads as follows.

Theorem 2$'$. If there exist $0 < c_1 < c_2 < c_3 < c_4$ such that

(a$'$) $f(z) \le \frac{225}{11} c_i$ on $[0, c_i]$, $i = 2, 3$.

(b$'$) $f(z) \ge \frac{960}{29} c_i$ on $[\frac{11}{16} c_i, c_i]$, $i = 1, 4$.

Then the problem (e_1)–(b_1) has at least two positive solutions.

It is easy to check that if we choose

$$c_1 = \frac{2}{10}, \quad c_2 = \frac{9}{10}, \quad c_3 = \frac{11}{10}, \quad \text{and} \qquad c_4 = 7,$$

then all the conditions in Theorem 2′ are satisfied. Thus, the problem (e_1)–(b_1) has at least two positive solutions.

References

[1] R. P. Agarwal, Focal Boundary Value Problems for Differential and Difference Equations, Kluwer Academic, Dordrecht, 1998.

[2] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, Positive Solutions of Differential, Difference, and Integral Equations, Kluwer Academic, Dordrecht, 1998.

[3] R. Agarwal and F. H. Wong, Existence of positive solutions for higher order boundary value problems, Nonlinear Studies **5** (1998), 15–24.

[4] R. I. Avery, J. M. Davis, and J. Henderson, Three symmetric positive solutions for Lidstone problems by a generalization of the Leggett-Williams theorem, Electron. J. Differential Equations, Vol. 2000 (2000), No. 40, pp. 1-15.

[5] J. Baxley and L. J. Haywood, Nonlinear boundary value problems with multiple solutions, Nonlinear Anal. **47** (2001), 1187-1198.

[6] J. Baxley and L. J. Haywood, Multiple positive solutions of nonlinear boundary value problems, Dynam. Contin. Discrete Impuls. Systems, to appear.

[7] C. J. Chyan and J. Henderson, Multiple solutions for (n, p) boundary value problems, Dynamic Systems Appl., to appear.

[8] J. M. Davis, P. Eloe, and J. Henderson, Triple positive solutions and dependence on higher order derivatives, J. Math. Anal. Appl., to appear.

[9] E. Dulácska, Soil Settlement Effects on Buildings, Developments in Geotechnical Engineering Vol. 69, Elsevier, Amsterdam, 1992.

[10] P. W. Eloe and J. Henderson, Positive solutions and nonlinear multipoint conjugate eigenvalue problems, Electron. J. Differential Equations Vol. 1997 (1997), No. 3, pp. 1–11.

[11] J. R. Graef and J. Henderson, Double solutions of boundary value problems for $2m^{th}$–order differential equations and difference equations, Comput. Math. Appl. **45** (2003), 873–885.

[12] J. R. Graef, C. Qian, and B. Yang, Multiple symmetric positive solutions of a class of boundary value problems for higher order ordinary differential equations, Proc. Amer. Math. Soc. **131** (2003), 577–585.

[13] J. R. Graef, C. Qian, and B. Yang, A three point boundary value problem for nonlinear fourth order differential equations, to appear.

[14] J. R. Graef, C. Qian, and B. Yang, Positive solutions of a three point boundary value problem for nonlinear differential equations, to appear.

[15] J. R. Graef and B. Yang, Existence and nonexistence of positive solutions of fourth order nonlinear boundary value problems, Appl. Anal. **74** (2000), 201–214.

[16] X. He and W. Ge, Triple solutions for second order three-point boundary value problems, J. Math. Anal. Appl. **268** (2002), 256–265.

[17] J. Henderson and H. B. Thompson, Multiple symmetric positive solutions for a second order boundary value problem, Proc. Amer. Math. Soc. **128** (2000), 2373–2379.

[18] M. A. Krasnosel'skii, Positive Solutions of Operator Equations, Noordhoff, Groningen, 1964.

[19] A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, Fourth Ed., Dover Publications, New York, 1944.

[20] R. Ma, Positive solutions of a nonlinear three-point boundary value problem, Electron. J. Differential Equations, Vol. 1998 (1998), No. 34, pp. 1–8.

[21] E. H. Mansfield, The Bending and Stretching of Plates, International Series of Monographs on Aeronautics and Astronautics, Vol. 6, Pergamon, New York, 1964.

[22] J. Prescott, Applied Elasticity, Dover Publications, New York, 1961.

[23] Y. N. Raffoul, Positive solutions of three point nonlinear second order boundary value problem, Electron. J. Qual. Theory Differ. Equ. Vol. 2002 (2002), No. 15, pp. 1–11.

[24] W. Soedel, Vibrations of Shells and Plates, Dekker, New York, 1993.

[25] S. P. Timoshenko, Theory of Elastic Stability, McGraw–Hill, New York, 1961

[26] J. R. L. Webb, Remarks on positive solutions of some three point boundary value problems, to appear.

[27] P. J. Y. Wong, Triple positive solutions of conjugate boundary value problems, Comput. Math. Appl. **36** (1998), 19–35.

(Received August 13, 2003)