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Local analytic solutions to some nonhomogeneous problems with p-Laplacian*

Gabriella Bognár
University of Miskolc
Miskolc-Egyetemváros, 3515 Hungary

Abstract

Applying the Briot-Bouquet theorem we show that there exists an unique analytic solution to the equation $\left(t^{n-1}\Phi_p\left(y'\right)\right)'+(-1)^i t^{n-1}\Phi_q(y)=0$, on $(0,\ a)$, where $\Phi_r(y):=|y|^{r-1}y$, $0<r,p,q\in R^+$, $i=0,1$, $1\le n\in N$, a is a small positive real number. The initial conditions to be added to the equation are $y(0)=A\neq 0$, $y'(0)=0$, for any real number A. We present a method how the solution can be expanded in a power series for near zero.

1 Preliminaries

We consider the quasilinear differential equation

$$\Delta_p u+(-1)^i\ |u|^{q-1}u=0,\ u=u(x),\ x\in\mathbf{R}^n,$$

where $n\ge 1$, p and q are positive real numbers, $i=0,1$ and Δ_p denotes the $p-$Laplacian $\left(\Delta_p u=\operatorname{div}(|\nabla u|^{p-1}\nabla u)\right)$. If $n=1$, then the equation is reduced to

$$\left(\Phi_p\left(y'\right)\right)'+(-1)^i\Phi_q(y)=0,$$

where for $r\in\{p,q\}$

$$\Phi_r(y):=\begin{cases}|y|^{r-1}y, & \text{for } y\in\mathbf{R}\setminus\{0\}\\ 0, & \text{for } y=0.\end{cases}$$

We note that function Φ_r is an odd function. For $n>1$ we restrict our attention to radially symmetric solutions. The problem under consideration is reduced to

$$\left(t^{n-1}\Phi_p\left(y'\right)\right)'+(-1)^i t^{n-1}\Phi_q(y)=0,\ \text{on } (0,\ a) \tag{1}$$

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where $a > 0$. A solution of (1) means a function $y \in C^1(0,\ a)$ for which $t^{n-1}\Phi_p(y') \in C^1(0,\ a)$ and (1) is satisfied. We shall consider the initial values

$$\begin{aligned} y(0) = A \neq 0, \\ y'(0) = 0, \end{aligned} \tag{2}$$

for any $A \in \mathbf{R}$.

For the existence and uniqueness of radial solutions to (1) we refer to [9]. If $n = 1$ and $i = 0$, then it was showed that the initial value problem $(1) - (2)$ has a unique solution defined on the whole $\mathbf{R}$ (see [3], and [4]), moreover, its solution can be given in closed form in terms of incomplete gamma functions [4]. If $n = 1$, $i = 0$, Lindqvist gives some properties of the solutions [8]. If $n = 1$ and $p = q = 1$, then (1) is a linear differential equation, and its solutions are well-known:
if $i = 0$, the solution $(1) - (2)$ with $A = 1$ is the cosine function,
if $i = 1$, the solution $(1) - (2)$ with $A = 1$ is the hyperbolic cosine function,
and both the cosine and hyperbolic cosine functions can be expanded in power series.

In the linear case, when $n = 2$, $p = q = 1$, $i = 0$, the solution of $(1)-(2)$ with $A = 1$ is $J_0(t)$, the Bessel function of first kind with zero order, and for $n = 3$, $p = q = 1$, $i = 0$ then the solution of $(1) - (2)$ with $A = 1$ is $j_0(t) = \sin t/t$, called the spherical Bessel function of first kind with zero order.

In the cases above, for special values of parameteres n, p, q, i, we know the solution in the form of power series.

The type of singularities of $(1) - (2)$ was classified in [1] in the case when $i = 0$, and $p = q$. If $n = 1$, then a solution of (1) is not singular.

Our purpose is to show the existence of the solution of problem $(1) - (2)$ in power series form near the origin. We intend to examine the local existence of an analytic solution to problem $(1) - (2)$ and we give a constructive procedure for calculating solution y in power series near zero. Moreover we present some numerical experiments.

2 Existence of an unique solution

We will consider a system of certain differential equations, namely, the special Briot-Bouquet differential equations. For this type of differential equations we refer to the book of E. Hille [6] and E. L. Ince [7].

Theorem 1 *(Briot-Bouquet Theorem) Let us assume that for the system of equations*

$$\left.\begin{aligned} \xi \tfrac{dz_1}{d\xi} &= u_1(\xi, z_1(\xi), z_2(\xi)), \\ \xi \tfrac{dz_2}{d\xi} &= u_2(\xi, z_1(\xi), z_2(\xi)), \end{aligned}\right\} \tag{3}$$

where functions u_1 and u_2 are holomorphic functions of ξ, $z_1(\xi)$, and $z_2(\xi)$ near the origin, moreover $u_1(0,0,0) = u_2(0,0,0) = 0$, then a holomorphic solution *of* (3) *satisfying the initial conditions* $z_1(0) = 0$, $z_2(0) = 0$ *exists if none of the eigenvalues of the matrix*

$$\begin{bmatrix} \left.\frac{\partial u_1}{\partial z_1}\right|_{(0,0,0)} & \left.\frac{\partial u_1}{\partial z_2}\right|_{(0,0,0)} \\ \left.\frac{\partial u_2}{\partial z_1}\right|_{(0,0,0)} & \left.\frac{\partial u_2}{\partial z_2}\right|_{(0,0,0)} \end{bmatrix}$$

is a positive integer.

For a proof of Theorem 1 we refer to [2].

The differential equation (1) has singularity around $t = 0$ for the case $n > 1$. Theorem 1 ensures the existence of formal solutions $z_1 = \sum_{k=0}^{\infty} a_k \xi^k$ and $z_2 = \sum_{k=0}^{\infty} b_k \xi^k$ for system (3), and also the convergence of formal solutions.

We apply the method Parades and Uchiyama [10].

Theorem 2 *For any* $p \in (0, +\infty)$, $q \in (0, +\infty)$, $i = 0, 1$, $n \in \mathbf{N}$ *the initial value problem* (1) $y(0) = A$, $y'(0) = 0$ *has an unique analytic solution of the form* $y(t) = Q\left(t^{1+1/p}\right)$ *in* $(0, a)$ *for small real value of* a, *where* Q *is a holomorphic solution to*

$$Q'' = \frac{(-1)^{i+1}}{p\ (1+1/p)^{p+1}} t^{-\frac{p+1}{p}} \frac{\Phi_q(Q)}{|Q'|^{p-1}} - \frac{n}{p\ \alpha} t^{-(1+1/p)} Q'$$

near zero satisfying $Q(0) = A$, $Q'(0) = \frac{p}{p+1} \Phi_{1/p}\left[(-1)^{i+1} \Phi_q(A)/n\right]$.

Proof. We shall now present a formulation of (1) as a system of Briot-Bouquet type differential equations (3). Let us take solution of (1) in the form

$$y(t) = Q\left(t^{\alpha}\right),\ t \in (0, a),$$

where function $Q \in C^2(0, a)$ and α is a positive constant. Substituting $y(t) = Q\left(t^{\alpha}\right)$ into (1) we get that Q satisfies

$$Q''\left(t^{\alpha}\right) = \frac{(-1)^{i+1}}{p\ \alpha^{p+1}} t^{-(\alpha-1)(p+1)} \frac{\Phi_q(Q)}{|Q'|^{p-1}} - \frac{n - 1 + p\,(\alpha - 1)}{p\ \alpha} t^{-\alpha} Q'$$

and introducing variable ξ by $\xi = t^{\alpha}$ we have

$$Q''(\xi) = \frac{(-1)^{i+1}}{p\ \alpha^{p+1}} \xi^{-\frac{(\alpha-1)}{\alpha}(p+1)} \frac{\Phi_q(Q)}{|Q'|^{p-1}} - \frac{n - 1 + p\,(\alpha - 1)}{p\ \alpha} \xi^{-1} Q'. \tag{4}$$

Here, we introduce function Q as follows

$$Q(\xi) = \gamma_0 + \gamma_1 \xi + z(\xi), \tag{5}$$

where $z \in C^2(0,a)$, $z(0)=0$, $z'(0)=0$. Therefore Q has to fulfill the properties $Q(0)=\gamma_0$, $Q'(0)=\gamma_1$, $Q'(\xi)=\gamma_1+z'(\xi)$, $Q''(\xi)=z''(\xi)$. From initial condition $y(0)=A$ we have that

$$\gamma_0 = A.$$

We restate (4) as a system of equations:

$$\left.\begin{array}{c} z_1(\xi)=z(\xi) \\ z_2(\xi)=z'(\xi) \end{array}\right\} \text{ with } \left.\begin{array}{c} z_1(0)=0 \\ z_2(0)=0 \end{array}\right\},$$

according to (4) we get that

$$\begin{aligned} z''(\xi) &= \frac{(-1)^{i+1}}{p\ \alpha^{p+1}}\xi^{-\frac{(\alpha-1)}{\alpha}(p+1)}\frac{\Phi_q\left(\gamma_0+\gamma_1\xi+z(\xi)\right)}{\left|\gamma_1+z'(\xi)\right|^{p-1}} \\ &\quad -\frac{n-1+p\left(\alpha-1\right)}{p\ \alpha}\xi^{-1}\left(\gamma_1+z'(\xi)\right). \end{aligned}$$

We generate the system of equations

$$\left.\begin{array}{c} u_1(\xi,z_1(\xi),z_2(\xi))=\xi\ z_1'(\xi) \\ u_2(\xi,z_1(\xi),z_2(\xi))=\xi\ z_2'(\xi) \end{array}\right\}$$

as follows

$$\left.\begin{array}{l} u_1(\xi,z_1(\xi),z_2(\xi))=\xi\ z_2 \\ u_2(\xi,z_1(\xi),z_2(\xi))=\frac{(-1)^{i+1}}{p\ \alpha^{p+1}}\xi^{\frac{1-p(\alpha-1)}{\alpha}}\frac{\Phi_q(\gamma_0+\gamma_1\xi+z_1(\xi))}{|\gamma_1+z_2(\xi)|^{p-1}} \\ \qquad\qquad -\frac{n-1+p(\alpha-1)}{p\ \alpha}\left(\gamma_1+z_2(\xi)\right) \end{array}\right\}.$$

In order to satisfy conditions $u_1(0,0,0)=0$ and $u_2(0,0,0)=0$ we must get zero for the power of ξ in the right-hand side of the second equation:

$$\frac{1-p(\alpha-1)}{\alpha}=0,$$

i.e., $\alpha=\frac{1}{p}+1$. To ensure $u_2(0,0,0)=0$ we have the connection

$$n\ \Phi_p(\gamma_1)+\left(\frac{p}{p+1}\right)^p(-1)^i\Phi_q(\gamma_0)=0,$$

i.e.,

$$\gamma_1=(-1)^{i+1}\frac{p}{p+1}\Phi_{1/p}\left((-1)^{i+1}\frac{\Phi_q\left(\gamma_0\right)}{n}\right). \tag{6}$$

Therefore, taking into consideration that Φ_r is an even function for any $r \in \{p,q\}$, we obtain

$$\gamma_1=\begin{cases} \frac{p}{p+1}A^{q/p}(-1)^{i+1}\frac{1}{n^{1/p}} & \text{if } A>0, \\ \frac{p}{p+1}\left|A\right|^{q/p}(-1)^{i}\frac{1}{n^{1/p}} & \text{if } A<0. \end{cases} \tag{7}$$

From initial conditions $y(0) = A \neq 0$, $y'(0) = 0$, and (5) it follows that $\gamma_0 = A$.

For u_1 and u_2 we find that

$$\left.\frac{\partial u_1}{\partial z_1}\right|_{(0,0,0)} = 0, \qquad \left.\frac{\partial u_1}{\partial z_2}\right|_{(0,0,0)} = 0,$$

$$\left.\frac{\partial u_2}{\partial z_1}\right|_{(0,0,0)} = -\frac{p^p q \; |\gamma_0|^{q-1}}{(p+1)^{p+1} \, |\gamma_1|^{p-1}}, \qquad \left.\frac{\partial u_2}{\partial z_2}\right|_{(0,0,0)} = -\frac{n\,p}{p+1}.$$

Therefore the eigenvalues of matrix

$$\begin{bmatrix} \partial u_1/\partial z_1 & \partial u_1/\partial z_2 \\ \partial u_2/\partial z_1 & \partial u_2/\partial z_2 \end{bmatrix}$$

at $(0, 0, 0)$ are 0 and $-np/(p + 1)$. Since both eigenvalues are non-positive, applying Theorem 1 we get the existence of unique analytic solutions z_1 and z_2 at zero. Thus we get the analytic solution $Q(\xi) = \gamma_0 + \gamma_1\xi + z(\xi)$ satisfying (4) with $Q(0) = \gamma_0$, $Q'(0) = \gamma_1$, where $\gamma_0 = A$ and γ_1 is determined in (7). ■

Corollary 3 *From Theorem 2 it follows that solution $y(t)$ for* (1) *has an expansion near zero of the form* $y(t) = \sum_{k=0}^{\infty} a_k t^{k\left(\frac{1}{p}+1\right)}$ *satisfying* $y(0) = A$ *and* $y'(0) = 0$.

3 Determination of local solution

We give a method for the determination of power series solution of (1) – (2). For simplicity, we take $A = 1$. Thus initial conditions

$$y(0) = 1,$$
$$y'(0) = 0$$

are considered. We seek a solution of the form

$$y(t) = a_0 + a_1\, t^{\frac{1}{p}+1} + a_2\, t^{2\left(\frac{1}{p}+1\right)} + \ldots, \quad t > 0, \tag{8}$$

with coefficients $a_k \in \mathbf{R}$, $k = 0, 1, \ldots$. From Section 2 we get that $a_0 = \gamma_0 = 1$ and $a_1 = \gamma_1 = \frac{p}{p+1}(-1)^{i+1}\frac{1}{n^{1/p}}$. Near zero $y(t) > 0$ and $y'(t) < 0$ for $i = 0$, $y'(t) > 0$ for $i = 1$. Therefore

$$\Phi_q(y(t)) = y^q(t) = \left(a_0 + a_1\, t^{\frac{1}{p}+1} + a_2\, t^{2\left(\frac{1}{p}+1\right)} + \ldots\right)^q.$$

After differentiating (8), we get

$$y'(t) = t^{\frac{1}{p}} \left[a_1\left(\frac{1}{p}+1\right) + 2a_2\left(\frac{1}{p}+1\right)\, t^{\frac{1}{p}+1} + 3a_3\left(\frac{1}{p}+1\right)\, t^{2\left(\frac{1}{p}+1\right)} + \ldots\right],$$

and hence

$$\Phi_p(y'(t)) = (-1)^{i+1}\,(y'(t))^p$$
$$= (-1)^{i+1} t\ \left[a_1\left(\frac{1}{p}+1\right) + 2a_2\left(\frac{1}{p}+1\right)\ t^{\frac{1}{p}+1} + 3a_3\left(\frac{1}{p}+1\right)\ t^{2\left(\frac{1}{p}+1\right)} + \ldots\right]^p.$$

For $y^q(t)$ and $(y'(t))^p$

$$y^q(t) = A_0 + A_1\ t^{\frac{1}{p}+1} + A_2\ t^{2\left(\frac{1}{p}+1\right)} + \ldots \tag{9}$$
$$(y'(t))^p = t\ \left[B_0 + B_1\ t^{\frac{1}{p}+1} + B_2\ t^{2\left(\frac{1}{p}+1\right)} + \ldots\right], \tag{10}$$

where coefficients A_k and B_k can be expressed in terms of a_k $(k = 0, 1, \ldots)$.

Using (10) we obtain

$$\left(t^{n-1}\Phi_p\left(y'\right)\right)' = \left((-1)^{i+1} t^n\ \left[B_0 + B_1\ t^{\frac{1}{p}+1} + B_2\ t^{2\left(\frac{1}{p}+1\right)} + \ldots\right]\right)'$$
$$= (-1)^{i+1} t^{n-1}\left[B_0 n + B_1(n + \frac{1}{p} + 1)t^{\frac{1}{p}+1} + B_2(n + 2\left(\frac{1}{p}+1\right))\ t^{2\left(\frac{1}{p}+1\right)} + \ldots\right],$$

and substituing it to the equation (1) with (9) we get

$$(-1)^{i+1} t^{n-1}\left[B_0 n + B_1\left(n + \frac{1}{p} + 1\right) t^{\frac{1}{p}+1} + B_2\left(n + 2\left(\frac{1}{p}+1\right)\right)\ t^{2\left(\frac{1}{p}+1\right)} + \ldots\right]$$
$$+ (-1)^i\, t^{n-1}\left[A_0 + A_1\ t^{\frac{1}{p}+1} + A_2\ t^{2\left(\frac{1}{p}+1\right)} + \ldots\right] = 0.$$

Comparing the coefficients of the proper power of t we find

$$\begin{aligned}
B_0 n - A_0 &= 0,\\
B_1(n + \frac{1}{p} + 1) - A_1 &= 0,\\
B_2(n + 2\left(\frac{1}{p}+1\right)) - A_2 &= 0,\\
&\vdots\\
B_k(n + k\left(\frac{1}{p}+1\right)) - A_k &= 0, \qquad (11)\\
&\vdots
\end{aligned}$$

Applying the J. C. P. Miller formula (see [5]) for the determination of A_k and B_k $(k = 0, 1, \ldots)$ we have.

$$A_k = \frac{1}{k}\sum_{j=0}^{k-1}\left[(k-j)\,q - j\right] A_j a_{k-j}, \tag{12}$$
$$B_k = \frac{p}{a_1 k(p+1)}\sum_{j=0}^{k-1}\left[(k-j)\,p - j\right] B_j a_{k-j+1}\left[(k-j+1)\left(\frac{1}{p}+1\right)\right] \tag{13}$$

for any $k > 0$.

From initial condition $y(0) = 1$ we get $a_0 = 1$, $A_0 = 1$, and therefore

$$B_0 = \frac{1}{n}.$$

From (11) for $i = 1$ we get $B_1(n + \frac{1}{p} + 1) - A_1 = 0$, and evaluating A_1 from (12) and B_1 from (13) we find

$$B_0 = \left[a_1\left(\frac{1}{p} + 1\right)\right]^p,$$

thus

$$a_1 = \frac{p}{p+1}(-1)^{i+1}\frac{1}{n^{1/p}}.$$

Similarly, we determine coefficients a_k for all $k = 0, 1, ...$ from (11), (12) and (13).

Example 4 *Solve* (1) – (2) *for n=2; i=0; p=0.5; q=1.*

The solution of the differential equation $(t\Phi_{0.5}(y'))' + t\Phi_1(y) = 0$ with conditions $y(0) = 0$, $y'(0) = 1$ near zero we evaluate by MAPLE from (11), (12) and (13). We obtain

$$y(t) = 1 - 0.2222222222t^3 + 0.0370370370t^6$$
$$-0.0047031158t^9 + 0.0005443421t^{12} + ...$$

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