*Electronic Journal of Qualitative Theory of Differential Equations*
Proc. 7th Coll. QTDE, 2004, No. **15** 1-9;
**http://www.math.u-szeged.hu/ejqtde/**

# On Stability for Numerical Approximations of Stochastic Ordinary Differential Equations*

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**Abstract**

Stochastic ordinary differential equations (SODE) represent physical phenomena driven by stochastic processes. Like for deterministic differential equations, various numerical schemes are proposed for SODE (see references). We will consider several concepts of stability and connection between them.

A.M.S. Classification codes. (1991) 60H10,65C05,65C20,65U05

Keywords. Stochastic ordinary differential equations, numerical methods, stability

## 1 Introduction

We consider the stochastic initial value problem (SIVP) for the Itô ordinary differential equations (SODE) given by

$$dX(t) = a(t, X(t))dt + \sum_{j=0}^{m} b_j(t, X(t))dW_j(t), \text{ for } 0 \leq t < +\infty, \ X(t) \in \mathbf{R}, \tag{1}$$

where

$$a : [0, +\infty) \times \mathbf{R} \to \mathbf{R}, \ b_j : [0, +\infty) \times \mathbf{R} \to \mathbf{R}, \ X(0) = x, \ (\in \mathbf{R}),$$

where $W_j(t)$ is one-dimensional Brownian motion, $1 \leq j \leq m$. Let $\mathcal{F}_t$ denote the increasing family of $\sigma$-algebras (filtration) generated by the Brownian motion $W_j(s), s \leq t$. Details about this stochastic object and corresponding calculus can be found in [1, 6].

It is known that in deterministic models we consider parameters which are completely known, though in the original problem one often has insufficient information on parameter values. These may fluctuate due to some external or internal 'noise', which is random- at least appears to be so. In such a way we move from deterministic problems to stochastic problems (stochastic ordinary differential equations (SODE)). Explicit solutions are not usually known for equation (1), so they must be solved numerically so we have to do a qualitative investigation on the boundedness and stability of their solutions. In the literature there are some works on that topic such as [2, 4, 5, 8].

In this paper, we shall be interested in obtaining stability properties for numerical approximations of strong solutions of SODE. The concepts of numerical stability in the quadratic mean-square sense, stochastical numerical stability, asymptotical numerical stability in the quadratic mean-square sense, asymptotical stochastical numerical stability are introduced. This paper is organized as follows: in Section 2 we define stability concepts for SODE, though in Section 3 we analyse stability properties for numerical approximations of SODE, Section 4 is devoted to the asymptotic stability properties for the same. In Section 5 we illustrate our results for geometric Brownian motion and Ornstein-Uhlenbeck process.

*Research supported by the Central Research Found of the Hungarian Academy of Sciences (Grant No. T26176) and The Ministry of Science and Tecnology, Croatia (Grant. No O37011)
This paper is in final form and no version of it will be submitted for publication elsewhere.
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## 2 Stability for SODE

We consider an Itô equation (1) with a steady solution $X_t \equiv 0$. This means that $a(t,0) = b_j(t,0) = 0, 1 \leq j \leq m$ holds. $X^{t_0,0}(t)$ means that $X^{t_0,0}(t_0) = 0$.

$\| \cdot \|$ denotes an Eucledean norm in $R^d$, or this is equivalent to absolute value in $R$.

The following definitions are due to Hasminski:

**Definition 2.1** *The steady state solution* $X^{t_0,0}(t) \equiv 0$ *of the equation (1) is said to be* **stochastically stable** *if for any* $\epsilon > 0$ *and* $t_0 \geq 0$

$$\lim_{x_0 \to 0} P(\sup_{t \geq t_0} \| X^{t_0,x_0}(t) \| \geq \epsilon) = 0.$$

**Definition 2.2** *The steady state solution* $X^{t_0,0}(t) \equiv 0$ *of the equation (1) is said to be* **stochastically asymptotically stable** *if, in addition to being stochastically stable,*

$$\lim_{x_0 \to 0} P(\lim_{t \to +\infty} \| X^{t_0,x_0}(t) \| = 0) = 1.$$

Notice, that in view of the 0-1 law the above probability is either equal to 1, or to 0.

For the general SODE (1) Kloeden and Platen (cf. [7]) gave the following definition:

**Definition 2.3** *The steady state solution* $X^{t_0,0}(t) \equiv 0$ *of the equation (1) is* **stable in the** *pth* **mean** *if for*

$$(\forall \epsilon > 0)\ (\forall t_0 > 0),\ (\exists \delta = \delta(t_0, \epsilon) > 0)$$

*such that*

$$E \| X^{t_0,x_0}(t) \|^p < \epsilon$$

*for all* $t \geq t_0$ *and* $\| x_0 \| < \delta$.

**Definition 2.4** *The steady state solution* $X^{t_0,0}(t) \equiv 0$ *of the equation (1) is* **asymptotically stable in the** *pth* **mean** *if in addititon, there exists a* $\delta = \delta(t_0)$ *such that*

$$\lim_{t \to +\infty} E \| X^{t_0,x_0}(t) \|^p = 0 \textit{ for all } \| x_0 \| < \delta.$$

The most frequently used case $p = 2$ is called the mean-square case and in sequel we focus our investigation to mean-square stability.

We suppose that the equation (1) has a unique, mean-square bounded strong solution $X(t)$.

## 3 Numerical Stability

Doing numerics for SODE it means that we have to define discretization.

The most usual deterministic time discretization of a bounded time interval $[0,T], T > 0$ is of the form $0 = t_0 < t_1 < \ldots < t_N = T$, where $N$ is a natural number. The differences $\Delta_n = t_{n+1} - t_n$ are defined as step sizes.

We recall the definition of the general time discretization from [7]: for a given maximum step size $\delta \in (0, \delta_0)$ we define a time discretization

$$(\tau)_\delta = \{\tau_n : n = 0, 1, \ldots\}$$

as a sequence of time instants $\{\tau_n : n = 0, 1, \ldots\}$ which may be random, satisfying

$$\begin{aligned} 0 = \tau_0 < \tau_1 < \quad \ldots \quad &< \tau_N < \ldots < +\infty, \\ \sup_n \{\tau_{n+1} \quad - \quad &\tau_n\} \leq \delta, \\ n_t \quad < \quad &+\infty \end{aligned}$$

w.p.1 for all $t \in R^+$, where $\tau_{n+1}$ is $\mathcal{F}_{\tau_n}$-measurable for each $n = 0, 1, 2, \ldots$ and $n_t$ is defined as follows

$$n_t = \max\{n = 0, 1, \ldots : \tau_n \leq t\}.$$

Here we give the definition of a time discrete approximation as in [7].

**Definition 3.1** *We shall say that a right continuous with left hand limits process* $\mathbf{Y} = (Y(t), t \geq 0)$ *is a time discrete approximation of the solution of equation (1) with maximum step size* $\delta \in (0, \delta_0)$ *if it is based on a time discretization* $(\tau)_\delta$ *such that* $Y(\tau_n)$ *is* $\mathcal{F}_{\tau_n}-$ *measurable and* $Y(\tau_{n+1})$ *can be expressed as a function of* $Y(\tau_0), Y(\tau_1), \ldots, Y(\tau_n), \tau_0, \tau_1, \ldots, \tau_n, \tau_{n+1}$ *and a finite number* $r$ *of* $\mathcal{F}_{\tau_{n+1}}-$ *measurable random variables* $Z_{n,j}$ *for* $j = 1, \ldots, r$ *and each* $n = 0, 1, \ldots$

In the sequel $Y_n^\Delta$ always denotes the approximation of $X(t_n)$ using a given numerical scheme with maximum step size $\Delta$. The most simplest form of the discretization is an equidistant discretization. We will omit the superscript $\Delta$ if there is no confusion.

The used label *Const* is independent of $\Delta$ and it may depend on $T$. In the sequel *Const* always denotes constants of this property.

Let the general multi-dimensional one-step scheme for a given time discretization be defined as

$$Y_{n+1} = Y_n + \psi(\tau_n, Y_{n+1}, Y_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}). \tag{2}$$

Here $Z_{n,1}, \ldots, Z_{n,r}$ are stochastic variables depending on $\Delta$ and measurable with respect to $\mathcal{F}_{t_{n+1}}$ evaluated at points of the partition $t_m, m = n, n+1$. Function $\psi$ is defined by

$$\begin{aligned} \psi &= \Delta \sum_{j=0}^{1} \beta_{1,j} \hat{f}_{n+j,1} + \Delta^2 \sum_{j=0}^{1} \beta_{2,j} \hat{f}_{n+j,2} + \ldots + \Delta^k \sum_{j=0}^{1} \beta_{k,j} \hat{f}_{n+j,k} + \\ & \quad Z_{n,1}\beta_{0,1} f_{n,1} + \ldots + Z_{n,r}\beta_{0,r} f_{n,r}, \end{aligned}$$

where $\hat{f}_{n+j,i} = \hat{f}_i(t_{n+j}, Y_{n+j}), f_{n+j,i} = f_i(t_{n+j}, Y_{n+j})$, and functions $f_i, \hat{f}_i$ depend on functions $a$, $b_j, j = 1, \ldots, m$ and derivatives of them.

To this end let

$$\phi(t_n, Y_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}),$$

be defined implicitly by

$$\phi(t_n, Y_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}) = \psi(t_n, \phi + Y_n, Y_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}). \tag{3}$$

The one-step formula (2) can then be written as

$$Y_{n+1} = Y_n + \phi(t_n, Y_n, \Delta_n, Z_{n,1}, \ldots, Z_{n,r}). \tag{4}$$

We have several concepts of the stability for numerical approximations of SODE. The concept of stochastical numerical stability is defined in [7].

**Definition 3.2** *Let* $\mathbf{Y}^\Delta$ *denotes a time discrete approximation (i. e. numerical solution) with an maximum step size* $\Delta > 0$, *starting at time* $0$ *at* $Y_0^\Delta$, $\hat{\mathbf{Y}}^\Delta$ *denotes the corresponding approximation (constructed using the same driving Brownian path) starting at time* $0$ *at* $\hat{Y}_0^\delta$. *We shall say that a time discrete* $\mathbf{Y}^\Delta$ *is* **stochastically numerically stable** *for a given stochastic differential equation if for any finite interval* $[0, T]$ *there exists a positive constant* $\Delta_0$ *such that for every* $\epsilon > 0$ *and every* $\Delta \in (0, \Delta_0)$

$$\lim_{\|Y_0^\Delta - \hat{Y}_0^\Delta\| \to 0} \sup_{0 \leq t \leq T} P(\| Y_{n_t}^\Delta - \hat{Y}_{n_t}^\Delta \| \geq \epsilon) = 0.$$

In paper [5] there was established the connections, which guarantee numerical stability, namely, the next theorem was proven.

**Theorem 3.3 (Stochastical numerical stability)** *Suppose that the time discretization is equidistant with step size* $\Delta$ *and the increment function* $\phi$ *a one-step approximation*

$$Y_{n+1} = Y_n + \phi(t_n, Y_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}) \tag{5}$$

*of the SODE (1) satisfies*

$$\begin{aligned} & (E \| E(\phi(t_n, Y_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}) - \phi(t_n, \hat{Y}_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}) | \mathcal{F}_{t_n}) \|^2)^{\frac{1}{2}} \\ \leq \; & Const \cdot (E(\| Y_n - \hat{Y}_n \|^2)^{\frac{1}{2}} \Delta, \end{aligned} \tag{6}$$

$$
\begin{aligned}
& (E \parallel \phi(t_n, Y_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) - \phi(t_n, \hat{Y}_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) - \\
- \quad & E(\phi(t_n, Y_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) | \mathcal{F}_{t_n}) + E(\phi(t_n, \hat{Y}_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) | \mathcal{F}_{t_n}) \parallel^2)^{\frac{1}{2}} \\
\leq \quad & Const \cdot (E \parallel Y_n - \hat{Y}_n \|^2)^{\frac{1}{2}} \Delta^{\frac{1}{2}},
\end{aligned} \tag{7}
$$

*where $Y_n$ and $\hat{Y}_n$ be two numerical approximation of the equation (1) of the form (8) and having initial values $Y_0$ and $\hat{Y}_0$, respectively.*

*Then a one-step approximation* $(\mathbf{Y})$ *is stochastically numerically stable.*

Namely, it means that the conditions (9) and (10) imply continuous dependence of initial values in probability.

The next Definition is given in [2].

**Definition 3.4** *Let $\mathbf{Y}^\Delta$ denotes a time discrete approximation (i. e. numerical solution) with a maximum step size $\Delta > 0$, starting at time $0$ at $Y_0^\Delta$, $\hat{\mathbf{Y}}^\Delta$ denotes the corresponding approximation (constructed using the same driving Brownian path) starting at time $0$ at $\hat{Y}_0^\Delta$. We shall say that a time discrete $\mathbf{Y}^\Delta$ is* **numerically stable in the quadratic mean-square sense** *for a given stochastic differential equation if for any finite interval $[0, T]$ there exists a positive constant $\Delta_0$ such that for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon, \Delta_0)$ and every $\Delta \in (0, \Delta_0)$*

$$
E \parallel Y_0^\Delta - \hat{Y}_0^\Delta \parallel^2 \leq \delta
$$

*then*

$$
E \parallel Y_{n_t}^\Delta - \hat{Y}_{n_t}^\Delta \parallel^2 \leq \epsilon
$$

*holds for every positive integers $n_t \leq N$.*

This means that numerical approximations continuously depend on initial values in mean-square sense.

**Remark 3.5** *If we examine our proof of the Theorem 3.6 given in paper [5], we notice that (using the same notions as there) there was proven the next estimation*

$$
D(t) = \sup_{0 \leq s \leq t} E(\parallel Y_{n_s}^\Delta - \hat{Y}_{n_s}^\Delta \parallel^2) \leq Const E \parallel Y_0^\Delta - \hat{Y}_0^\Delta \parallel^2, \forall t, 0 \leq t \leq T.
$$

*Now, using the definition above we notice that we actually prove the numerical stability in the quadratic mean-square sense.*

**Theorem 3.6 (Numerical stability in mean-square sense)** *Suppose that the time discretization is equidistant with step size $\Delta$ and the increment function $\phi$ a one-step approximation*

$$
Y_{n+1} = Y_n + \phi(t_n, Y_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) \tag{8}
$$

*of the SODE (1) satisfies*

$$
\begin{aligned}
& (E \parallel E(\phi(t_n, Y_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) - \phi(t_n, \hat{Y}_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) | \mathcal{F}_{t_n}) \parallel^2)^{\frac{1}{2}} \\
\leq \quad & Const \cdot (E(\parallel Y_n - \hat{Y}_n \|^2)^{\frac{1}{2}} \Delta,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
& (E \parallel \phi(t_n, Y_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) - \phi(t_n, \hat{Y}_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) - \\
- \quad & E(\phi(t_n, Y_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) | \mathcal{F}_{t_n}) + E(\phi(t_n, \hat{Y}_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) | \mathcal{F}_{t_n}) \parallel^2)^{\frac{1}{2}} \\
\leq \quad & Const \cdot (E \parallel Y_n - \hat{Y}_n \|^2)^{\frac{1}{2}} \Delta^{\frac{1}{2}},
\end{aligned} \tag{10}
$$

*where $Y_n$ and $\hat{Y}_n$ be two numerical approximations of the equation (1) of the form (8) and having initial values $Y_0$ and $\hat{Y}_0$, respectively.*

*Then a one-step approximation* $(\mathbf{Y})$ *is numerically stable in the quadratic mean-square sense.*

# 4 Asymptotical Numerical Stability

As in deterministic case, the numerical stability in the quadratic mean-square sense for one-step numerical approximations does not tell us how to pick an appropriate step size $\Delta$. We will investigate some other concepts because of that.

We notice the propagated error of a numerically stable scheme which is theoretically still under control, may in fact, become so unrealistically large as to make approximation useless for some practical purposes.

As an example in the case of the simulation of the first exit times we do not know the appropriate time interval in advance. Because of that we need to control the error propagation for an arbitrarily large interval, namely, for an interval $[t_0, +\infty)$. The concept of asymptotic stochastic numerical stability is defined in [7].

**Definition 4.1** *We shall say that a time discrete approximation* $\mathbf{Y}^\Delta$ *is* **asymptotically stochastically numerically stable** *for a given stochastic ordinary differential equation if it is stochastically numerically stable and there exists a positive constant* $\Delta_a$ *such that for each* $\epsilon > 0$ *and* $\Delta \in (0, \Delta_a)$

$$\lim_{\| Y_0^\Delta - \hat{Y}_0^\Delta \| \to 0} \lim_{T \to +\infty} P( \sup_{0 \le t \le T} \| Y_{n_t}^\Delta - \hat{Y}_{n_t}^\Delta \| \ge \epsilon) = 0,$$

*where we have used the same notation as in Definition (3.2).*

**Definition 4.2** *We shall say that a time discrete approximation* $\mathbf{Y}^\Delta$ *is* **asymptotically numerically stable in the quadratic mean-square sense** *for a given stochastic ordinary differential equation if it is numerically stable in the quadratic mean-square sense and there exists a positive constant* $\Delta_a$ *such that for each* $\epsilon > 0$ *and* $\Delta \in (0, \Delta_a)$

$$\lim_{E\| Y_0^\Delta - \hat{Y}_0^\Delta \|^2 \to 0} \lim_{T \to +\infty} E( \sup_{0 \le t \le T} \| Y_{n_t}^\Delta - \hat{Y}_{n_t}^\Delta \|^2) = 0,$$

*where we have used the same notation as in Definition (3.2).*

**Theorem 4.3** *A time discrete approximation* $\mathbf{Y}^\Delta$ *which is asymptotically numerically stable in the quadratic mean-square sense is asymptotically stochastically numerically stable.*

**Proof of Theorem** Using Chebysev inequality we get

$$P( \sup_{0 \le t \le T} \| Y_{n_t}^\Delta - \hat{Y}_{n_t}^\Delta \| \ge \epsilon) \le \frac{E(\sup_{0 \le t \le T} \| Y_{n_t}^\Delta - \hat{Y}_{n_t}^\Delta \|^2)}{\epsilon^2}.$$

and using the definition for asymptotical numerical stability in quadratic mean-square sense we will easily prove asymptotical stochastical numerical stability property. $\Box$

Here we give a fundamental theorem on the asymptotic numerical stability in the quadratic mean-square sense:

**Theorem 4.4 (Asymptotic numerical stability in the quadratic mean-square sense)** *Suppose that the time discretization is equidistant with step size* $\Delta$ *and the increment function* $\phi$ *of a one-step approximation*

$$Y_{n+1} = Y_n + \phi(t_n, Y_n, \Delta, Z_{n,1}, \dots, Z_{n,r}) \tag{11}$$

*of the SODE (1) satisfies*

$$\begin{aligned} & E(\phi(t_n, x, \Delta, Z_{n,1}, \dots, Z_{n,r}) - \phi(t_n, y, \Delta, Z_{n,1}, \dots, Z_{n,r}) | \mathcal{F}_{t_n}) \\ = \; & C_1(\Delta) \cdot (x - y), \end{aligned} \tag{12}$$

$$\begin{aligned} & E \| \phi(t_n, x, \Delta, Z_{n,1}, \dots, Z_{n,r}) - \phi(t_n, y, \Delta, Z_{n,1}, \dots, Z_{n,r}) \|^2 \\ \le \; & C_2(\Delta) \cdot \| x - y \|^2, \end{aligned} \tag{13}$$

*where* $x$ *and* $y$ *are real numbers from* $R$ *and* $C_1(\Delta)$ *and* $C_2(\Delta)$ *real continuous functions depending on* $\Delta$*, and it may depend on* $x$ *and* $y$*. If there exists* $\Delta_a > 0$ *such that for all* $\Delta \in (0, \Delta_a)$ *the next inequality*

$$0 < 1 + 2C_1(\Delta) + C_2(\Delta) < 1,$$

*is valid, then a one-step approximation* $(\mathbf{Y})$ *is asymptotically numerically stable in the quadratic mean-square sense.*

**Proof of Theorem** In the proof of the theorem we use the next theorem with $a_n(\Delta) = (1 + 2C_1(\Delta) + C_2(\Delta))^n$. $\square$

**Remark 4.5** *We notice that $C_2(\Delta)$ must be positive constant but for the validity of the condition above we need that $C_1(\Delta)$ be negative constant.*

**Theorem 4.6** *Suppose that the time discretization is equidistant with step size $\Delta$. Let $\mathbf{Y}$ denotes a time discrete approximation (i. e. numerical solution) starting at time $0$ at $Y_0$, $\hat{\mathbf{Y}}$ denotes the corresponding approximation (constructed using the same driving Brownian path) starting at time $0$ at $\hat{Y}_0$. If there exists $\Delta_0$ such that for every $\Delta \in (0, \Delta_0)$*

$$E \parallel Y_n - \hat{Y}_n \parallel^2 \leq a_n(\Delta) E \parallel Y_0 - \hat{Y}_0 \parallel^2,$$

*for every $n = 0, 1, 2, \ldots,$ where $a_n(\Delta)$ is continuous and positive function of $\Delta$ such that $\sum_{n=0}^{+\infty} a_n(\Delta) = b(\Delta)$ converges and $\{a_n(\Delta) : n = 0, 1, 2, \ldots; \Delta \in (0, \Delta_0)\}$ is bounded then a time discrete approximation $\mathbf{Y}$ is asymptotically numerically stable in the quadratic mean-square sense.*

**Proof of Theorem** We notice the validity of the next relation:

$$E(\sup_{0\leq t\leq T} \parallel Y_{n_t} - \hat{Y}_{n_t} \parallel^2) \quad \leq \quad \sum_{n=0}^{n_T} E(\parallel Y_n - \hat{Y}_n \parallel^2)$$

Now, we take limit as $T \to +\infty$, so

$$\begin{aligned} \lim_{E\|Y_0-\hat{Y}_0\|^2\to 0} \lim_{T\to+\infty} E(\sup_{0\leq t\leq T} \parallel Y_{n_t} \quad &- \quad \hat{Y}_{n_t} \parallel^2) \leq \\ &\leq \quad \lim_{E\|Y_0-\hat{Y}_0\|^2\to 0} \sum_{n=0}^{+\infty} E(\parallel Y_n - \hat{Y}_n \parallel)^2) \\ &\leq \quad \lim_{E\|Y_0-\hat{Y}_0\|^2\to 0} \sum_{n=0}^{+\infty} a_n(\Delta) E(\parallel Y_0 - \hat{Y}_0 \parallel)^2. \end{aligned}$$

Using the conditions of the Theorem we get

$$\lim_{E\|Y_0-\hat{Y}_0\|^2\to 0} E(\parallel Y_0 - \hat{Y}_0 \parallel)^2 \left( \sum_{n=0}^{+\infty} a_n(\Delta) \right) = 0.$$

We need to prove the numerical stability in the quadratic mean-square sense of the time discrete approximation $(\mathbf{Y})$ under this condition.

Using the well-known relations from the convergence of the $\sum_{n=0}^{+\infty} a_n(\Delta)$ we get that $\lim_{n\to+\infty} a_n(\Delta) = 0$ and the existence of the constant $K > 0$ such that $|a_n(\Delta)| < K$ for all $n = 0, 1, 2, \ldots$. We estimate

$$\begin{aligned} \sup_{0\leq t\leq T} E(\parallel Y_{n_t} - \hat{Y}_{n_t} \parallel^2) \quad &\leq \quad E(\parallel Y_0 - \hat{Y}_0 \parallel)^2 \{ \sup_{0\leq t\leq T} a_{n_t}(\Delta) \} \\ &\leq \quad K \cdot E(\parallel Y_0 - \hat{Y}_0 \parallel)^2. \end{aligned}$$

For every $\epsilon > 0$ there is a $\delta = \delta(\epsilon, \Delta_0)$ and every $\Delta \in (0, \Delta_0)$

$$E \parallel Y_0 - \hat{Y}_0 \parallel^2 \leq \delta$$

then

$$E \parallel Y_{n_t} - \hat{Y}_{n_t} \parallel^2 \leq \epsilon$$

holds for every positive integers $n_t \leq N$.

Now the numerical stability in the quadratic mean-square sense for $(\mathbf{Y})$ is proved. $\square$

## 5 Examples

**Example 5.1** *As an example (cf. eg. [9]), let us consider the one-dimensional stochastic equation*

$$dX(t) = \nu X(t)dt + \mu X(t)dW(t), \ \ X(0) = x_0, t \geq 0, \tag{14}$$

*where $\nu$ and $\mu$ are complex numbers.*

The exact solution of (14) is

$$X(t) = exp\left( (\nu - \frac{\mu^2}{2})t + \mu W(t) \right) x_0,$$

which is sometimes called geometric Brownian motion. It has the second moment

$$E|X(t)|^2 = exp((2Re(\nu) + |\mu|^2)t)|x_0|^2.$$

Schurz [9] and Saito and Mitsui [8] showed that the zero solution of the equation (14) is asymptotically mean-square stable if and only if

$$2Re(\nu) + |\mu|^2 < 0,$$

as we can see the form of the second moment of $X(t)$.

Generally, in the vector case, when we apply a numerical scheme $(Y_n)$ to the equation (14) and take the mean-square norm, we obtain a one-step difference equation of the form

$$E \parallel Y_{n+1} \parallel^2 = R(\overline{\Delta}, k) E \parallel Y_n \parallel^2, \tag{15}$$

where $\overline{\Delta} = \Delta\nu$ and $k = \frac{-\mu^2}{\nu}$.

Y. Saito and T. Mitsui in their work [8] called the function $R(\overline{\Delta}, k)$ as the stability function of the scheme. In this case $E \parallel Y_n \parallel^2 \rightarrow 0$ as $n \rightarrow +\infty$ iff

$$|R(\overline{\Delta}, k)| < 1.$$

They gave the next definition in their work [8]:

**Definition 5.2** *The scheme is said to be $MS$-stable for those values of $\overline{\Delta}$ and $k$ satisfying*

$$|R(\overline{\Delta}, k)| < 1. \tag{16}$$

*The set $\mathcal{R}$ given by*

$$\mathcal{R} = \{(\overline{\Delta}, k) : |R(\overline{\Delta}, k)| < 1\}$$

*is analogously called the domain of MS-stability of the scheme.*

Consider the recursive form of one-step schemes given in [8]

$$Y_{n+1} = Y_n + \phi(t_n, Y_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}).$$

We notice that in that case the increment function is linear, so

$$\phi(t_n, Y_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}) - \phi(t_n, \hat{Y}_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}) = \phi(t_n, Y_n - \hat{Y}_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}).$$

It means

$$Y_{n+1} - \hat{Y}_{n+1} = Y_n - \hat{Y}_n + \phi(t_n, Y_n - \hat{Y}_n, \Delta, Z_{n,1}, \ldots, Z_{n,r}).$$

Then the next equation

$$E \parallel Y_{n+1} - \hat{Y}_{n+1} \parallel^2 = R(\overline{\Delta}, k) E \parallel Y_n - \hat{Y}_n \parallel^2$$

is valid, where the function $R(\overline{\Delta}, k)$ is the same as in the case of MS-stability. The condition $|R(\overline{\Delta}, k)| < 1$ for MS-stability using theorem 4.6 guarantees the asymptotical numerical stability in quadratic mean-square sense. It means that any MS-stable one-step approximation is asymptotical numerical stable in quadratic mean-square sense, which implies asymptotical stochastical numerical stability.

**Example 5.3** *As we know from the theory of ODE, the numerical stability conditions of one-step methods for deterministic differential equations does not tell us how to pick an appropriate step size* $\Delta$*. Because of that we consider a class of test equations. These are the complex-valued linear differential equations*

$$\frac{dx}{dt} = \lambda x,$$

*with* $\lambda = \lambda_r + i\lambda_i$*.*

Usually, the one-step numerical approximations with $Re(\lambda) < 0$ in the recursive form

$$Y_{n+1} = G(\lambda\Delta)Y_n$$

is given.

We shall call the set of complex number $\lambda\Delta$ with $Re(\lambda) < 0$ and $|G(\lambda\Delta)| < 1$ the region of absolute stability of the scheme.

**Definition 5.4** *We shall say that a numerical method is A-stable if its region of absolute stability contains all of the left half of the complex plane, that is all* $\lambda\Delta$ *with* $Re(\lambda) < 0$ *and* $\Delta > 0$*.*

In stochastic case, we will consider the same complex-valued linear differential equations with an additive noise

$$dX(t) = \lambda X(t)dt + dW(t), \tag{17}$$

where the parameter $\lambda$ is complex number with real part $Re(\lambda) < 0$ and $\mathbf{W}$ is a real-valued standard Wiener process.

We have almost the same recursive form as in the above case

$$Y_{n+1} = G(\lambda\Delta)Y_n + Z_n, \tag{18}$$

for $n = 0, 1, 2, \ldots$ where G is a mapping of the complex plane $C$ into itself and $Z_0, Z_1, \ldots$ are random variable which do not depend on $\lambda$ or $Y_0, Y_1, \ldots$

We have the same definition of A-stability as in deterministic case, namely

**Definition 5.5** *We shall call the set of complex numbers* $\lambda\Delta$ *with* $Re(\lambda) < 0$ *and* $|G(\lambda\Delta)| < 1$ *the region of absolute stability of the scheme (18).*

*If this region coincides with the left half of the complex plane, we say the scheme is A-stable.*

The general recursive one-step approximation for a stochastic differential equation

$$dX(t) = \lambda X(t)dt + dW(t)$$

with a real constant $\lambda, \lambda < 0$ has an increment function

$$\phi(t_n, Y_n, \Delta, Z_n) = G(\lambda\Delta)Y_n - Y_n + Z_n,$$

with $E(Z_n) = 0$ and $E(Z_n^2) = \Delta$.

Now we estimate the conditions

$$E(\phi(t_n, Y_n, \Delta, Z_n) - \phi(t_n, \hat{Y}_n, \Delta, Z_n)|\mathcal{F}_{t_n}) = (G(\lambda\Delta) - 1)(Y_n - \hat{Y}_n),$$

$$E \parallel \phi(t_n, Y_n, \Delta, Z_n) - \phi(t_n, \hat{Y}_n, \Delta, Z_n) \parallel^2 = (G(\lambda\Delta) - 1)^2 \cdot E \parallel Y_n - \hat{Y}_n \|^2.$$

Using this conditions we get

$$E \parallel Y_{n+1} - \hat{Y}_{n+1} \parallel^2 = (1 + 2(G(\lambda\Delta) - 1) + (G(\lambda\Delta) - 1)^2)(E \parallel Y_n - \hat{Y}_n \parallel^2),$$

and now we get the condition

$$1 + 2(G(\lambda\Delta) - 1) + (G(\lambda\Delta) - 1)^2 < 1,$$

which implies $|G(\lambda\Delta)|^2 < 1$, and it means that $|G(\lambda\Delta)| < 1$. This condition is the same as in the case of A-stability. Our conclusion is that we have the condition which guarantees A-stability and implies asymptotical numerical stability in quadratic mean-square sense and asymptotical stochastical numerical stability.

In the simpliest case, for the Euler-Maruyama scheme for the test stochastic ordinary differential equation

$$dX_t = aX_t dt + dW(t),$$

we get the condition

$$0 < (1 + 2a\Delta + a^2\Delta^2) < 1,$$

which gives us two condition, namely $a < 0$ and $\Delta < \frac{-2}{a}$. It means that for an asymptotical numerical stability in quadratic mean-square sense we have the same bounds on step size as for A-stability for Euler-scheme and A-stability for Euler-Maruyama scheme.

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(Received September 30, 2003)