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ON THE ASYMPTOTIC STABILITY OF A CLASS OF PERTURBED ORDINARY DIFFERENTIAL EQUATIONS WITH WEAK ASYMPTOTIC MEAN REVERSION

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Abstract. In this paper we consider the global and local stability and instability of solutions of a scalar nonlinear differential equation with non–negative solutions. The differential equation is a perturbed version of a globally stable autonomous equation with unique zero equilibrium where the perturbation is additive and independent of the state. It is assumed that the restoring force is asymptotically negligible as the solution becomes large, and that the perturbation tends to zero as time becomes indefinitely large. It is shown that solutions are always locally stable, and that solutions either tend to zero or to infinity as time tends to infinity. In the case when the perturbation is integrable, the zero solution is globally asymptotically stable. If the perturbation is non–integrable, and tends to zero faster than a critical rate which depends on the strength of the restoring force, then solutions are globally stable. However, if the perturbation tends to zero more slowly than this critical rate, and the initial condition is sufficiently large, the solution tends to infinity. Moreover, for every initial condition, there exists a perturbation which tends to zero more slowly than the critical rate, for which the solution once again escapes to infinity. Some extensions to general scalar equations as well as to finite–dimensional systems are also presented, as well as global convergence results using Liapunov techniques.

1. Introduction and Connection with the Literature

In this paper we consider the global and local stability and instability of solutions of the perturbed scalar differential equation

$$x'(t) = -f(x(t)) + g(t), \quad t \geq 0; \quad x(0) = \xi. \tag{1.1}$$

It is presumed that the underlying unperturbed equation $y'(t) = -f(y(t))$ for $t \geq 0$ has a globally stable and unique equilibrium at zero. It is a natural question to ask whether stability is preserved in the case when g is asymptotically small. In the case when g is integrable, it is known that

$$\lim_{t\to\infty} x(t,\xi) = 0, \quad \text{for all } \xi \neq 0. \tag{1.2}$$

However, when g is not integrable, and $f(x) \to 0$ as $x \to \infty$ examples of equations are known for $x(t,\xi) \to \infty$ as $t \to \infty$. However, if we know only that $g(t) \to 0$ as $t \to \infty$, but that $\liminf_{|x|\to\infty} |f(x)| > 0$, then all solutions obey (1.2).

In this paper, we investigate the asymptotic behaviour of solutions of (1.1) under the assumption that $f(x) \to 0$ as $x \to \infty$ and $g \notin L^1(0,\infty)$, but that $g(t) \to 0$ as $t \to \infty$. In order to characterise critical rates of decay of g for which stability still

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pertains we stipulate that $\xi > 0$ and $g(t) > 0$ for all $t \geq 0$, so that solutions always lie above the zero equilibrium.

As might be expected, such a critical rate depends on the rate at which $f(x)$ tends to zero as $x \to \infty$, and the more rapidly that f decays, the more rapidly that g needs to decay in order to guarantee that x obeys (1.2). Furthermore, regardless of how rapidly f decays to zero, there are still a class of non–integrable g for which solutions obey (1.2), and regardless of how slowly g tends to zero, there are a class of f for which $f(x) \to 0$ as $x \to \infty$ for which (1.2) still pertains.

More precisely, if we define by F the invertible function

$$F(x) = \int_1^x \frac{1}{f(u)}\, du, \quad x > 0,$$

it is shown that provided f is ultimately decreasing on $[0, \infty)$, and g decays to zero more rapidly than the non–integrable function $f \circ F^{-1}$, then solutions are globally stable (i.e., they obey (1.2)). This rate of decay of g is essentially the slowest possible, for it can be shown in the case when f decays either very slowly or very rapidly, that for every initial condition there exists a perturbation g which tends to zero more slowly than $f \circ F^{-1}$, for which solutions of (1.1) actually obey $x(t) \to \infty$ as $t \to \infty$. Moreover it can be shown under a slight strengthening of the decay hypothesis on g that for every g decaying more slowly than $f \circ F^{-1}$ that all solutions of (1.1) obey $x(t) \to \infty$ as $t \to \infty$, provided the initial condition is large enough. In the intermediate case when f tends to zero like $x^{-\beta}$ for $\beta > 0$ as $x \to \infty$ (modulo a slowly varying factor) a similar situation pertains, except that the critical rate of decay to zero of g is $\lambda f \circ F^{-1}$, where $\lambda > 1$ is a constant which depends purely on β.

The question addressed in this paper is classical; under the assumptions in this paper, we note that the autonomous differential equation

$$x'(t) = -f(x(t)) \tag{1.3}$$

is the unique positive limiting equation of the differential equation (1.1) if either $g(t) \to 0$ as $t \to \infty$ or if $g \in L^1(0, \infty)$. Therefore the problem studied here is connected strongly with work which relates the asymptotic behaviour of original non–autonomous equations to their limiting equations. Especially interesting work in this direction is due to Artstein in a series of papers [4, 5, 6]. Among the major conclusions of his work show that in some sense asymptotic stability and attracting regions of the limiting equation are synonomous with the asymptotic stability and attracting regions of the original nonautonomous equation. However, these results do not apply directly to the problems considered here, because the non–autonomous differential equation (1.1) does not have zero as a solution. Moreover, equation (1.1) does not exhibit the property that its limiting equation is not an ordinary differential equation, so the extension of the limiting equation theory expounded in e.g., [4] is not needed to explain the difference in the asymptotic behaviour between the original equation and its limiting equation. Other interesting works on asymptotically autonomous equations in this direction include Strauss and Yorke [13, 14] and D'Anna, Maio and Moauro [8].

Another approach which seems to generate good results one involving Liapunov functions. Since the equation (1.1) is non–autonomous, we are inspired by the works of LaSalle (especially [11] and [10]), in which ideas from Liapunov's direct method, as well inspiration from the limiting equation approach are combined. In our case, however, it seems that the only possible ω–limit set is zero, the equilibrium point of the limiting equation, and once more the fact that zero is not an equilibrium of (1.1) makes it difficult to determine a t–independent lower bound on the derivative of the Liapunov function. Some Liapunov–like results are presented here in order to compare the results with those achieved using comparison approaches. However, the methods using comparison arguments to which the bulk of this paper is devoted, seem at this point to generate a more precise characterisation of the asymptotic behaviour of (1.1).

The motivation for this work originates from work on the asymptotic behaviour of stochastic differential equations with state independent perturbations, for which the underlying deterministic equation is globally asymptotically stable. In the case when f has relatively strong mean reversion, it is shown in [3], for a sufficiently rapidly decaying noise intensity, that solutions are still asymptotically stable, but that slower convergence leads to unbounded solutions. A complete categorisation of the asymptotic behaviour in the linear case is given in [1]. It appears that the situation in the scalar case for Itô stochastic equations differs from the ordinary case (see [2]), even in the case when there is weak mean–reversion, but the situation in finite dimensions may differ. The Liapunov–like approach we have applied here is also partly inspired by work of Mao, who presented work on a version of LaSalle's invariance principle for Itô stochastic equations in [12], partly because the intrinsically non–autonomous character of the stochastic equation leads the author to allows for the presence of an integrable t–dependent function on the righthand side of the inequality for the "derivative" of the Liapunov function. A similar relaxation of the conditions on the "derivative" of the Liapunov function for Itîo equations can be seen in [9, Chapter 7.4] of Hasminskii when considering the asymototic behaviour of so–called damped stochastic differential equations, which also form the subject of [3, 1, 2] cited above.

The paper is organised as follows. Section 2 contains preliminaries, introduces the equation to be studied, and states explicitly the hypotheses to be studied. Section 3 lists the main results of the paper. In Section 4 a number of examples are given which illustrate the main results. Section 5 considers extensions to the results indicated above to include finite–dimensional equations or equations in which the perturbation changes sign. A Liapunov–style stability theorem is given in Section 6, along with some examples. The proofs of the results are given in the remaining Sections 7–13.

2. Mathematical Preliminaries

2.1. **Notation.** In advance of stating and discussing our main results, we introduce some standard notation. We denote the maximum of the real numbers x and y by $x \vee y$. Let $C(I;J)$ denote the space of continuous functions $f : I \to J$ where I and J are intervals contained in $\mathbb{R}$. Similarly, we let $C^1(I;J)$ denote the space of differentiable functions $f : I \to J$ where $f' \in C(I;J)$. We denote by $L^1(0,\infty)$ the space of Lebesgue integrable functions $f : [0,\infty) \to \mathbb{R}$ such that

$$\int_0^\infty |f(s)|\, ds < +\infty.$$

If I, J and K are intervals in $\mathbb{R}$ and $f : I \to J$ and $g : J \to K$, we define the composition $g \circ f : I \to K : x \mapsto (g \circ f)(x) := g(f(x))$. If $g : [0,\infty) \to \mathbb{R}$ and $h : [0,\infty) \to (0,\infty)$ are such that

$$\lim_{x\to\infty} \frac{g(x)}{h(x)} = 1,$$

we sometimes write $g(x) \sim h(x)$ as $x \to \infty$.

2.2. **Regularly varying functions.** In this short section we introduce the class of slowly growing and decaying functions called regularly varying functions. The results and definition given here may all be found in e.g., Bingham, Goldie and Teugels [7].

We say that a function $h : [0, \infty) \to (0, \infty)$ is regularly varying at infinity with index $\alpha \in \mathbb{R}$ if

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = \lambda^\alpha.$$

We write $h \in \mathrm{RV}_\infty(\alpha)$.

We record some useful and well–known facts about regularly varying functions that will be used throughout the paper. If h is invertible, and $\alpha \neq 0$ we have that $h^{-1} \in \mathrm{RV}_\infty(1/\alpha)$. If h is continuous, and $\alpha > -1$ it follows that the function $H : [0, \infty) \to \mathbb{R}$ defined by

$$H(x) = \int_1^x h(u)\, du, \quad x \geq 0$$

obeys $H \in \mathrm{RV}_\infty(\alpha + 1)$ and in fact we have that

$$\lim_{x\to\infty} \frac{H(x)}{xh(x)} = \frac{1}{\alpha + 1}.$$

If $h_1 \in \mathrm{RV}_\infty(\alpha_1)$ and $h_2 \in \mathrm{RV}_\infty(\alpha_2)$, then the composition $h_1 \circ h_2$ is in $\mathrm{RV}_\infty(\alpha_1\alpha_2)$.

2.3. **Set-up of problem and statement and discussion of hypotheses.** We consider the perturbed ordinary differential equation

$$x'(t) = -f(x(t)) + g(t), \quad t > 0; \quad x(0) = \xi. \tag{2.1}$$

We suppose that

$$f \in C(\mathbb{R}; \mathbb{R}); \quad xf(x) > 0, \quad x \neq 0; \quad f(0) = 0. \tag{2.2}$$

and that g obeys

$$g \in C([0, \infty); \mathbb{R}). \tag{2.3}$$

To simplify the existence and uniqueness of a continuous solutions on $[0, \infty)$, we assume that

$$f \text{ is locally Lipschitz continuous.} \tag{2.4}$$

In the case when g is identically zero, it follows under the hypothesis (2.2) that the solution x of (2.1) i.e.,

$$x'(t) = -f(x(t)), \quad t > 0; \quad x(0) = \xi, \tag{2.5}$$

obeys

$$\lim_{t\to\infty} x(t; \xi) = 0 \text{ for all } \xi \neq 0. \tag{2.6}$$

Clearly $x(t) = 0$ for all $t \geq 0$ if $\xi = 0$. The convergence phenomenon captured in (2.6) for the solution of (2.1) is often called *global convergence* (or *global stability* for the solution of (2.5)), because the solution of the perturbed equation (2.1) converges to the zero equilibrium solution of the underlying unperturbed equation (2.5). We see that if g obeys

$$g \in L^1(0, \infty), \tag{2.7}$$

then (2.2) still suffices to ensure that the solution x of (2.1) obeys (2.6). On the other hand if we assume only that

$$\lim_{t\to\infty} g(t) = 0, \tag{2.8}$$

but that $g \notin L^1(0,\infty)$, (2.2) is not sufficient to ensure that x obeys (2.6). Under (2.8), it is only true in general that

$$\lim_{t\to\infty} x(t,\xi) = 0, \quad \text{for all } |\xi|, \sup_{t\geq 0} |g(t)| \text{ sufficiently small.} \tag{2.9}$$

This convergence phenomenon is referred to as *local stability with respect to perturbations*, and is established in this paper.

An example which show that some solutions of (2.1) even obey

$$\lim_{t\to\infty} x(t) = \infty \tag{2.10}$$

in the case when g obeys (2.8) but $g \notin L^1(0,\infty)$ and when f obeys (2.2) but the restoring force $f(x)$ as $x \to \infty$ is so weak that

$$\lim_{x\to\infty} f(x) = 0 \tag{2.11}$$

are presented in Appleby, Gleeson and Rodkina [3].

However, when (2.11) is strengthened so that in addition to (2.2), f also obeys

$$\text{There exists } \phi > 0 \text{ such that } \phi := \liminf_{|x|\to\infty} |f(x)|, \tag{2.12}$$

then the condition (2.8) on g suffices to ensure that the solution x of (2.1) obeys (2.6). See also [3]. For this reason, we restrict our focus in this paper to the case when f obeys (2.11).

The question therefore arises: if f obeys (2.11), is the condition (2.7) *necessary* in order for solutions of (2.1) to obey (2.6), or does a weaker condition suffice. In this paper we give a relatively sharp characterisation of conditions on g under which solutions of (2.1) obey (2.6) or (2.10). In general, we focus on the case where $g \notin L^1(0,\infty)$, once we have shown that x obeys (2.6) when $g \in L^1(0,\infty)$.

To capture these critical rates of decay of the perturbation g, we constrain it obey

$$g(t) > 0, \quad t \geq 0, \tag{2.13}$$

Our purpose here is not to simplify the analysis, but rather to try to obtain a good lower bound on a critical rate of decay of the perturbation. To see why choosing g to be positive may help in this direction, suppose momentarily that $g(t)$ tends to zero in such a way that it experiences relatively large but rapid fluctuations around zero. In this case, it is possible that the "positive" and "negative" fluctuations cancel. Therefore an upper bound on the rate of decay of the perturbation to zero, which must majorise the amplitude of the fluctuations of g, is likely to give a conservative estimate on the rate of decay. Hence it may be difficult to ascertain whether a given upper bound on the rate of decay of g is sharp in this case. Similarly, we constrain the initial condition ξ to obey

$$\xi > 0, \tag{2.14}$$

as this in conjunction with the positivity of g and the condition (2.2) on f will prevent the solution of (2.1) from oscillating around the zero equilibrium of (2.5): indeed these conditions force $x(t) > 0$ for all $t \geq 0$. This positivity enables us to get lower as well as upper bounds on the solution.

Many stability results in the case when ξ and g do not satisfy these sign constraints can be inferred by applying a comparison argument to a related equation which does possess a positive initial condition and g. Details of some representative results, and extensions of our analysis to systems of equations is given in Section 5.

To determine the critical rate of decay to zero of g, we introduce the invertible function F, given by

$$F(x) = \int_1^x \frac{1}{f(u)}\,du, \quad x > 0. \tag{2.15}$$

Roughly speaking, we show here that provided $g(t)$ decays to zero according to

$$\limsup_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} < 1, \tag{2.16}$$

and

$$\text{There exists } x^* \geq 0 \text{ such that } f \text{ is non–increasing on } (x^*, \infty) \tag{2.17}$$

then the solution x of (2.1) obeys (2.6). The condition (2.16) forces $g(t) \to 0$ as $t \to \infty$. To see this note that the fact that f obeys (2.17), and (2.2) implies that $F(t) \to \infty$ as $t \to \infty$ and therefore $F^{-1}(t) \to \infty$ as $t \to \infty$. Since f obeys (2.11), we have $f(F^{-1}(t)) \to 0$ as $t \to \infty$. This implies that $g(t) \to 0$ as $t \to \infty$. We note also that (2.16) allows for g to be non–integrable, because $t \mapsto f(F^{-1}(t))$ is non–integrable, owing to the identity

$$\int_0^t f(F^{-1}(s))\,ds = \int_{F^{-1}(0)}^{F^{-1}(t)} f(u)\cdot F'(u)\,du = F^{-1}(t) - 1,$$

which tends to $+\infty$ as $t \to \infty$. Careful scrutiny of the proofs reveals that the condition (2.17) can be relaxed to the hypothesis that f is asymptotic to a function which obeys (2.17). However, for simplicity of exposition, we prefer the stronger (2.17) when it is required.

On the other hand, the condition (2.16) is sharp when f decays either very rapidly or very slowly to zero. We make this claim precise. When f decays so rapidly that

$$f \circ F^{-1} \in \mathrm{RV}_\infty(-1) \tag{2.18}$$

or f decays to zero so slowly that

$$f \in \mathrm{RV}_\infty(0) \tag{2.19}$$

then *for every* $\xi > 0$ *there exists* a g which obeys

$$\limsup_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} > 1, \tag{2.20}$$

for which the solution x of (2.1) obeys (2.10). In fact we can construct explicitly such a g. Moreover, under either (2.18) or (2.19), it follows that *for every* g for which

$$\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} > 1, \tag{2.21}$$

there exists $\bar{x} > 0$ such that the solution x of (2.1) obeys (2.10) for all $\xi > \bar{x}$. We observe that (2.21) implies that $g \notin L^1(0,\infty)$. We note that the condition (2.18) automatically implies that f obeys (2.11) and also that f is asymptotic to a function which obeys (2.17).

In the case when f decays to zero "polynomially" we can still characterise quite precisely the critical rate of decay. Once again, what matters is the relative rate of convergence of $g(t)$ and of $f(F^{-1}(t))$ to 0 as $t \to \infty$. Suppose that f obeys

$$\text{There exists } \beta > 0 \text{ such that } f \in \mathrm{RV}_\infty(-\beta). \tag{2.22}$$

This condition automatically implies that f obeys (2.11) and moreover that it is asymptotic to a function which obeys (2.17). In the case that

$$\limsup_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} < \lambda(\beta) := \beta^{\frac{1}{\beta+1}} \left(1 + \beta^{-1}\right), \tag{2.23}$$

and f obeys (2.17), we have that the solution x of (2.1) obeys (2.6). On the other hand if f obeys (2.22), then *for every* $\xi > 0$ *there exists* a g which obeys

$$\limsup_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} \geq \lambda(\beta), \tag{2.24}$$

where $\lambda(\beta)$ is defined in (2.23) for which the solution x of (2.1) obeys (2.10). Moreover, when f obeys (2.22), it follows that *for every* g for which

$$\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} > \lambda(\beta), \tag{2.25}$$

that *there exists* $\bar{x} > 0$ such that the solution x of (2.1) obeys (2.10) for all $\xi > \bar{x}$. We note that (2.25) implies that $g \notin L^1(0,\infty)$.

In the next section, we state precisely the results proven in the paper, referring to the above hypotheses. Although the hypotheses (2.19), (2.18) and (2.22) do not cover all possible modes of convergence of $f(x) \to 0$ as $x \to \infty$, we find in practice that collectively they cover many functions f which decay monotonically to zero.

3. Precise Statement of Main Results

In this section we list our main results, and demonstrate that for any non–integrable g that it is possible to find an f for which solutions of (2.1) are globally stable. We also find the maximal size of perturbation g which is permissible for a given f so that solutions of (2.1) are globally stable.

3.1. List of main results.

In our first result, we show that when $g \in L^1(0,\infty)$, then x obeys (2.6) even when f obeys (2.11).

Theorem 1. *Suppose that f obeys* (2.2) *and that g obeys* (2.3) *and* (2.7)*. Let x be the unique continuous solution of* (2.1)*. Then x obeys* (2.6)*.*

As a result of Theorem 1 we confine attention when f obeys (2.11) to the case in which g is not integrable. We assume instead that $g(t) \to 0$ as $t \to \infty$ and try to identify the appropriate non–integrable and f–dependent pointwise rate of decay which ensures that x obeys (2.6). Our first result shows that the non–negativity of g and global stability of the zero solution of the underlying equation (2.5) ensure that solutions x of the perturbed equation (2.1) obey either $\lim_{t\to\infty} x(t) = 0$ or $\lim_{t\to\infty} x(t) = \infty$.

Theorem 2. *Suppose that g obeys* (2.3), (2.8)*, and g is non–negative. Suppose that f obeys* (2.2) *and that x is the unique continuous solution x of* (2.1)*. Then either* $\lim_{t\to\infty} x(t) = 0$ *or* $\lim_{t\to\infty} x(t) = +\infty$*.*

Of course, Theorem 2 does not tell us into which category of asymptotic behaviour a particular initial value problem will fall, or whether either asymptotic behaviour is possible under certain asymptotic assumptions on f and g.

We first show that when the initial condition ξ is sufficiently small and $\sup_{t\geq 0} g(t)$ is sufficiently small (in addition to g obeying (2.8)), then the zero solution of the underlying unperturbed equation is *locally* stable and we have that the solution x of (2.1) obeys $x(t) \to 0$ as $t \to \infty$.

Theorem 3. *Suppose that f obeys* (2.2) *and that g obeys* (2.8)*. Then for every $\epsilon > 0$ sufficiently small there exists a number $x_1(\epsilon) > 0$ such that $g(t) \le \epsilon$ for all $t \ge 0$ and $\xi \in (0, x_1(\epsilon))$ implies $x(t, \xi) \to 0$ as $t \to \infty$.*

In the case when $f(x) \to 0$ as $x \to \infty$ and f is ultimately monotone, our most general *global* stability result states that if g decays to zero so rapidly that (2.16) is true, then we have that the solution x of (2.1) obeys $x(t) \to 0$ as $t \to \infty$.

Moreover, instead of the pointwise rate of decay (2.16), we can provide a slightly sharper condition, that is if g decays to zero so rapidly that

$$\limsup_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} < 1, \tag{3.1}$$

then we have that the solution x of (2.1) obeys $x(t) \to 0$ as $t \to \infty$.

Theorem 4. *Suppose that f obeys* (2.2) *and g obeys* (2.3)*. Suppose that x is the unique continuous solution of* (2.1)*. Suppose that f obeys* (2.11) *and* (2.17) *and let F be defined by* (2.15)*. If g and f are such that* (3.1) *holds, then the solution x of* (2.1) *obeys* (2.6)*.*

Therefore we can think of the following Theorem as a Corollary of Theorem 4.

Theorem 5. *Suppose that f obeys* (2.2) *and g obeys* (2.3)*. Suppose that x is the unique continuous solution of* (2.1)*. Suppose that f obeys* (2.11) *and* (2.17) *and let F be defined by* (2.15)*. If g and f are such that* (2.16) *holds, then the solution x of* (2.1) *obeys* (2.6)*.*

We have some partial converses to this result. If it is supposed that for every f which decays to zero so slowly that $f \in \text{RV}_\infty(0)$, and for every initial condition $\xi > 0$ there exists g which violates (2.16) (and *a fortiori* obeys (2.20)) for which the solution of (2.1) obeys $x(t) \to \infty$ as $t \to \infty$.

Theorem 6. *Suppose that f obeys* (2.2) *and g obeys* (2.3)*. Suppose that x is the unique continuous solution of* (2.1)*. Suppose that f obeys* (2.11) *and* (2.19) *and let F be defined by* (2.15)*. For every $\xi > 0$ there is a g which obeys* (2.20) *such that the solution x of* (2.1) *obeys* (2.10)*.*

Moreover, we have that the solution $x(\cdot, \xi)$ of (2.1) obeys $x(t, \xi) \to \infty$ as $t \to \infty$ for any g obeying an asymptotic condition slightly stronger than the negation of (2.20), provided the initial condition ξ is sufficiently large. More precisely the asymptotic condition on g is (2.21).

Theorem 7. *Suppose that f obeys* (2.2)*, g obeys* (2.3)*, and that f obeys* (2.19) *and g and f obey* (2.21)*. Suppose that x is the unique continuous solution of* (2.1)*. Then there exists $\bar{x} > 0$ such that for all $\xi > \bar{x}$ we have $\lim_{t\to\infty} x(t, \xi) = \infty$.*

Similar converses to Theorem 4 exist in the case that $f(x)$ decays so rapidly to zero as $x \to \infty$ that $f \circ F^{-1}$ is in $\text{RV}_\infty(-1)$. We first note that for every initial condition, a destabilising perturbation can be found.

Theorem 8. *Suppose that f obeys* (2.2) *and g obeys* (2.3)*. Suppose that x is the unique continuous solution of* (2.1)*. Suppose that f obeys* (2.11) *and* (2.18) *where F is defined by* (2.15)*. For every $\xi > 0$ there is a g which obeys* (2.20) *such that the solution x of* (2.1) *obeys* (2.10)*.*

Once again, if the initial condition is sufficiently large, and g obeys an asymptotic condition slightly stronger than the negation of (2.20) (viz., (2.21)), then once again solutions tend to infinity.

Theorem 9. *Suppose that f obeys* (2.2)*, g obeys* (2.3)*, and that f obeys* (2.18) *and g and f obey* (2.21)*. Suppose that x is the unique continuous solution of* (2.1)*. Then there exists $\bar{x} > 0$ such that for all $\xi > \bar{x}$ we have $\lim_{t\to\infty} x(t,\xi) = \infty$.*

In the case where f is in $\mathrm{RV}_\infty(-\beta)$ for some $\beta > 0$ we have the following case distinction. If g decays to zero so slowly that (2.23) holds, then $x(t) \to 0$ as $t \to \infty$. Moreover, analogously to Theorem 4, instead of the pointwise rate of decay (2.23), if we impose the weaker condition

$$\limsup_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} \leq \lambda < \lambda(\beta), \tag{3.2}$$

then we have that the solution x of (2.1) obeys $x(t) \to 0$ as $t \to \infty$.

Theorem 10. *Suppose that f obeys* (2.2) *and g obeys* (2.3)*. Suppose that x is the unique continuous solution of* (2.1)*. Suppose that there is $\beta > 0$ such that f obeys* (2.17) *and* (2.22) *and let F be defined by* (2.15)*. If g and f are such that* (3.2) *holds, then the solution x of* (2.1) *obeys* (2.6)*.*

Therefore the following Theorem is a direct collorary of Theorem 10.

Theorem 11. *Suppose that f obeys* (2.2) *and g obeys* (2.3)*. Suppose that x is the unique continuous solution of* (2.1)*. Suppose that there is $\beta > 0$ such that f obeys* (2.17) *and* (2.22) *and let F be defined by* (2.15)*. If g and f are such that* (2.23) *holds, then the solution x of* (2.1) *obeys* (2.6)*.*

The condition (2.23), which is sufficient for stability in the case when $f \in \mathrm{RV}_\infty(-\beta)$ is weaker than (2.16). However, it is difficult to relax it further. For every f in $\mathrm{RV}_\infty(-\beta)$ and every initial condition ξ it is possible to find a g which violates (2.23) (and therefore obeys (2.24)) for which the solution obeys $x(t) \to \infty$ as $t \to \infty$.

Theorem 12. *Suppose that f obeys* (2.2) *and g obeys* (2.3)*. Suppose that x is the unique continuous solution of* (2.1)*. Suppose that there is $\beta > 0$ such that f obeys* (2.22) *and let F be defined by* (2.15)*. Then for every $\xi > 0$ there is a g which obeys* (2.24) *such that the solution x of* (2.1) *obeys* (2.10)*.*

On the other hand, we have that the solution $x(\cdot,\xi)$ of (2.1) obeys $x(t,\xi) \to \infty$ as $t \to \infty$ for any g obeying an asymptotic condition slightly stronger than the negation of (2.24), provided the initial condition ξ is sufficiently large. More precisely the asymptotic condition on g is (2.25), where $\lambda(\beta)$ is as defined by (2.23).

Theorem 13. *Suppose that f obeys* (2.2)*, g obeys* (2.3)*, and that f obeys* (2.22) *and g and f obey* (2.25)*. Suppose that x is the unique continuous solution of* (2.1)*. Then there exists $\bar{x} > 0$ such that for all $\xi > \bar{x}$ we have $\lim_{t\to\infty} x(t,\xi) = \infty$.*

3.2. **Minimal conditions for global stability.** In this short subsection we address two questions: given any non–integrable g, we show that it is possible to find an f for which the solution of (2.1) is globally stable. And given an f, we determine how large is the largest possible perturbation g that is permissible so that the solution is globally stable.

We also consider two extreme cases: when g just fails to be integrable $g \in \mathrm{RV}_\infty(-1)$, and when g tends to zero so slowly that $g \in \mathrm{RV}_\infty(0)$. In the case when g just fails to be integrable (so that $g \in \mathrm{RV}_\infty(-1)$), we can choose an f which decays to zero so rapidly that $f \circ F^{-1} \in \mathrm{RV}_\infty(-1)$ while at the same time ensuring that solutions of (2.1) are globally asymptotically stable. On the other hand, if g

decays to zero so slowly that $g \in \mathrm{RV}_\infty(0)$, we choose f to decay to zero slowly also while preserving global stability. In particular, it transpires that f is in $\mathrm{RV}_\infty(0)$.

Consider first the general question. Suppose that $g(t) \to 0$ as $t \to \infty$ in such a way that $g \notin L^1(0,\infty)$. If moreover g is ultimately decreasing, the next Proposition show that it is possible to find an f,which satisfies all the conditions of Theorem 4, so that the solution f of (2.1) x obeys (2.6). Therefore, there is no rate of decay of g to zero, however slow, that cannot be stabilised by an f for which $f(x) \to 0$ as $x \to \infty$. Therefore, it is possible for g to be very far from being integrable, and $f(x) \to 0$ as $x \to \infty$, but provided that this rate of decay of f is not too fast, then solutions of (2.1) can still be globally stable.

Proposition 1. *Suppose that g is positive, continuous and obeys* (2.8) *and $g \notin L^1(0,\infty)$. Let $\lambda > 0$. Then there exists a continuous f which obeys* (2.2), (2.11) *and also obeys*

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = \lambda. \tag{3.3}$$

Moreover, if g is decreasing on $[\tau,\infty)$ for some $\tau \ge 0$, then f obeys (2.17).

Proof. Suppose that f is such that $f(0) = 0$, $f(x) > 0$ for all $x \in (0,1]$ and that

$$\lim_{x\to 1^-} f(x) = \frac{1}{\lambda} g(0) > 0.$$

Define

$$G_\lambda(x) = \frac{1}{\lambda}\int_1^x g(s)\,ds, \quad x \ge 0. \tag{3.4}$$

Then G_λ is increasing and therefore G_λ^{-1} exists. Moreover since $g \notin L^1(0,\infty)$, we have that $G_\lambda(x) \to \infty$ as $x \to \infty$, so $G_\lambda^{-1}(x) \to \infty$ as $x \to \infty$. Define also

$$f(x) = \frac{1}{\lambda} g(G_\lambda^{-1}(x - 1 + G_\lambda(0))), \quad x \ge 1.$$

For $x \ge 1$ we have that $x - 1 + G_\lambda(0) \ge G_\lambda(0)$, so $G_\lambda^{-1}(x - 1 + G_\lambda(0)) \ge 0$. Therefore f is well–defined. Moreover, since g is positive, we have that $f(x) > 0$ for all $x > 0$. Note that $f(1) = g(0)/\lambda$, and g and G_λ are continuous, we have that $f : [0,\infty) \to [0,\infty)$ is continuous. Since $g(t) \to 0$ as $t \to \infty$ and $G_\lambda^{-1}(t) \to \infty$ as $t \to \infty$, it follows that $f(x) \to 0$ as $x \to \infty$. We see also that if g is ultimately decreasing, that f must obey (2.17), because G_λ^{-1} is increasing.

Finally, notice that

$$F(x) = \int_1^x \frac{1}{f(u)}\,du = \int_0^{G_\lambda^{-1}(x-1+G_\lambda(0))} \frac{1}{1/\lambda \cdot g(s)} \frac{1}{\lambda} g(s)\,ds = G_\lambda^{-1}(x - 1 + G_\lambda(0)).$$

Therefore for $x \ge 1$ we have $g(F(x)) = \lambda f(x)$. Now $F(x) \ge 0$ for $x \ge 1$, so we have $g(y) = \lambda f(F^{-1}(y))$ for $y \ge 0$, so clearly we have that (3.3) holds. □

Suppose next that g tends to zero arbitrarily slowly (restricted to the class of $\mathrm{RV}_\infty(0)$). Then it is possible to find an f (also in $\mathrm{RV}_\infty(0)$) which satisfies all the conditions of Theorem 4, so that x obeys (2.6).

Proposition 2. *Suppose that $g \in RV_\infty(0)$ is continuous, positive and decreasing and obeys* (2.8). *Define*

$$G(t) = \int_1^t g(s)\,ds, \quad t \ge 0. \tag{3.5}$$

Let $\lambda > 0$. Suppose that f is continuous and obeys (2.2)*, as well as*

$$f(x) \sim \frac{1}{\lambda} g(G^{-1}(x)), \quad x \to \infty. \tag{3.6}$$

Then f obeys (2.11)*, f is asymptotic to a decreasing function, $f \in RV_\infty(0)$ and* (3.3)*.*

As an example, suppose that $n \in \mathbb{N}$ and that $g(x) \sim 1/(\log_n x)$ as $x \to \infty$. It can then be shown that $G^{-1}(x) \sim x \log_n x$ as $x \to \infty$. Therefore we have

$$g(G^{-1}(x)) \sim \frac{1}{\log_n x}$$

Hence if $f(x) \sim \lambda^{-1}/\log_n x$ as $x \to \infty$, we have that g and f obey (3.3).

Remark 1. If f tends to zero very slowly, we can still have g tending to zero very slowly, and yet have solutions of (2.1) obeying (2.6). Indeed, suppose that $f \in \mathrm{RV}_\infty(0)$. Then $F \in \mathrm{RV}_\infty(1)$ so $F^{-1} \in \mathrm{RV}_\infty(1)$. Therefore $f \circ F^{-1} \in \mathrm{RV}_\infty(0)$. Hence if g obeys (3.3) with $\lambda < 1$, we have that $g \in \mathrm{RV}_\infty(0)$.

Remark 2. We note that if f tends to zero very rapidly, so that $f \circ F^{-1}$ is in $\mathrm{RV}_\infty(-1)$, then g must be dominated by a function in $\mathrm{RV}_\infty(-1)$. Therefore, if f tends to zero very rapidly, it can be seen that g must be close to being integrable. This is related to the fact that however rapidly f tends to zero (in the sense that $f \circ F^{-1}$ is in $\mathrm{RV}_\infty(-1)$), it is always possible to find non–integrable g for which solutions of (2.1) are globally asymptotically stable and obey (2.6).

Remark 3. Suppose conversely that $g \in \mathrm{RV}_\infty(-1)$ in such a way that $g \notin L^1(0,\infty)$. Then we can find an f which decays so quickly to zero as $x \to \infty$ that $f \circ F^{-1} \in \mathrm{RV}_\infty(-1)$ while f and g obey (3.3). Therefore, if g tends to zero in such a way that it is close to being integrable (but is non–integrable), then solutions of (2.1) are globally asymptotically stable provided f exhibits very weak mean reversion.

To see this let $\lambda > 0$. Then it can be shown in a manner similar to Proposition 1 that if f is defined by

$$f(x) = \frac{1}{\lambda} g(G_\lambda^{-1}(x)), \quad x \geq 1$$

where G_λ is defined by (3.4), then f and g obey (3.3). Moreover, if F is defined by (2.15), for this choice of f we have $F(x) = G_\lambda^{-1}(x) - G_\lambda^{-1}(1)$ for $x \geq 1$. Rearranging yields $F^{-1}(x) = G_\lambda(x + G')$ for $x \geq 0$, where we define $G' := G_\lambda^{-1}(1)$. Hence

$$f(F^{-1}(x)) = \frac{1}{\lambda} g(G_\lambda^{-1}(F^{-1}(x))) = \frac{1}{\lambda} g(x + G').$$

Since $g \in \mathrm{RV}_\infty(-1)$ it follows that $f \circ F^{-1} \in \mathrm{RV}_\infty(-1)$.

Example 14. In the case when $g(t) \sim 1/(t \log t)$ as $t \to \infty$, we have

$$G_\lambda(t) \sim \frac{1}{\lambda} \log_2 t, \quad \text{as } t \to \infty.$$

Therefore can see (formally) that $\log G_\lambda^{-1}$ behaves asymptotically like $e^{\lambda t}$ and that $G_\lambda^{-1}(t)$ behaves like $\exp(e^{\lambda t})$ as $t \to \infty$. Hence a good candidate for f is

$$f(x) = \frac{1}{\lambda} e^{-\lambda x} \exp(-e^{\lambda x}), \quad x \geq 1.$$

Then, with $x' = \exp(e^{\lambda})$, we have $F(x) = \exp(e^{\lambda x}) - x'$. Therefore we have $F^{-1}(x) = \log_2(x + x')/\lambda$. Hence

$$f(F^{-1}(x)) = \frac{1}{\lambda}\frac{1}{x+x'}\frac{1}{\log(x+x')}.$$

Therefore we have that g and f obey (3.3). Note moreover that $f \circ F^{-1}$ is in $\mathrm{RV}_\infty(-1)$.

4. Examples

In this section we give examples of equations covered by Theorems 2—13 above.

Example 15. Let $a > 0$ and $\beta > 0$. Suppose that $f(x) = ax(1+x)^{-(\beta+1)}$ for $x \geq 0$. Then f obeys (2.2) and (2.17). We have that $f \in \mathrm{RV}_\infty(-\beta)$. Now as $x \to \infty$ we have

$$F(x) = \int_1^x \frac{1}{f(u)}\,du \sim \int_1^x 1/au^\beta\,du = \frac{1/a}{\beta+1}x^{\beta+1}.$$

Then $F^{-1}(x) \sim [a(1+\beta)x]^{1/(\beta+1)}$ as $x \to \infty$. Therefore as $x \to \infty$ we have

$$f(F^{-1}(x)) \sim a[a(1+\beta)x]^{-\beta/(\beta+1)} = a^{1/(\beta+1)}(1+\beta)^{-\beta/(\beta+1)}x^{-\beta/(\beta+1)}.$$

Suppose that

$$\limsup_{t\to\infty} \frac{g(t)}{a^{1/(\beta+1)}(1+\beta)^{-\beta/(\beta+1)}t^{-\beta/(\beta+1)}} < \beta^{1/(\beta+1)}(1+\beta^{-1})$$

Then for every $\xi > 0$ we have $x(t,\xi) \to 0$ as $t \to \infty$. On the other hand, for every $\xi > 0$, there is a g which obeys

$$\limsup_{t\to\infty} \frac{g(t)}{a^{1/(\beta+1)}(1+\beta)^{-\beta/(\beta+1)}t^{-\beta/(\beta+1)}} \geq \beta^{1/(\beta+1)}(1+\beta^{-1})$$

such that $x(t,\xi) \to \infty$ as $t \to \infty$. Finally, for every g which obeys

$$\liminf_{t\to\infty} \frac{g(t)}{a^{1/(\beta+1)}(1+\beta)^{-\beta/(\beta+1)}t^{-\beta/(\beta+1)}} > \beta^{1/(\beta+1)}(1+\beta^{-1})$$

there is an $\bar{x} > 0$ such that for all $\xi > \bar{x}$ we have $x(t,\xi) \to \infty$ as $t \to \infty$.

Example 16. Let $a > 0$ and suppose that

$$f(x) = \frac{ax}{(1+x)\log(e+x)}, \quad x \geq 0.$$

Then f obeys (2.2) and (2.17). Moreover, we have that $f \in \mathrm{RV}_\infty(0)$. Hence as $x \to \infty$ we have

$$F(x) \sim \int_1^x \frac{1}{a}\log(e+u)\,du \sim \frac{1}{a}x\log x.$$

Therefore we have $F^{-1}(x) \sim ax/\log x$ as $x \to \infty$. Thus as $x \to \infty$ we have

$$f(F^{-1}(x)) \sim a/\log F^{-1}(x) \sim a/\log x.$$

Therefore if

$$\limsup_{t\to\infty} g(t)\log t < a,$$

we have $x(t,\xi) \to 0$ for all $\xi > 0$. On the other hand for every $\xi > 0$ there is a g which obeys

$$\limsup_{t\to\infty} g(t)\log t > a,$$

for which $x(t,\xi)\to\infty$. Finally, for every g which obeys

$$\liminf_{t\to\infty} g(t)\log t > a,$$

there is a $\bar{x}>0$ such that for all $\xi>\bar{x}$ we have $x(t,\xi)\to\infty$.

Example 17. Let $a>0$, $\beta>0$ and $\delta>0$ and suppose that

$$f(x) = axe^{-\delta x^\beta}, \quad x\geq 1,$$

where $f(0)=0$, $f(x)>0$ for $x\in(0,1)$ and f is continuous on $[0,1)$ with $\lim_{x\to 1^-} f(x) = ae^{-\delta}$. Then f obeys (2.2) and (2.17). By L'Hôpital's rule we have

$$\lim_{x\to\infty}\frac{F(x)}{e^{\delta x^\beta}/x^\beta} = \frac{1}{a}\lim_{x\to\infty}\frac{x^{-1}}{-\beta x^{-\beta-1}+\delta\beta x^{-1}} = \frac{1}{a\delta\beta}.$$

Therefore we have

$$\lim_{x\to\infty}\frac{x}{e^{\delta F^{-1}(x)^\beta}/F^{-1}(x)^\beta} = \frac{1}{a\delta\beta}.$$

From this it can be inferred that

$$\lim_{x\to\infty}\frac{e^{\delta F^{-1}(x)^\beta}/F^{-1}(x)^\beta}{x} = a\delta\beta.$$

Now we have $e^{\delta F^{-1}(x)^\beta}\sim a\delta\beta x F^{-1}(x)^\beta$ as $x\to\infty$. Therefore as $x\to\infty$ we get

$$xf(F^{-1}(x)) = xaF^{-1}(x)/e^{\delta F^{-1}(x)^\beta} \sim \frac{xaF^{-1}(x)}{a\delta\beta xF^{-1}(x)^\beta} = \frac{1}{\delta\beta}\cdot F^{-1}(x)^{1-\beta}.$$

It remains to estimate the asymptotic behaviour of $F^{-1}(x)$ as $x\to\infty$. Since $\delta F^{-1}(x)^\beta - \beta\log F^{-1}(x) - \log x \to \log(a\delta\beta)$ as $x\to\infty$, we therefore obtain

$$\lim_{x\to\infty}\frac{\delta F^{-1}(x)^\beta}{\log x} = 1.$$

Hence

$$\lim_{x\to\infty}\frac{F^{-1}(x)^{1-\beta}}{(\log x)^{(1-\beta)/\beta}} = \left(\frac{1}{\delta}\right)^{(1-\beta)/\beta}.$$

Thus $(F^{-1})^{1-\beta}$ is in $\mathrm{RV}_\infty(0)$ and thus $f\circ F^{-1}\in\mathrm{RV}_\infty(-1)$. Moreover as $x\to\infty$ we have

$$f(F^{-1}(x)) \sim \frac{1}{\delta\beta}\cdot\frac{1}{x}\cdot F^{-1}(x)^{1-\beta} \sim \frac{1}{\beta\delta^{1/\beta}}\cdot\frac{1}{x}\cdot\frac{1}{(\log x)^{-1/\beta+1}}.$$

Therefore if

$$\limsup_{t\to\infty}\frac{g(t)}{\frac{1}{\beta\delta^{1/\beta}}\cdot\frac{1}{t}\cdot\frac{1}{(\log t)^{-1/\beta+1}}} < 1,$$

we have $x(t,\xi)\to 0$ for all $\xi>0$. On the other hand for every $\xi>0$ there is a g which obeys

$$\limsup_{t\to\infty}\frac{g(t)}{\frac{1}{\beta\delta^{1/\beta}}\cdot\frac{1}{t}\cdot\frac{1}{(\log t)^{-1/\beta+1}}} > 1,$$

for which $x(t,\xi)\to\infty$. Finally, for every g which obeys

$$\liminf_{t\to\infty}\frac{g(t)}{\frac{1}{\beta\delta^{1/\beta}}\cdot\frac{1}{t}\cdot\frac{1}{(\log t)^{-1/\beta+1}}} > 1,$$

there is a $\bar{x}>0$ such that for all $\xi>\bar{x}$ we have $x(t,\xi)\to\infty$.

5. Extensions to General Scalar Equations and Finite–Dimensional Equations

We have formulated and discussed our main results for scalar equations where the solutions remain of a single sign. This restriction has enabled us to achieve sharp results on the asymptotic stability and instability. However, it is also of interest to investigate asymptotic behaviour of equations of a similar form in which changes in the sign of g lead to changes in the sign of the solution, or to equations in finite dimensions. In this section, we demonstrate that results giving sufficient conditions for global stability can be obtained for these wider classes of equation, by means of appropriate comparison arguments. In this section, we denote by $\langle x, y\rangle$ the standard innerproduct of the vectors $x, y \in \mathbb{R}^d$, and let $\|x\|$ denote the standard Euclidean norm of $x \in \mathbb{R}^d$ induced from this innerproduct.

5.1. **Finite–dimensional equations.** In this section, we first discuss appropriate hypotheses under which the d–dimensional ordinary differential equation

$$x'(t) = -\phi(x(t)) + \gamma(t), \quad t > 0; \quad x(0) = \xi \in \mathbb{R}^d \tag{5.1}$$

will exhibit asymptotically convergent solutions under conditions of weak asymptotic mean reversion. Here, we assume that $\phi : \mathbb{R}^d \to \mathbb{R}^d$ and that $\gamma : [0,\infty) \to \mathbb{R}^d$. Therefore, if there is a solution x, $x(t) \in \mathbb{R}^d$ for any $t \geq 0$ for which x exists. In order to simplify matters, we assume once again that ϕ is locally Lipschitz on $\mathbb{R}^d$ and that γ is continuous, as these assumptions guarantee the existence of a unique continuous solution, defined on $[0, T)$ for some $T > 0$. In order that solutions be global (i.e., that $T = +\infty$, we need to show that there does not exist $T < +\infty$ such that

$$\lim_{t\uparrow T} \|x(t)\| = +\infty.$$

In the scalar setting, this is ensured by the global stability condition (2.2). We need a natural analogue of this condition, as well as the condition that 0 is the unique solution of the underlying unperturbed equation

$$z'(t) = -\phi(z(t)), \quad t > 0; \quad z(0) = \xi. \tag{5.2}$$

A suitable and simple condition which achieves all these ends is

$$\phi \text{ is locally Lipschitz continuous, } \phi(0) = 0, \quad \langle \phi(x), x\rangle > 0 \text{ for all } x \neq 0. \tag{5.3}$$

We also find it convenient to introduce a function φ_0 given by

$$\varphi_0(x) = \begin{cases} \inf_{\|u\|=x} \frac{\langle u,\phi(u)\rangle}{\|u\|}, & x > 0, \\ 0, & x = 0. \end{cases} \tag{5.4}$$

It turns out that the function φ_0 is important in several of our proofs. For this reason, we list here its relevant properties.

Lemma 1. *Let $\varphi_0 : [0,\infty) \to \mathbb{R}$ be the function defined in* (5.4). *Then*

$$\varphi_0(x) = \inf_{\|u\|=1} \langle u, \phi(xu)\rangle, \quad x \geq 0. \tag{5.5}$$

If ϕ obeys (5.3)*, then $\varphi_0(0) = 0$, $\varphi_0(x) > 0$ for $x > 0$ and φ_0 is locally Lipschitz continuous. Moreover, if $\phi(x) \to 0$ as $\|x\| \to \infty$, then $\varphi_0(x) \to 0$ as $x \to \infty$.*

In the scalar case when ϕ is an odd function, we note that φ_0 collapses to ϕ itself. The proof of Lemma 1 is presented in the final section.

We consolidate the facts collected above regarding solutions of (5.2) and (5.1) into two propositions. Their proofs are standard, and are also relegated to the end.

Proposition 3. *Suppose that ϕ obeys* (5.3)*. Then $x = 0$ is the unique equilibrium solution of* (5.2)*. Moreover, the initial value problem* (5.2) *has a unique continuous solution defined on* $[0, \infty)$ *and for all initial conditions $z(t) \to 0$ as $t \to \infty$.*

Proposition 4. *Suppose that ϕ obeys* (5.3)*. Then, the initial value problem* (5.2) *has a unique continuous solution defined on* $[0, \infty)$.

5.2. **Extension of Results.** In order to compare solutions of finite–dimensional equations with scalar equations to which results in Section 3 can be applied, we make an additional hypotheses on ϕ.

$$\begin{gathered}\varphi : [0, \infty) \to [0, \infty) \text{ is locally Lipschitz continuous where} \\ \langle x, \phi(x) \rangle \geq \varphi(||x||) \text{ for all } x \in \mathbb{R}^d \setminus \{0\}, \quad \varphi(0) = 0, \quad \varphi(x) > 0 \text{ for all } x > 0.\end{gathered} \tag{5.6}$$

Under (5.3), we observe by Lemma 1 that the function φ_0 introduced in (5.4) can play the role of φ in (5.6). Our comparison theorem is now stated.

Theorem 18. *Suppose that ϕ obeys* (5.3) *and* (5.6)*, and that γ is a continuous function. Let x be the unique continuous solution of* (5.1)*. Let $\epsilon > 0, \eta > 0$ and suppose that $x_{\epsilon,\eta}$ is the unique continuous solution of*

$$x'_{\eta,\epsilon}(t) = -\varphi(x_{\eta,\epsilon}(t)) + \|\gamma(t)\| + \frac{\epsilon}{2}e^{-t}, \quad t > 0; \quad x_{\eta,\epsilon}(0) = \|x(0)\| + \frac{\eta}{2}. \tag{5.7}$$

Then for every $\epsilon > 0, \eta > 0$, $\|x(t)\| \leq x_{\eta,\epsilon}(t)$ for all $t \geq 0$.

The proof is deferred to the end.

5.2.1. *Scalar equations.* We now consider the ramifications of Theorem 18 for scalar differential equations. Notice first that the function φ_0 introduced in (5.4) is very easily computed. Due to (5.5), we have that

$$\varphi_0(x) = \inf_{\|u\|=1} u\phi(xu) = \min_{u=\pm 1} u\phi(xu) = \min(\phi(x), -\phi(-x)). \tag{5.8}$$

We restate the hypothesis (5.3) for ϕ in scalar form:

$$\phi : \mathbb{R} \to \mathbb{R} \text{ is locally Lipschitz continuous, } x\phi(x) > 0 \text{ for } x \neq 0,\ \phi(0) = 0. \tag{5.9}$$

The following results are then direct corollaries of results in Section 3 and Theorem 18.

Theorem 19. *Suppose that ϕ obeys* (5.9) *and γ is continuous and in $L^1(0, \infty)$. Then the unique continuous solution x of* (5.1) *obeys $x(t) \to 0$ as $t \to \infty$.*

Proof. Let $\epsilon > 0$. Define $g(t) = |\gamma(t)| + \epsilon e^{-t}/2$ for $t \geq 0$. Then by hypothesis, g is continuous and positive on $[0, \infty)$, and $g \in L^1(0, \infty)$. By (5.9) and Lemma 1, the function φ_0 defined in (5.8) is locally Lipschitz continuous and obeys $\varphi_0(0) = 0$ and $\varphi_0(x) > 0$ for $x > 0$. Therefore for any $\epsilon > 0$ and $\eta > 0$, we may apply Theorem 1 to the solution $x_{\eta,\epsilon}$ of (5.7) and conclude that $x_{\eta,\epsilon}(t) \to 0$ as $t \to \infty$. By Theorem 18 we have that $x(t) \to 0$ as $t \to \infty$. □

Theorem 20. *Suppose that ϕ obeys* (5.9) *and γ is continuous and $\gamma(t) \to 0$ as $t \to \infty$. Then for every $\epsilon > 0$ sufficiently small there exists a number $x_1(\epsilon) > 0$ such that $|\gamma(t)| \leq \epsilon/2$ for all $t \geq 0$ and $|\xi| < x_1(\epsilon)/2$ implies that the unique continuous solution x of* (5.1) *obeys $x(t, \xi) \to 0$ as $t \to \infty$.*

Proof. Let $\epsilon > 0$. Define $g(t) = |\gamma(t)| + \epsilon e^{-t}/2$ for $t \geq 0$. Then by hypothesis, g is continuous and positive on $[0, \infty)$, obeys $g(t) \to 0$ as $t \to \infty$, and also $g(t) < \epsilon$ for all $t \geq 0$. By (5.9) and Lemma 1, the function φ_0 defined in (5.8) is locally Lipschitz continuous and obeys $\varphi_0(0) = 0$ and $\varphi_0(x) > 0$ for $x > 0$. There exists $\epsilon_0 > 0$ sufficiently small so that the set $\inf\{x > 0 : \varphi_0(x) = 2\epsilon_0\}$ is non–empty. For $\epsilon \in (0, \epsilon_0)$ define $x_1(\epsilon) = \inf\{x > 0 : \varphi_0(x) = 2\epsilon\}$. Then $\varphi_0(x) < 2\epsilon$ for all $x \in [0, x_1(\epsilon))$. Fix $\eta(\epsilon) = x_1(\epsilon) > 0$. Since $|\xi| < x_1(\epsilon)/2$, we have that $|x_{\eta(\epsilon),\epsilon}(0)| = |x(0)| + \eta(\epsilon)/2 < x_1(\epsilon)$. Suppose there is a finite $T_1(\epsilon) = \inf\{t > 0 : x_{\eta(\epsilon),\epsilon}(t) = x_1(\epsilon)\}$. Then $x'_{\eta(\epsilon),\epsilon}(T_1(\epsilon)) \geq 0$. Also

$$0 \leq x'_{\eta(\epsilon),\epsilon}(T_1(\epsilon)) = -\varphi_0(x_{\eta(\epsilon),\epsilon}(T_1(\epsilon))) + g(T_1(\epsilon)) \leq -\varphi_0(x_1(\epsilon)) + \epsilon = -\epsilon < 0,$$

a contradiction. Hence we have that $x_{\eta(\epsilon),\epsilon}(t) < x_1(\epsilon)$ for all $t \geq 0$. Now by Lemma 2 it follows that $x_{\eta(\epsilon),\epsilon}(t) \to 0$ as $t \to \infty$. Therefore, by Theorem 18, we have that $|x(t)| < x_1(\epsilon)$ for all $t \geq 0$ and that $x(t) \to 0$ as $t \to \infty$. □

Theorem 21. *Suppose that ϕ obeys* (5.9) *and γ is continuous and obeys $\gamma(t) \to 0$ as $t \to \infty$. Suppose also that φ_0 given by* (5.8) *is decreasing on (x^*, ∞) for some $x^* > 0$. If Φ_0 is defined by*

$$\Phi_0(x) = \int_1^x \frac{1}{\varphi_0(u)}\, du,$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t |\gamma(s)|\, ds}{\Phi_0^{-1}(t)} < 1$$

then the unique continuous solution x of (5.1) *obeys $x(t) \to 0$ as $t \to \infty$.*

Proof. Let $\epsilon > 0$. Define $g(t) = |\gamma(t)| + \epsilon e^{-t}/2$ for $t \geq 0$. Then by hypothesis, g is continuous and positive on $[0, \infty)$, obeys $g(t) \to 0$ as $t \to \infty$, and also $g(t) < \epsilon$ for all $t \geq 0$. By (5.9) and Lemma 1, the function φ_0 defined in (5.8) is locally Lipschitz continuous and obeys $\varphi_0(0) = 0$ and $\varphi_0(x) > 0$ for $x > 0$. Therefore for every $\epsilon > 0$ and $\eta > 0$ the equation (5.7) is of the form of (2.1) with φ_0 in the role of f and Φ_0 in the role of F. Notice that the monotonicity of φ_0 implies that $\Phi_0(x) \to \infty$ as $x \to \infty$, and therefore that $\Phi_0^{-1}(x) \to \infty$ as $x \to \infty$. Therefore by hypothesis, we have

$$\limsup_{t\to\infty} \frac{\int_0^t g(s)\, ds}{\Phi_0^{-1}(t)} = \limsup_{t\to\infty} \frac{\int_0^t |\gamma(s)|\, ds}{\Phi_0^{-1}(t)} + \frac{\int_0^t \epsilon e^{-s}\, ds}{\Phi_0^{-1}(t)} = \limsup_{t\to\infty} \frac{\int_0^t |\gamma(s)|\, ds}{\Phi_0^{-1}(t)} < 1.$$

Therefore, by Theorem 4 we have that $x_{\eta,\epsilon}(t) \to 0$ as $t \to \infty$, and hence by Theorem 18, it follows that $x(t) \to 0$ as $t \to \infty$. □

A result analogous to Theorem 10 can be formulated even when γ changes sign. We state the result but do not provide a proof.

Theorem 22. *Suppose that ϕ obeys* (5.9) *and γ is continuous and obeys $\gamma(t) \to 0$ as $t \to \infty$. Suppose also that φ_0 given by* (5.8) *is in $RV_\infty(-\beta)$ for $\beta > 0$. If Φ_0 is defined by*

$$\Phi_0(x) = \int_1^x \frac{1}{\varphi_0(u)}\, du,$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t |\gamma(s)|\, ds}{\Phi_0^{-1}(t)} < \lambda(\beta) = \beta^{1/(\beta+1)}(1 + \beta^{-1}),$$

then the unique continuous solution x of (5.1) *obeys $x(t) \to 0$ as $t \to \infty$.*

5.2.2. *Finite–dimensional results.* In this section, we often request that the function φ introduced in (5.6) obeys a monotonicity restriction.

$$x \mapsto \varphi(x) \text{ is decreasing on } (x^*, \infty) \text{ for some } x^* > 0. \tag{5.10}$$

Results analogous to Theorems 19, 20, 21 and 22 can be stated for finite–dimensional systems. The proofs are very similar to those of the corresponding scalar results, and are therefore omitted.

Theorem 23. *Suppose that ϕ obeys (5.3) and γ is continuous and in $L^1(0,\infty)$. Then the unique continuous solution x of (5.1) obeys $x(t) \to 0$ as $t \to \infty$.*

Theorem 24. *Suppose that ϕ obeys (5.3) and that γ is continuous and $\gamma(t) \to 0$ as $t \to \infty$. Then for every $\epsilon > 0$ sufficiently small there exists a number $x_1(\epsilon) > 0$ such that $\|\gamma(t)\| \le \epsilon/2$ for all $t \ge 0$ and $\|\xi\| < x_1(\epsilon)/2$ implies that the unique continuous solution x of (5.1) obeys $x(t,\xi) \to 0$ as $t \to \infty$.*

Theorem 25. *Suppose that ϕ obeys (5.3) and that ϕ and φ obey (5.6) and (5.10). Suppose that γ is continuous and that $\gamma(t) \to 0$ as $t \to \infty$. If Φ is defined by*

$$\Phi(x) = \int_1^x \frac{1}{\varphi(u)}\,du,$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t \|\gamma(s)\|\,ds}{\Phi^{-1}(t)} < 1$$

then the unique continuous solution x of (5.1) obeys $x(t) \to 0$ as $t \to \infty$.

Theorem 26. *Suppose that ϕ obeys (5.3) and that ϕ and φ obey (5.6). Suppose also that φ is in $RV_\infty(-\beta)$ for $\beta > 0$. Suppose that γ is continuous and that $\gamma(t) \to 0$ as $t \to \infty$. If Φ is defined by*

$$\Phi(x) = \int_1^x \frac{1}{\varphi(u)}\,du,$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t \|\gamma(s)\|\,ds}{\Phi^{-1}(t)} < \beta^{1/(\beta+1)}(1+\beta^{-1}),$$

then the unique continuous solution x of (5.1) obeys $x(t) \to 0$ as $t \to \infty$.

6. A Liapunov Result

The main result of this section shows that if f has a certain rate of decay to zero, and g decays more rapidly than a certain rate which depends on f, then solutions of (2.1) can be shown to tend to 0 as $t \to \infty$ by means of a Liapunov–like technique. The results are not as sharp as those obtained in Section 3, and do not have anything to say about instability, but nonetheless the conditions do seem to identify, albeit crudely, the critical rate for g at which global stability is lost.

The conditions of the theorem appear forbidding in general, and the reader may doubt it is possible to construct auxiliary functions with the desired properties. However, by considering examples in which f decays either polynomially or exponentially, we demonstrate that the result can be applied in practice, and that the claims made above regarding the sharpness of the result are not unjustified.

Theorem 27. *Suppose that f obeys (2.2) and (2.4) and that $g \in C([0,\infty);(0,\infty))$ and $g(t) \to 0$ as $t \to \infty$. Let $\Theta \in C([0,\infty);[0,\infty))$ be a twice differentiable and increasing function such that $\Theta(0) = 0$. Define $\psi(x) = x\Theta^{-1}(x)$ for $x > 0$ and $\psi(0) = 0$, and suppose that ψ is an increasing and convex function on $(0,\infty)$ with $\lim_{x\to 0^+} x\psi'(x) = 0$. Define also $\theta : [0,\infty) \to [0,\infty)$ by*

$$\theta(x) = x(\psi')^{-1}(x) - (\psi \circ (\psi')^{-1})(x), \quad x > 0; \quad \theta(0) = 0.$$

Suppose that $\Theta\circ f \notin L^1(0,\infty)$ and that $\theta\circ g \in L^1(0,\infty)$. Then the unique continuous solution x of (2.1) obeys $x(t) \to 0$ as $t \to \infty$.

Proof. Since Θ is increasing, ψ is a well–defined function. Moreover, as Θ is twice differentiable, it follows that Θ^{-1} is twice differentiable, and therefore we have that $x \mapsto \psi'(x)$ is a continuous function and that $\psi''(x)$ is well–defined for all $x > 0$. In fact, by the assumption that ψ is increasing and convex, we have that $\psi'(x) > 0$ and that $\psi''(x) > 0$ for all $x > 0$. Let $\Psi : [0,\infty) \to \mathbb{R}$ be defined by $\Psi(x) = \psi'(x)$ for $x > 0$ and $\Psi(0) = 0$. Then Ψ is an increasing and continuous function on $[0,\infty)$ with $\Psi(0) = 0$. Therefore, by Young's inequality, for every $a, b > 0$ we have

$$ab \le \int_0^a \Psi(s)\,ds + \int_0^b \Psi^{-1}(s)\,ds = \psi(a) + H(b), \tag{6.1}$$

using the fact that ψ is continuous from the left at zero with $\psi(0) = 0$, and the definition

$$H(x) = \int_0^x \Psi^{-1}(s)\,ds, \quad x \ge 0. \tag{6.2}$$

Now for $x > 0$, using the fact that ψ is twice differentiable, and that $\psi'(0+) = 0$, we have

$$H(x) = \int_0^x \Psi^{-1}(s)\,ds = \int_0^x (\psi')^{-1}(s)\,ds = \int_{0+}^{(\psi')^{-1}(x)} w\psi''(u)\,dw.$$

Now, by integration by parts, and the definition of θ, we have

$$\begin{aligned}
H(x) &= \int_{0+}^{(\psi')^{-1}(x)} w\psi''(w)\,dw \\
&= (\psi')^{-1}(x)\psi'((\psi')^{-1}(x)) - \lim_{w\to 0^+} w\psi'(w) - \int_{0+}^{(\psi')^{-1}(x)} \psi'(w)\,dw \\
&= (\psi')^{-1}(x)\psi'((\psi')^{-1}(x)) - \lim_{w\to 0^+} w\psi'(w) - \psi((\psi')^{-1}(x)) - \lim_{w\to 0^+} \psi(w) \\
&= \theta(x),
\end{aligned}$$

since $\psi(w) \to 0$ as $w \to 0^+$ and $w\psi'(w) \to 0$ as $w \to 0^+$ by hypothesis. Therefore by (6.1) and the fact that $\psi(a) = a\Theta^{-1}(a)$ for $a > 0$, we have

$$ab \le a\Theta^{-1}(a) + \theta(b), \quad \text{for all } a, b > 0. \tag{6.3}$$

We notice also that the definition of H forces $\theta(x) = H(x) > 0$ for all $x > 0$, and since Ψ^{-1} is a positive and increasing function, it follows that θ will be increasing and convex on $(0,\infty)$.

Now, define

$$I(x) = \int_0^x \Theta(f(s))\,ds, \quad x \ge 0 \tag{6.4}$$

Notice that $I(x) > 0$ for $x > 0$ because $\Theta(x) > 0$ and $f(x) > 0$ for $x > 0$. Also, $\Theta \circ f \notin L^1(0,\infty)$ is equivalent to $I(x) \to \infty$ as $x \to \infty$. Define also

$$V(t) = I(x(t)), \quad t \geq 0. \tag{6.5}$$

Since $\Theta \circ f$ is continuous on $[0,\infty)$ and the solution x of (2.1) is in $C^1(0,\infty)$, it follows that $V \in C^1(0,\infty)$ and moreover

$$V'(t) = \Theta(f(x(t)))x'(t) = -f(x(t))\Theta(f(x(t))) + g(t)\Theta(f(x(t))), \quad t > 0. \tag{6.6}$$

By hypothesis, $g(t) > 0$ for all $t \geq 0$. Also, it is a consequence of our hypotheses that $x(t) > 0$ for all $t > 0$, and so by (2.2) that $f(x(t)) > 0$ for all $t \geq 0$. Since $\Theta(0) = 0$ and Θ is increasing on $(0,\infty)$ by hypothesis, it follows that $\Theta(f(x(t))) > 0$ for all $t \geq 0$. Therefore we can apply (6.3) with $b := g(t) > 0$ and $a = \Theta(f(x(t))) > 0$ to get

$$\begin{aligned}\Theta(f(x(t)))g(t) &\leq \Theta(f(x(t)))\Theta^{-1}(\Theta(f(x(t)))) + \theta(g(t)) \\ &= f(x(t))\Theta(f(x(t))) + (\theta \circ g)(t), \quad t \geq 0.\end{aligned}$$

Inserting this estimate into (6.6) we get

$$V'(t) = -f(x(t))\Theta(f(x(t))) + g(t)\Theta(f(x(t))) \leq (\theta \circ g)(t), \quad t > 0.$$

Therefore by (6.4) and (6.5) we get

$$I(x(t)) = V(t) = V(0) + \int_0^t V'(s)\,ds \leq V(0) + \int_0^t (\theta \circ g)(s)\,ds = I(\xi) + \int_0^t (\theta \circ g)(s)\,ds$$

for all $t \geq 0$. Since $\theta \circ g \in L^1(0,\infty)$ by hypothesis, we have that there is a finite $K > 0$ such that

$$I(x(t)) \leq I(\xi) + \int_0^\infty (\theta \circ g)(s)\,ds =: K, \quad t \geq 0.$$

The positivity of K is guaranteed by the fact that $I(x) > 0$ for $x > 0$, and the fact that $\theta(x) > 0$ for $x > 0$ and $g(t) > 0$ for $t > 0$. Suppose now that $\limsup_{t\to\infty} x(t) = +\infty$, so by the continuity of $t \mapsto x(t)$, there is an increasing sequence of times $t_n \to \infty$ such that $x(t_n) = n$. Then $I(n) \leq K$ for all $n \in \mathbb{N}$ sufficiently large. Since $I(n) \to +\infty$ as $n \to \infty$, we have $\infty = \lim_{n\to\infty} I(n) \leq K < +\infty$, a contradiction. Therefore, it follows that $\limsup_{t\to\infty} x(t)$ is finite and non–negative. Therefore by (4), we have that $x(t) \to 0$ as $t \to \infty$, as required. □

The next result is a corollary of Theorem 27 which is of utility when $f(x)$ decays like a power of x for large x. In this case, we know from our earlier analysis that g must also exhibit a power law decay. Our Liapunov–like result also reflects this fact.

Corollary 1. *Suppose that f obeys (2.2) and (2.4), and $g \in C([0,\infty);(0,\infty))$ satisfies $g(t) \to 0$ as $t \to \infty$. Suppose that there is $\alpha > 0$ such that $f^\alpha \notin L^1(0,\infty)$ and $g^{1+\alpha} \in L^1(0,\infty)$. Then x, the unique continuous solution of (2.1), obeys $x(t) \to 0$ as $t \to \infty$.*

Proof. Suppose for all $x \geq 0$ that $\Theta(x) = x^\alpha$, where $\alpha > 0$. Then Θ is increasing on $(0,\infty)$ with $\Theta^{-1}(x) = x^{1/\alpha}$ for $x \geq 0$. Moreover, we have that Θ is in $C^2(0,\infty)$. Now, define $\psi(x) = x^{1+1/\alpha}$ for $x \geq 0$. Then $\psi(0) = 0$, $\psi'(x) = (1+1/\alpha)x^{1/\alpha} > 0$ for $x > 0$ and $\psi''(x) = \alpha^{-1}(1+\alpha^{-1})x^{1/\alpha-1} > 0$ for $x > 0$. Thus ψ is increasing and convex with $\lim_{x\to 0^+} x\psi'(x) = 0$. With $\psi'(x) = \Psi(x) = (1+1/\alpha)x^{1/\alpha}$ for $x > 0$, and $\Psi(0) = 0$, we have $\Psi^{-1}(x) = K_\alpha x^\alpha$ for $x \geq 0$, where $K_\alpha = 1/(1+\alpha^{-1})^\alpha > 0$. Therefore for $x \geq 0$, we have that $\theta(x) = \int_0^x \Psi^{-1}(s)\,ds = K_\alpha(1+\alpha)^{-1}x^{1+\alpha}$. Thus

$g^{1+\alpha} \in L^1(0,\infty)$ implies that $\theta \circ g \in L^1(0,\infty)$. Moreover $\Theta \circ f = f^\alpha \notin L^1(0,\infty)$. Therefore, all the hypotheses of Theorem 27 are satisfied, and so $x(t) \to 0$ as $t \to \infty$, as claimed. □

An example illustrates the close connection between Corollary 1 and Theorem 10. In fact we see that the results are consistent in many cases.

Example 28. Suppose that there is $\beta > 0$ such that $f(x) \sim x^{-\beta}$ as $x \to \infty$ and that $g^{1+1/\beta} \in L^1(0,\infty)$. Let $\alpha = 1/\beta > 0$. Then $f^\alpha(x) \sim x^{-1}$ as $x \to \infty$, and thus $f^\alpha \notin L^1(0,\infty)$ and $g^{1+\alpha} \in L^1(0,\infty)$. Thus, by Corollary 1, we have that $x(t) \to 0$ as $t \to \infty$.

A condition that implies $g^{1+1/\beta} \in L^1(0,\infty)$ but $g \notin L^1(0,\infty)$ is $g(t) \sim t^{-\eta}$ as $t \to \infty$ for $\eta \in (\beta/(\beta+1), 1)$. Then

$$\int_0^t g(s)\,ds \sim \frac{1}{\eta} t^{1-\eta}, \text{ as } t \to \infty$$

while

$$F(x) = \int_1^x 1/f(u)\,du \sim \int_1^x u^\beta\,du = \frac{1}{1+\beta} x^{1+\beta}, \text{ as } x \to \infty.$$

Therefore $F^{-1}(x) = C_\beta x^{1/(\beta+1)}$ as $x \to \infty$. Hence

$$\lim_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} = \frac{1}{C_\beta \eta} \lim_{t\to\infty} \frac{t^{1-\eta}}{t^{1/(\beta+1)}} = 0.$$

By Theorem 10, we have that $x(t) \to 0$ as $t \to \infty$.

Therefore if $f(x) \sim x^{-\beta}$ for some $\beta > 0$ and $g(t) \sim t^{-\eta}$ as $t \to \infty$ for $\eta > \beta/(\beta+1)$, both Theorem 10 and Corollary 1 imply that $x(t) \to 0$ as $t \to \infty$. If $\eta > \beta/(\beta+1)$, we have that

$$\lim_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} = \frac{1}{C_\beta \eta} \lim_{t\to\infty} \frac{t^{1-\eta}}{t^{1/(\beta+1)}} = +\infty,$$

and so we know from Theorem 13 that $x(t,\xi) \to \infty$ as $t \to \infty$ for all initial conditions $\xi > 0$ that are sufficiently large. On the other hand, we see that the conditions of Corollary 1 do not hold if $\eta > \beta/(\beta+1)$, because $g^{1+1/\beta}(t) \sim t^{-\eta(\beta+1)/\beta}$ as $t \to \infty$, and so $g^{1+1/\beta} \notin L^1(0,\infty)$. Therefore, the conditions of Corollary 1 are quite sharp.

One reason to use the general form of Young's inequality in the proof of Theorem 27 is to enable us to prove stability results for differential equations in which g and f do not have power law asymptotic behaviour. The following example shows how Theorem 27 can be used in this situation.

Example 29. Suppose that $f(x) = e^{-x}$ for $x \geq 1$ and that $f(x) = xe^{-1}$ for $x \in [0,1]$. Suppose that $g/\log(1/g) \in L^1(0,\infty)$. Let Θ be such that $\Theta(0) = 0$, $\Theta(y) = 1/\log(1/y)$ for $0 < y \leq 1/e$.

If we now suppose that we can extend Θ on $[1/e,\infty)$ so that Θ is twice differentiable and increasing on $[1/e,\infty)$ and $y \mapsto y\Theta^{-1}(y)$ is convex on $(1,\infty)$, Theorem 27 allows us to conclude that $x(t) \to 0$ as $t \to \infty$.

Notice that $\Theta^{-1}(y) = e^{-1/y}$ for $0 < y \leq 1$. Therefore for $y > 0$, we may define $\psi(y) = y\Theta^{-1}(y)$ with $\psi(0) = 0$. Since Θ is increasing, Θ^{-1} is increasing, and so ψ is increasing, and by hypothesis, ψ is convex on $[1,\infty)$.

In particular, for $y \in (0,1]$ we have $\psi(y) = ye^{-1/y}$. Then $\psi'(y) = (1 + y^{-1})e^{-1/y} > 0$ for $y \in (0,1)$ and

$$\psi''(y) = (1+y^{-1})e^{-1/y} \cdot \frac{1}{y^2} - \frac{1}{y^2}e^{-1/y} = \frac{1}{y^3}e^{-1/y} > 0$$

for $y \in (0,1)$. Therefore ψ is increasing and convex on $(0,\infty)$. Also, we have the limit $\lim_{y\to 0^+} y\psi'(y) = 0$. Now for x sufficiently small

$$\theta(x) = \int_0^{(\psi')^{-1}(x)} y\psi''(y)\,dy = \int_0^{(\psi')^{-1}(x)} \frac{1}{y^2}e^{-1/y}\,dy = \int_{1/(\psi')^{-1}(x)}^{\infty} e^{-u}\,du,$$

so $\theta(x) = e^{-1/(\psi')^{-1}(x)}$ for $x > 0$ sufficiently small. Now, using the formula for ψ', we have for $x > 0$ sufficiently small

$$x = \left(1 + \frac{1}{(\psi')^{-1}(x)}\right) e^{-1/(\psi')^{-1}(x)}.$$

Therefore we have $\log 1/x \sim 1/(\psi')^{-1}(x)$ as $x \to 0^+$, from which the limit

$$\lim_{x\to 0^+} \frac{\theta(x)}{x/\log(1/x)} = \lim_{x\to 0^+} \frac{e^{-1/(\psi')^{-1}(x)}}{x/\log(1/x)} = \lim_{x\to 0^+} \frac{x/\left(1+\frac{1}{(\psi')^{-1}(x)}\right)}{x/\log(1/x)} = 1$$

can be inferred. Since $g(t) \to 0$ as $t \to \infty$ and $g/\log(1/g) \in L^1(0,\infty)$, we have that $\theta \circ g \in L^1(0,\infty)$. Also, because $\Theta(f(x)) = 1/x$ for $x \geq 1$ we have that $\Theta \circ f \notin L^1(0,\infty)$. Therefore all the hypotheses of Theorem 27 hold, and we conclude that $x(t) \to 0$ as $t \to \infty$.

In the case when $f(x) = e^{-x}$ for $x \geq 1$ and $g(t) \sim Ct^{-\eta}$ as $t \to \infty$ for any $\eta > 1$ and $C > 0$ we have that $g(t)/\log(1/g(t)) \sim t^{-\eta}/\log t$ as $t \to \infty$, and so $g/\log(1/g) \in L^1(0,\infty)$ and $\Theta \circ f \notin L^1(0,\infty)$. Therefore, by Theorem 27 we have that $x(t) \to 0$ as $t \to \infty$. If $\eta \leq 1$, then $g/\log(1/g) \notin L^1(0,\infty)$, and so the argument above does not apply.

On the other hand, we have for $x \geq 1$ that $F(x) = \int_1^x e^u\,du = e^x - e$, and so $F^{-1}(x) \sim \log(x)$ as $x \to \infty$. Then for $\eta > 1$, $g \in L^1(0,\infty)$, and so

$$\lim_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} = 0.$$

Therefore, by Theorem 4, we have that $x(t) \to 0$ as $t \to \infty$. If $\eta = 1$, we have that $\int_0^t g(s)\,ds \to C\log t$ as $t \to \infty$ and so

$$\lim_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} = C.$$

If $C < 1$, then $x(t) \to 0$ as $t \to \infty$; if $C > 1$ we have that $x(t) \to \infty$ for all initial conditions sufficiently large. If $\eta < 1$, we have that $\int_0^t g(s)\,ds$ grows polynomially fast as $t \to \infty$, and therefore

$$\lim_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} = +\infty.$$

Therefore, for all initial conditions sufficiently large, we have $x(t) \to \infty$ as $t \to \infty$.

This discussion once again shows how the results from Section 3 are consistent with the Liapunov stability result Theorem 27, and that moreover, Theorem 27 is quite sharp. The sharp results from Section 3 show that global asymptotic convergence holds for all $\eta > 1$, but that for $\eta \leq 1$, we can have $x(t) \to \infty$ for some initial conditions. On the other hand, Theorem 27 guarantees the global convergence of solutions for $\eta > 1$, but does not apply if $\eta \leq 1$.

7. Proof of Theorem 1

For all $t \geq 0$, $x(t) = \xi - \int_0^t f(x(s))ds + \int_0^t g(s)ds \leq \xi + \int_0^\infty g(s)ds := K$. Suppose $\liminf_{t\to\infty} x(t) = x^* > 0$. Clearly $x^* \leq K$. Now, as $f(x) > 0$ for $x > 0$

$$\inf_{x\in[\frac{x^*}{2},K]} f(x) := \phi > 0.$$

Therefore there exists $T > 0$ such that $x(t) \geq x^*/2$ for all $t \geq T$. Thus $x^*/2 \leq x(t) \leq K$ for all $t \geq T$ and so $f(x(t)) \geq \phi$ for all $t \geq T$. Therefore as $g \in L^1(0,\infty)$, for $t \geq T$ we have

$$\begin{aligned} x(t) &= x(T) - \int_T^t f(x(s))ds + \int_T^t g(s)ds \\ &\leq x(T) - \phi(t-T) + \int_T^\infty g(s)ds. \end{aligned}$$

Thus, as $\phi > 0$, we have $\liminf_{t\to\infty} x(t) = -\infty$, a contradiction. Therefore

$$\liminf_{t\to\infty} x(t) = 0 \tag{7.1}$$

Since $g \in L^1(0,\infty)$, for every $\epsilon > 0$, there is $T_1(\epsilon) > 0$ such that

$$\int_t^\infty g(s)ds < \epsilon \quad \text{for all } t > T_1(\epsilon).$$

(7.1) implies that there exists $t_n \to \infty$ such that $\lim_{n\to\infty} x(t_n) = 0$. Thus for every $\epsilon > 0$ there exists an $N_1(\epsilon) \in \mathbb{N}$ such that $x(t_n) \leq \epsilon$ for all $n \geq N_1(\epsilon)$. Clearly there exists $N_2(\epsilon)$ such that $t_{N_2(\epsilon)} \geq T_1(\epsilon) + 1$. Let $N_3(\epsilon) = \max[N_1(\epsilon), N_2(\epsilon)]$. Then $t_{N_3(\epsilon)} > T_1(\epsilon)$ and as $N_3(\epsilon) \geq N_1(\epsilon)$, $x(t_{N_3(\epsilon)}) \leq \epsilon$. Let $T_2(\epsilon) = t_{N_3(\epsilon)}$. Then for $t \geq T_2(\epsilon)$, we have

$$\begin{aligned} x(t) &= x(t_{N_3(\epsilon)}) - \int_{t_{N_3(\epsilon)}}^t f(x(s))ds + \int_{t_{N_3(\epsilon)}}^t g(s)\,ds \\ &\leq \epsilon + \int_{t_{N_3(\epsilon)}}^t g(s)\,ds \leq \epsilon + \int_{t_{N_3(\epsilon)}}^\infty g(s)\,ds < 2\epsilon. \end{aligned}$$

Thus for every $\epsilon > 0$, there is a $T_2(\epsilon) > 0$ such that $x(t) < 2\epsilon$ for all $t \geq T_2(\epsilon)$. Hence $x(t) \to 0$ as $t \to \infty$.

8. Finite liminf implies zero limit and Proof of Theorem 2

In this section, we show that whenever x had a finite liminf, it must have a zero limit.

Lemma 2. *Suppose that g obeys* (2.3), (2.8)*, and g is non–negative. Suppose that f obeys* (2.2) *and that the solution x of* (2.1) *obeys*

$$\liminf_{t\to\infty} x(t) \leq x^* \tag{8.1}$$

for some $x^* > 0$. Then x obeys (2.6).

A consequence of Lemma 2 is that only two types of behaviour are possible for solutions of (2.1). Either solutions tend to zero, or they tend to infinity. This is nothing other than Theorem 2.

Proof of Theorem 2. Suppose that there exists $x^* > 0$ such that x obeys (8.1). Then by Lemma 2 it follows that $x(t) \to 0$ as $t \to \infty$. On the other hand, if there does not exist $x^* > 0$ such that $\liminf_{t\to\infty} x(t) \le x^*$, it follows that $\liminf_{t\to\infty} x(t) = +\infty$, which implies $x(t) \to \infty$ as $t \to \infty$. □

It remains to establish Lemma 2. In order to do so, we start by proving that (8.1) implies that x is bounded above.

Lemma 3. *Suppose that g obeys* (2.8)*, f obeys* (2.2) *and that the solution x of* (2.1) *obeys* (8.1)*. Then*

$$\limsup_{t\to\infty} x(t) \le 2x^*.$$

Proof. Suppose that $\limsup_{t\to\infty} x(t) > 2x^*$. Since f obeys (2.2), we may define $f^* = \min_{x\in[5x^*/4,3x^*/2]} f(x) > 0$. Let $\epsilon < f^*/2$. Since $g(t) \to 0$ as $t \to \infty$, there is $T_1(\epsilon) > 0$ such that $g(t) \le \epsilon$ for all $t \ge T_1(\epsilon)$. Let $T_2(\epsilon) = \inf\{t > T_1(\epsilon) : x(t) = 5x^*/4\}$ and $T_3(\epsilon) = \inf\{t > T_2(\epsilon) : x(t) = 3x^*/2\}$. Then $x'(T_3) \ge 0$. Since $T_3 > T_2 > T_1$ we have

$$0 \le x'(T_3) = -f(x(T_3)) + g(T_3) = -f(3x^*/2) + g(T_3) \le -f^* + \epsilon < -f^* + f^*/2 < 0,$$

a contradiction. □

We next show that x has a zero liminf.

Lemma 4. *Suppose that g obeys* (2.8)*, f obeys* (2.2) *and that the solution x of* (2.1) *obeys* (8.1)*. Then*

$$\liminf_{t\to\infty} x(t) = 0.$$

Proof. Suppose that $\liminf_{t\to\infty} x(t) = c > 0$. By Lemma 3 it follows also that $c \le \limsup_{t\to\infty} x(t) \le 2x^*$. Therefore there exists $T_1 > 0$ such that $0 < c/2 \le x(t) \le 4x^*$ for all $t \ge T_1$. Define $c_1 = \min_{x\in[c/2,4x^*]} f(x) > 0$. Then $f(x(t)) \ge c_1$ for all $t \ge T_1$. Since $g(t) \to 0$ as $t \to \infty$, it follows that there exists $T_2 > 0$ such that that $g(t) \le c_1/2$ for all $t \ge T_2$. Let $T_3 = \max(T_1, T_2)$. Then for all $t \ge T_3$ we have

$$x'(t) = -f(x(t)) + g(t) \le -c_1 + \frac{c_1}{2} = -\frac{c_1}{2}.$$

Therefore we have that $x(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $x(t) \ge 0$ for all $t \ge 0$. □

We are now in a position to prove Lemma 2.

Proof of Lemma 2. By Lemma 3 we have that $\limsup_{t\to\infty} x(t) \le 2x^*$ and by Lemma 4 we have that $\liminf_{t\to\infty} x(t) = 0$. Therefore we have $x(t) < x^{**}$ for all $t \ge 0$. Suppose that there is $c \in (0, x^{**})$ such that $\limsup_{t\to\infty} x(t) > c$. Fix $\eta \in (0, c)$. Since f obeys (2.2) and g obeys (2.8) we may define

$$0 < \epsilon_1(\eta) = \min_{x\in[\eta,x^{**}]} f(x),$$
$$T(\eta) = \sup\{t > 0 : g(t) > \epsilon_1(\eta)/2\}.$$

Define $T_1(\eta) = \inf\{t > T(\eta) : x(t) = \eta\}$. There exists $T^* > T_1(\eta)$ such that $x(t) > c > \eta$. Let $T_2 = \sup\{t < T^* : x(t) = \eta\}$. Then $T_2 \ge T_1$ and there is a $\delta > 0$

such that $x(t) > \eta$ for all $t \in (T_2, T_2 + \delta)$. However, for $t \in (T_2, T_2 + \delta)$ we have

$$\begin{aligned} x(t) &= x(T_2) - \int_{T_2}^t f(x(s))\,ds + \int_{T_2}^t g(s)\,ds \\ &\le x(T_2) - \int_{T_2}^t \epsilon_1(\eta)\,ds + \int_{T_2}^t \frac{\epsilon_1(\eta)}{2}\,ds \\ &= x(T_2) - (t - T_2)\frac{\epsilon_1(\eta)}{2} < x(T_2) = \eta, \end{aligned}$$

which contradicts the definition of T_2. Therefore we have that $\lim_{t\to\infty} x(t) = 0$, as required. □

9. Proof of Theorem 4, 5, 10 and 11

9.1. **Proof of Theorem 4.** It is seen from Lemma 2 above that if we can show that there is an $x^* > 0$ such that the solution x of (2.1) obeys (8.1), then x obeys (2.6).

Lemma 5. *Suppose that f obeys* (2.17) *and* (2.2) *and that g is continuous. Suppose that F is given by* (2.15) *and that f and g obey* (3.1)*. Let x be the unique continuous solution of* (2.1)*. Then it obeys* (8.1).

Proof. Since g and f obey (3.1), there exists $\lambda < 1$ such that

$$\limsup_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} = \lambda < 1.$$

Choose $\epsilon \in (0, 2/3)$ so small that $\lambda(1+\epsilon) < 1-\epsilon/2$. Therefore for every $\epsilon \in (0, 2/3)$ there exists $T(\epsilon) > 0$ such that

$$\frac{\int_0^t g(s)\,ds}{F^{-1}(t)} \le \lambda(1+\epsilon) < 1 - \epsilon/2, \quad t \ge T(\epsilon).$$

Therefore $\int_0^t g(s)\,ds \le (1-\epsilon/2)F^{-1}(t)$ for all $t \ge T(\epsilon)$. Since f obeys (2.2), by defining $x_\epsilon = x(T(\epsilon))$, for all $t \ge T(\epsilon)$ we have

$$\begin{aligned} x(t) &= x_\epsilon - \int_{T(\epsilon)}^t f(x(s))\,ds + \int_{T(\epsilon)}^t g(s)\,ds \\ &\le x_\epsilon + \int_{T(\epsilon)}^t g(s)\,ds \\ &< x_\epsilon + (1-\epsilon/2)F^{-1}(t) := G(t). \end{aligned}$$

Suppose, in contradiction to the desired conclusion, that $\liminf_{t\to\infty} x(t) = x_1 > x^*$. Then there exists $T_2 > 0$ such that for all $t \ge T_2$ we have $x(t) > x^*$. Let $T_3(\epsilon) = \max(T(\epsilon), T_2)$. Then for $x^* < x(t) < G(t)$, so by (2.17) we have $f(x(t)) \ge f(G(t))$. Hence for $t \ge T_3(\epsilon)$ we have

$$\begin{aligned} x(t) &= x(T_3) - \int_{T_3}^t f(x(s))\,ds + \int_{T_3}^t g(s)\,ds \\ &< x(T_3) - \int_{T_3}^t f(G(s))\,ds + (1-\epsilon/2)F^{-1}(t) \\ &= x(T_3) + (1-\epsilon/2)F^{-1}(T_3) + \int_{T_3}^t [-f(G(s) + (1-\epsilon/2)f(F^{-1}(s))]\,ds. \end{aligned}$$

Hence

$$x(t) < x(T_3)+(1-\epsilon/2)F^{-1}(T_3)+\int_{T_3}^t [-f(G(s))+(1-\epsilon/2)f(F^{-1}(s))]\, ds, \quad t \geq T_3(\epsilon). \tag{9.1}$$

We next show that

For every $\epsilon \in (0, 2/3)$ there exists $\theta_3(\epsilon) > 0$ such that

$$(1-\epsilon/2)f(\theta) - f(x_\epsilon + (1-\epsilon/2)\theta) < -\epsilon/4 f(\theta), \quad \text{for all } \theta > \theta_3(\epsilon). \tag{9.2}$$

Now define $T_4(\epsilon) = F(\theta_3(\epsilon))$ and let $T_5(\epsilon) = \max(T_3(\epsilon), T_4(\epsilon)) + 1$. Therefore for $t \geq T_5(\epsilon) > T_4(\epsilon) = F(\theta_3(\epsilon))$ we have $F^{-1}(t) > \theta_3(\epsilon)$. Thus by (9.2) we have

$$(1-\epsilon/2)f(F^{-1}(t)) - f(G(t)) < -\epsilon/4 f(F^{-1}(t)), \quad \text{for all } t \geq T_5(\epsilon).$$

Since $T_5(\epsilon) > T_3(\epsilon)$, by (9.1) we have

$$\begin{aligned} x(t) &< x(T_3) + (1-\epsilon/2)F^{-1}(T_3) + \int_{T_5}^t [-f(G(s)) + (1-\epsilon/2)f(F^{-1}(s))]\, ds, \\ &< x(T_3) + (1-\epsilon/2)F^{-1}(T_3) - \epsilon/4 \int_{T_5}^t f(F^{-1}(s))\, ds, \\ &= x(T_3) + (1-\epsilon/2)F^{-1}(T_3) - \epsilon/4[F^{-1}(t) - F^{-1}(T_5)], \end{aligned}$$

for all $t \geq T_5(\epsilon)$, therefore we have $\lim_{t\to\infty} x(t) = -\infty$. Since $\liminf_{t\to\infty} x(t) = x_1 > x^* > 0$ and $x'(t) < 0$ for all $t \geq T_5(\epsilon)$ it follows that $\lim_{t\to\infty} x(t) = x_1 > x^*$, a contradiction. Hence it follows that $\liminf_{t\to\infty} x(t) \leq x^*$.

It remains to prove (9.2). Since x_ϵ is fixed, for every $\epsilon \in (0, 4/3)$ there exists $\theta_1(\epsilon) > 0$ such that $-\epsilon\theta/4 < x_\epsilon < \epsilon\theta/4$ for all $\theta > \theta_1(\epsilon)$. Thus for $\theta > \theta_1(\epsilon)$ we have

$$0 < (1 - \frac{3\epsilon}{4})\theta < x_\epsilon + (1-\epsilon/2)\theta < (1-\epsilon/4)\theta.$$

Also, there exists $\theta_2(\epsilon) > 0$ such that $(1-3\epsilon/4)\theta_2(\epsilon) > x^*$. Define $\theta_3(\epsilon)$ by $\theta_3(\epsilon) = \max(\theta_1(\epsilon), \theta_2(\epsilon))$. Then for $\theta > \theta_3(\epsilon)$ we have

$$x^* < (1 - \frac{3\epsilon}{4})\theta < x_\epsilon + (1 - \frac{\epsilon}{2})\theta < (1-\epsilon/4)\theta < \theta.$$

Thus for $\theta > \theta_3(\epsilon)$, by (2.17) we have

$$f(x_\epsilon + (1-\epsilon/2)\theta) > f(\theta(1-\epsilon/4)) > f(\theta) > (1-\epsilon/4)f(\theta),$$

which proves (9.2). □

9.2. **Proof of Theorem 10.** It is seen from Lemma 2 above that if we can show that there is an $x^* > 0$ such that the solution x of (2.1) obeys (8.1), then x obeys (2.6). We next show that if g and f obey (3.2), then x does indeed obey (8.1).

Lemma 6. *Suppose that f obeys* (2.2), (2.17) *and* (2.22), *and that g is continuous. Suppose that F is given by* (2.15) *and that f and g obey* (3.2). *Let x be the unique continuous solution of* (2.1). *Then x obeys* (8.1).

In order to prove this result we require and auxiliary lemma.

Lemma 7. *Let $\beta > 0$. Let $\lambda \in (1, \lambda(\beta))$, where $\lambda(\beta)$ is given by* (2.23). *Define $\Lambda(0) = \lambda$ and*

$$\Lambda(n+1) = \lambda - \Lambda(n)^{-\beta}, \quad 0 \leq n \leq n', \quad n' := \inf\{n \geq 1 \,:\, \Lambda(n+1) \leq 0\}. \tag{9.3}$$

Then n' is finite and $0 < \Lambda(n+1) < \Lambda(n)$ for $n = 0, \ldots, n'-1$.

Proof. We first note that because $\Lambda(0) = \lambda > 1$, we have $\Lambda(1) > 0$, so we can only have $\Lambda(n+1) \leq 0$ for $n \geq 1$. Hence n' is appropriately defined. Suppose that n' is infinite. Then we have that $\Lambda(n) > 0$ for all $n \geq 0$.

Define $k_\lambda(x) = x - \lambda + x^{-\beta}$ for $x > 0$ and $h_\lambda(x) = x^{\beta+1} - \lambda x^\beta + 1$ for $x \geq 0$. Then for $x > 0$ we have $k_\lambda(x) = x^{-\beta} h_\lambda(x)$. Clearly we have $h'_\lambda(x) = x^{\beta-1}\left((\beta+1)x - \lambda\beta\right)$ for $x > 0$. Define $x_* = \beta\lambda/(\beta+1)$. Then $x_* \in (0, \lambda)$ and we have that h_λ is increasing on $(0, x_*)$ and decreasing on (x_*, ∞). Therefore for all $x > 0$ we have

$$h_\lambda(x) \geq h_\lambda(x_*) = x_*^\beta(x_* - \lambda) + 1 = \frac{\beta^\beta\lambda^\beta}{(\beta+1)^\beta}\left(\frac{\beta\lambda}{\beta+1} - \lambda\right) + 1 = 1 - \frac{\beta^\beta\lambda^{\beta+1}}{(\beta+1)^{1+\beta}}.$$

Since $\lambda < \lambda(\beta)$, it follows that the righthand side is positive, and so we have $h_\lambda(x) > 0$ for all $x > 0$. Hence $k_\lambda(x) > 0$ for all $x > 0$.

Since $\Lambda(n) > 0$ for all $n \geq 0$, we have $k_\lambda(\Lambda(n)) > 0$ for all $n \geq 0$. Therefore $\Lambda(n) > \lambda - \Lambda(n)^{-\beta}$ for all $n \geq 0$. But $\Lambda(n+1) = \lambda - \Lambda(n)^{-\beta}$ for all $n \geq 0$, so we have $\Lambda(n+1) < \Lambda(n)$ for all $n \geq 0$. Therefore we have that $\Lambda(n) \to L \geq 0$ as $n \to \infty$. Suppose that $L > 0$. Then we have $L = \lambda - L^{-\beta}$, or $L^{\beta+1} - \lambda L^\beta + 1 = 0$. But this implies that $h_\lambda(L) = 0$, a contradiction. Suppose that $L = 0$. Then we have

$$0 = \lim_{n\to\infty} \Lambda(n+1) = \lim_{n\to\infty} \lambda - \frac{1}{\Lambda(n)^\beta} = -\infty,$$

a contradiction. Therefore we must have that there is a finite $n' \geq 1$ such that $\Lambda(n) > 0$ for $n \leq n'$ and $\Lambda(n'+1) \leq 0$. Moreover, we note that $0 < \Lambda(n+1) < \Lambda(n)$ for $n = 0, \ldots, n'-1$. □

Proof of Lemma 6. Without loss of generality, we may take λ in (2.23) to obey $\lambda > 1$, i.e.,

$$\limsup_{t\to\infty} \frac{\int_0^t g(s)\,ds}{F^{-1}(t)} \leq \lambda < \lambda(\beta) \tag{9.4}$$

From (2.1) and (9.4), we have for all $\epsilon > 0$, there exists $T(\epsilon)$ such that for all $t > T(\epsilon)$:

$$\int_0^t g(s)\,ds \leq \lambda(1+\epsilon)F^{-1}(t), \quad t \geq T(\epsilon),$$

and so

$$x(t) \leq x(T(\epsilon)) + \int_{T(\epsilon)}^t g(s)\,ds \leq x(T(\epsilon)) + \lambda(1+\epsilon)F^{-1}(t).$$

Therefore we have

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \leq \lambda =: \Lambda(0) > 1,$$

where Λ is the sequence defined in Lemma 7, so there is a $T_0(\epsilon) > 0$ such that $x(t) \leq \lambda(1+\epsilon)F^{-1}(t)$ for $t \geq T_0(\epsilon)$. Suppose, in contradiction to the desired conclusion, that $\liminf_{t\to\infty} x(t) = x_1 > x^*$. Then there exists $T_1 > 0$ such that $x(t) > x^*$ for all $t \geq T_1$. We have $x^* < x(t) \leq \lambda(1+\epsilon)F^{-1}(t)$ for $t \geq \max(T_0(\epsilon), T_1)$, which implies that

$$-f(x(t)) \leq -f(\lambda(1+\epsilon)F^{-1}(t)), \quad t \geq \max(T_0(\epsilon), T_1).$$

Therefore for $T_2(\epsilon) = \max(T(\epsilon), T_0(\epsilon), T_1)$, we have

$$\begin{aligned} x(t) &\leq x(T_2) - \int_{T_2}^{t} f(\lambda(1+\epsilon)F^{-1}(s))\, ds + \int_{T_2}^{t} g(s)\, ds \\ &\leq x(T_2) - \int_{T_2}^{t} f(\lambda(1+\epsilon)F^{-1}(s) ds + \lambda(1+\epsilon)F^{-1}(t) \\ &= x(T_2) - \int_{F^{-1}(T_2)}^{F^{-1}(t)} \frac{f(\lambda(1+\epsilon)u)}{f(u)} du + \lambda(1+\epsilon)F^{-1}(t). \end{aligned}$$

Therefore

$$\frac{x(t)}{F^{-1}(t)} \leq \frac{x(T_2)}{F^{-1}(t)} - \frac{1}{F^{-1}(t)} \int_{F^{-1}(T_2)}^{F^{-1}(t)} \frac{f(\lambda(1+\epsilon)u)}{f(u)} du + \lambda(1+\epsilon).$$

Thus, as $f \in RV_\infty(-\beta)$ we have

$$\begin{aligned} \limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} &\leq - \lim_{x\to\infty} \frac{1}{x} \int_{F^{-1}(T_3)}^{x} \frac{f(\lambda(1+\epsilon)s)}{f(s)} ds + \lambda(1+\epsilon), \\ &= - \lim_{x\to\infty} \frac{f(\lambda(1+\epsilon)x)}{f(x)} + \lambda(1+\epsilon), \\ &= -[\lambda(1+\epsilon)]^{-\beta} + \lambda(1+\epsilon). \end{aligned}$$

Therefore by (9.3) we have

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \leq \lambda - \Lambda(0)^{-\beta} = \Lambda(1).$$

Introduce the n-th level hypothesis for $n = 0, \dots, n'$:

$$\Lambda(n) > 0, \quad \limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \leq \Lambda(n). \tag{9.5}$$

We have already that (9.5) is true for $n = 0$ and $n = 1$.

By Lemma 7, one of the following holds:

(a) There exists $n' \geq 1$ such that $\Lambda(n) > 0$ for $n \leq n'$ and $\Lambda(n'+1) < 0$;
(b) There exists $n' \geq 1$ such that $\Lambda(n) > 0$ for $n \leq n'$ and $\Lambda(n'+1) = 0$;

We show that (9.5) at level n implies (9.5) at level $n+1$ provided that $n = 0, \dots, n'-1$. Therefore as (9.5) is true at level 0, we have that (9.5) is true at level n'. Hence

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \leq \Lambda(n').$$

Since it is assumed that $x(t) > x^*$ for all $t \geq T_1$, for every $\epsilon > 0$ there exists a $T_3(\epsilon) = \max(T_1, T_2)$ such that $x^* < x(t) < \Lambda(n)(1+\epsilon)F^{-1}(t)$ for $t \geq T_3(\epsilon)$. We have

$$\begin{aligned} x(t) &= x(T_3) - \int_{T_3}^{t} f(x(s)) ds + \int_{T_3}^{t} g(s) ds, \\ &< x(T_3) - \int_{T_3}^{t} f(\Lambda(n)(1+\epsilon)F^{-1}(s)) ds + \lambda(1+\epsilon)F^{-1}(t), \\ &= x(T_3) - \int_{F^{-1}(T_3)}^{F^{-1}(t)} \frac{f(\Lambda(n)(1+\epsilon)u)}{f(u)} du + \lambda(1+\epsilon)F^{-1}(t). \end{aligned}$$

Therefore, we have

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \lambda(1+\epsilon) - (\Lambda(n)(1+\epsilon))^{-\beta}$$

Letting $\epsilon \to 0$ and using (9.3) yields

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \lambda - \Lambda(n)^{-\beta} = \Lambda(n+1),$$

which is simply (9.5) at level $n+1$.

We now consider the case distinctions $\Lambda(n'+1) < 0$ and $\Lambda(n'+1) = 0$. In the former case we have already shown that

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \Lambda(n'),$$

and this implies that

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \lambda - \Lambda(n')^{-\beta} = \Lambda(n'+1) < 0.$$

Since $F^{-1}(t) \to \infty$ as $t \to \infty$, therefore we have $\lim_{t\to\infty} x(t) = -\infty$, it follows that since for all $t > T_3$, $\liminf_{t\to\infty} x(t) = x_1 > x^* \ge 0$ and $x'(t) < 0$, it follows that $\lim_{t\to\infty} x(t) > x^*$, a contradiction. Hence we must have $\liminf_{t\to\infty} x(t) \le x^*$ and the proof is complete.

On the other hand, suppose that $\Lambda(n'+1) = 0$. Therefore we have $\Lambda(n') = \lambda^{-1/\beta} \in (0,1)$. Let $\epsilon' > 0$ be so small that $\epsilon' \in (0, \lambda^{1/\beta} - 1)$ and

$$(1+\epsilon')^\beta < \frac{1}{1-\lambda^{-(\beta+1)/\beta}}.$$

Now we have that

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \Lambda(n') = \lambda^{-1/\beta} < \lambda^{-1/\beta}(1+\epsilon') =: \lambda' < 1. \tag{9.6}$$

Now define

$$\lambda'' := \lambda - (\lambda')^{-\beta} = \lambda\left(1 - \frac{1}{(1+\epsilon')^\beta}\right) > 0, \tag{9.7}$$

Moreover as $(1+\epsilon')^{-\beta} > 1 - \lambda^{-(\beta+1)/\beta}$, we have $1-(1+\epsilon')^{-\beta} < \lambda^{-(\beta+1)/\beta}$, so

$$\lambda'' = \lambda\left(1 - \frac{1}{(1+\epsilon')^\beta}\right) < \lambda^{-1/\beta}.$$

Thus $\lambda'' \in (0, \lambda^{-1/\beta})$, and we can prove that (9.6) and (9.7) together imply

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \le \lambda''. \tag{9.8}$$

Proceeding as in the case when $\Lambda(n'+1) < 0$ we arrive once more at the conclusion that $\liminf_{t\to\infty} x(t) \le x^*$. □

10. Proof of Theorem 6, 8, 12

Lemma 8. *Let $\alpha > 0$. Define $g \in C([0,\infty);(0,\infty))$ by*

$$g(t) = (1+\alpha)f(F^{-1}(\alpha t + F(\xi/2))) + e^{-t}, \quad t \ge 0. \tag{10.1}$$

Then the solution of (2.1) obeys $x(t) \to \infty$ as $t \to \infty$.

Proof. Define $x_L(t) = F^{-1}(\alpha t + F(\xi/2))$ for $t \geq 0$. Then $x_L(0) = \xi/2 < x(0)$. Clearly for $t \geq 0$ we have

$$x_L'(t) + f(x_L(t)) - g(t) = (1+\alpha) f(F^{-1}(\alpha t + F(\xi/2))) - g(t) < 0.$$

Then $x_L(t) < x(t)$ for all $t \geq 0$. Since $x_L(t) \to \infty$ as $t \to \infty$, we have that $x(t) \to \infty$ as $t \to \infty$. □

Lemma 9. *Let g be defined by* (10.1).

(i) *If $f \circ F^{-1} \in RV_\infty(-1)$, then*

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = 1 + \alpha^{-1}.$$

(ii) *If $\beta \geq 0$ and $f \in RV_\infty(-\beta)$, then*

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = (1+\alpha)\alpha^{-\beta/(\beta+1)}. \tag{10.2}$$

Proof. Since $f \circ F^{-1} \in \mathrm{RV}_\infty(-1)$ we have that

$$\lim_{t\to\infty} \frac{(f \circ F^{-1})(\alpha t + F(\xi/2))}{(f \circ F^{-1})(\alpha t)} = 1.$$

Also as $f \circ F^{-1} \in \mathrm{RV}_\infty(-1)$ we

$$\lim_{t\to\infty} \frac{(f \circ F^{-1})(\alpha t)}{(f \circ F^{-1})(t)} = \alpha^{-1}.$$

Since $f \circ F^{-1} \in \mathrm{RV}_\infty(-1)$, we have $e^{-t}/(f \circ F^{-1})(t) \to 0$ as $t \to \infty$ and so g obeys

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = (1+\alpha) \lim_{t\to\infty} \frac{(f \circ F^{-1})(\alpha t + F(\xi/2))}{(f \circ F^{-1})(\alpha t)} \frac{(f \circ F^{-1})(\alpha t)}{(f \circ F^{-1})(t)} = 1 + \alpha^{-1}.$$

Note that when $f \in \mathrm{RV}_\infty(-\beta)$, we have $f \circ F^{-1} \in \mathrm{RV}_\infty(-\beta/(\beta+1))$, so

$$\lim_{t\to\infty} \frac{f(F^{-1}(\alpha t))}{f(F^{-1}(t))} = \alpha^{-\beta/(\beta+1)}.$$

Since $f \circ F^{-1}$ is in $\mathrm{RV}_\infty(-\beta/(\beta+1))$ we have that

$$\lim_{t\to\infty} \frac{(f \circ F^{-1})(\alpha t + F(\xi/2))}{(f \circ F^{-1})(\alpha t)} = 1.$$

Since $f \circ F^{-1} \in \mathrm{RV}_\infty(-\beta/(\beta+1))$, we have $e^{-t}/(f \circ F^{-1})(t) \to 0$ as $t \to \infty$ and so g obeys

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = (1+\alpha) \lim_{t\to\infty} \frac{f(F^{-1}(\alpha t + F(\xi/2))}{f(F^{-1}(\alpha t))} \frac{f(F^{-1}(\alpha t))}{f(F^{-1}(t))} = (1+\alpha)\alpha^{-\beta/(\beta+1)},$$

as required. □

Proof of Theorem 8. Let $\kappa > 1$. By hypothesis $f \circ F^{-1} \in \mathrm{RV}_\infty(-1)$, and let $\alpha = 1/(\kappa - 1) > 0$. If g is defined by (10.1), then by part (i) of Lemma 9 we have

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = \kappa > 1.$$

Moreover, by Lemma 8 we have that $x(t) \to \infty$. □

Proof of Theorem 6. Let $\kappa > 1$. By hypothesis $f \circ F^{-1} \in \text{RV}_\infty(0)$. Let $\alpha = \kappa - 1 > 0$. If g is defined by (10.1), then by part (ii) of Lemma 9 we have

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = \kappa > 1.$$

Moreover, by Lemma 8 we have that $x(t) \to \infty$. □

Proof of Theorem 12. Let $\kappa \geq (1+\beta)\beta^{-\beta/(\beta+1)}$. Since $f \in \text{RV}_\infty(-\beta)$ we have $f \circ F^{-1} \in \text{RV}_\infty(-\beta/(\beta+1))$. Since $\kappa \geq (1+\beta)\beta^{-\beta/(\beta+1)}$ there exists a unique $\alpha \in (0, \beta]$ such that.

$$(1+\alpha)\alpha^{-\beta/(\beta+1)} = \kappa.$$

Since g is defined by (10.1), then by part (ii) of Lemma 9 we have

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = (1+\alpha)\alpha^{-\beta/(\beta+1)} = \kappa.$$

Moreover, by Lemma 8 we have that $x(t) \to \infty$. □

11. Proof of Theorems 3, 7, 9 and 13

The proof of Theorem 3 is an easy consequence of Lemma 2, and is given next. We consider the proof of the other theorems in the second subsection.

11.1. **Proof of Theorem 3.** There exists $\epsilon_0 > 0$ sufficiently small so that the set $\inf\{x > 0 : f(x) = 2\epsilon_0\}$ is non–empty. For $\epsilon \in (0, \epsilon_0)$ define $x_1(\epsilon) = \inf\{x > 0 : f(x) = 2\epsilon\}$. Then $f(x) < 2\epsilon$ for all $x \in [0, x_1(\epsilon))$. Suppose also that $g(t) \leq \epsilon$ for all $t \geq 0$.

Let $x(0) < x_1(\epsilon)$. Suppose there is a finite $T_1(\epsilon) = \inf\{t > 0 : x(t) = x_1(\epsilon)\}$. Then $x'(T_1(\epsilon)) \geq 0$. Also

$$0 \leq x'(T_1(\epsilon)) = -f(x(T_1(\epsilon))) + g(T_1(\epsilon)) \leq -f(x_1(\epsilon)) + \epsilon = -\epsilon < 0,$$

a contradiction. Hence we have that $x(t) < x_1(\epsilon)$ for all $t \geq 0$. Now by Lemma 2 it follows that $x(t) \to 0$ as $t \to \infty$.

11.2. **Proof of Theorems 7, 9 and 13.** In order to prove Theorems 7, 9 and 13, it proves convenient to establish the following condition on g and f:

$$\text{There exists } \alpha > 0 \text{ such that } \liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(\alpha t))} > 1 + \alpha. \tag{11.1}$$

We now show that (11.1) is satisfied under the conditions on g and f given in Theorems 7, 9 and 13.

Lemma 10. *Suppose that f obeys* (2.2) *and that g obeys* (2.3)*.*

(i) *Suppose that f obeys* (2.18) *and that g and f obey* (2.21)*. Then g and f obeys* (11.1)*.*
(ii) *Suppose that f obeys* (2.19) *and that g and f obey* (2.21)*. Then g and f obeys* (11.1)*.*
(iii) *Suppose that f obeys* (2.22) *and that g and f obey* (2.25)*. Then g and f obeys* (11.1)*.*

Proof. For part (i), by (2.21), there is $\kappa > 1$ be given by

$$\kappa = \liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))}. \tag{11.2}$$

Then we may choose $\alpha > 1/(\kappa-1) > 0$. Hence by (2.21) and (2.18) we have

$$\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(\alpha t))} = \kappa / \lim_{t\to\infty} \frac{f(F^{-1}(\alpha t))}{f(F^{-1}(t))} = \frac{\kappa}{\alpha}.$$

Since $\alpha > 1/(\kappa-1)$, we have $\kappa\alpha > \alpha+1$, so (11.1) holds.

For part (ii), once again there is $\kappa > 1$ which obeys (11.2). Then we may choose $\alpha \in (0, \kappa-1) > 0$. Then $\alpha < \kappa - 1$. Hence by (2.21) and (2.19) we have

$$\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(\alpha t))} = \kappa / \lim_{t\to\infty} \frac{f(F^{-1}(\alpha t))}{f(F^{-1}(t))} = \kappa.$$

Since $\kappa > \alpha + 1$, (11.1) holds.

For part (iii), there is $\lambda > \lambda(\beta) = (1+\beta)\beta^{-\beta/(1+\beta)}$ such that

$$\lambda = \liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))}.$$

Let $\alpha = \beta > 0$. Since f is in $\mathrm{RV}_\infty(-\beta)$, it follows that $f \circ F^{-1}$ is in $\mathrm{RV}_\infty(-\beta/(\beta+1))$. Using this and the fact that f and g obey (2.25), we have

$$\begin{aligned}\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(\alpha t))} &= \lambda / \lim_{t\to\infty} \frac{f(F^{-1}(\alpha t))}{f(F^{-1}(t))} = \lambda / \alpha^{-\beta/(\beta+1)} \\ &= \lambda \beta^{\beta/(\beta+1)} > \lambda(\beta)\beta^{\beta/(\beta+1)} = 1 + \beta = 1 + \alpha,\end{aligned}$$

which proves (11.1). □

Lemma 11. *Suppose that f obeys* (2.2) *and that g obeys* (2.3). *Let x be the unique continuous solution of* (2.1). *Suppose that g and f obey* (11.1). *Then there exists $\bar{x} > 0$ such that for all $\xi > \bar{x}$, we have that* $\lim_{t\to\infty} x(t,\xi) = \infty$.

Proof. Define F by (2.15). By (11.1) there exists $\eta > 1 + \alpha$ such that

$$\liminf_{t\to\infty} \frac{g(t)}{f(F^{-1}(\alpha t))} = \eta.$$

Now for $\epsilon > 0$ sufficiently small we have $\eta(1-\epsilon) > (1+\alpha)(1+\epsilon)$. For such an $\epsilon > 0$ sufficiently small, there is $T(\epsilon) > 0$ such that

$$\frac{g(t)}{f(F^{-1}(\alpha t))} \geq \eta(1-\epsilon), \quad t \geq T(\epsilon),$$

and so

$$g(t) \geq (1+\epsilon)(1+\alpha) f(F^{-1}(\alpha t)), \quad t \geq T(\epsilon). \tag{11.3}$$

Next suppose that

$$\xi > F^{-1}((1+\alpha)T(\epsilon)), \quad \xi > F^{-1}(F(1) + T(\epsilon)). \tag{11.4}$$

Define

$$x_L(t) = F^{-1}(\alpha t), \quad t \geq T(\epsilon). \tag{11.5}$$

Define by y the solution of

$$y'(t) = -f(y(t)), \quad t \geq 0; \quad y(0) = \xi. \tag{11.6}$$

Since $g(t) \geq 0$ for all $t \geq 0$, we have $x'(t) \geq -f(x(t))$ for all $t \geq 0$. Then $x(t) \geq y(t)$ for all $t \geq 0$. Now, by (11.6) and (2.15) we have $y(t) = F^{-1}(F(\xi) - t)$ for all $t \in [0, T(\epsilon)]$, because the second part of (11.4) guarantees that $y(t) > 1$ for all $t \in [0, T(\epsilon)]$. Therefore by the first part of (11.4) and (11.5) we have

$$x_L(T(\epsilon)) = F^{-1}(\alpha T(\epsilon)) < F^{-1}(F(\xi) - T(\epsilon)) = y(T(\epsilon)) \leq x(T(\epsilon)). \tag{11.7}$$

Next note for $t \geq T(\epsilon)$ and by using (11.5) and (11.3) we have

$$\begin{aligned} x_L'(t) + f(x_L(t)) - g(t) &= (1+\alpha)f(F^{-1}(\alpha t)) - g(t) \\ &\leq (1+\alpha)f(F^{-1}(\alpha t)) - (1+\epsilon)(1+\alpha)f(F^{-1}(\alpha t)) \\ &= -\epsilon(1+\alpha)f(F^{-1}(\alpha t)) < 0. \end{aligned}$$

Therefore by this and (11.7) we have

$$x_L'(t) < -f(x_L(t)) + g(t), \quad t \geq T(\epsilon); \quad x_L(T(\epsilon)) < x(T(\epsilon)). \tag{11.8}$$

Hence $x_L(t) < x(t)$ for all $t \geq T(\epsilon)$. Therefore as $\alpha > 0$ we have $x_L(t) \to \infty$ as $t \to \infty$, and therefore it follows that $x(t) \to \infty$ as $t \to \infty$, as required. □

The proof of Theorem 7 is now a consequence of part (i) of Lemma 10 and Lemma 11. The proof of Theorem 9 is a consequence of part (ii) of Lemma 10 and Lemma 11. The proof of Theorem 13 is a consequence of part (iii) of Lemma 10 and Lemma 11.

12. Proof of Proposition 2

Note G is increasing. Moreover as $g \in \mathrm{RV}_\infty(0)$, we have $G \in \mathrm{RV}_\infty(1)$. Therefore $G^{-1} \in \mathrm{RV}_\infty(1)$, and so $g \circ G^{-1} \in \mathrm{RV}_\infty(0)$. By (3.6), we have that $f \in \mathrm{RV}_\infty(0)$. Since $G^{-1}(x) \to \infty$ as $x \to \infty$ and g obeys (2.8), we have from (3.6) that f obeys (2.11). Since g is decreasing and G^{-1} is increasing, $x \mapsto g(G^{-1}(x))$ is decreasing, and so by (3.6), f is asymptotic to a decreasing function.

Define $G_\lambda(t) = G(t)/\lambda$ for $t \geq 0$. Then G_λ^{-1} exists and we have $G_\lambda^{-1}(t) = G^{-1}(t/\lambda)$. Since $g \circ G^{-1} \in \mathrm{RV}_\infty(0)$, we have as $x \to \infty$ that

$$f(x) \sim \frac{1}{\lambda} g(G^{-1}(x)) \sim \frac{1}{\lambda} g(G^{-1}(x/\lambda)) = \frac{1}{\lambda} g(G_\lambda^{-1}(x)). \tag{12.1}$$

Now $g \in \mathrm{RV}_\infty(0)$ implies that $G_\lambda(x) \sim xg(x)/\lambda$ as $x \to \infty$. Since $G_\lambda^{-1}(x) \to \infty$ as $x \to \infty$, we have that $\lambda x \sim G_\lambda^{-1}(x) g(G_\lambda^{-1}(x))$ as $x \to \infty$. Therefore we have that as $x \to \infty$

$$f(x) \sim \frac{1}{\lambda} g(G_\lambda^{-1}(x)) \sim \frac{1}{\lambda} \cdot \frac{\lambda x}{G_\lambda^{-1}(x)} = \frac{x}{G_\lambda^{-1}(x)}.$$

Since f is in $\mathrm{RV}_\infty(0)$ we have as $x \to \infty$ that

$$F(x) = \int_1^x \frac{1}{f(u)}\, du \sim \frac{x}{f(x)} \sim G_\lambda^{-1}(x)$$

Since g is decreasing and g is in $\mathrm{RV}_\infty(0)$ we have that $g(F(x)) \sim g(G_\lambda^{-1}(x))$ as $x \to \infty$. Therefore by (12.1) we have that as $x \to \infty$

$$g(F(x)) \sim g(G_\lambda^{-1}(x)) \sim \lambda f(x).$$

Since $F \in \mathrm{RV}_\infty(1)$ we have that $F^{-1}(x) \to \infty$ as $x \to \infty$, and therefore it follows that (3.3) holds.

13. Proofs from Section 5

13.1. **Proof of Lemma 1.** For $x > 0$ we have that

$$\varphi_0(x) = \inf_{\|y\|=x} \langle \frac{y}{\|y\|}, \phi(y) \rangle = \inf_{\|u\|=1} \langle u, \phi(xu) \rangle.$$

Since $\phi(0) = 0$, we have that (5.5) holds. Moreover, (5.5) is equivalent to

$$-\varphi_0(x) = \sup_{\|u\|=1} -\langle u, \phi(xu) \rangle, \quad x \geq 0.$$

It is true for any $A, B : \mathbb{R}^d \to \mathbb{R}$ that

$$\left| \sup_{\|u\|=1} A(u) - \sup_{\|u\|=1} B(u) \right| \leq \sup_{\|u\|=1} |A(u) - B(u)|. \tag{13.1}$$

Let $x, y \in \mathbb{R}$ such that $|x| \vee |y| \leq n \in \mathbb{N}$. Since ϕ is locally Lipschitz continuous, for every $u \in \mathbb{R}^d$ with $\|u\| = 1$, we have

$$\|\phi(xu) - \phi(yu)\| \leq K_n|x - y| \tag{13.2}$$

for some $K_n > 0$. Therefore, for $|x| \vee |y| \leq n$, by using (13.1), the Cauchy–Schwarz inequality and (13.2) in turn, we get

$$\begin{aligned} |\varphi_0(x) - \varphi_0(y)| &= \left| \sup_{\|u\|=1} -\langle u, \phi(xu)\rangle - \sup_{\|u\|=1} -\langle u, \phi(yu)\rangle \right| \\ &\leq \sup_{\|u\|=1} |\langle u, \phi(yu)\rangle - \langle u, \phi(xu)\rangle| \\ &= \sup_{\|u\|=1} |\langle u, \phi(yu) - \phi(xu)\rangle| \\ &\leq \sup_{\|u\|=1} \|u\| \|\phi(yu) - \phi(xu)\| \\ &\leq K_n|x - y|, \end{aligned}$$

which establishes the local Lipschitz continuity of φ_0.

To show that $\varphi_0(x) > 0$ for $x > 0$, notice first by (5.3) that $\varphi_0(x) \geq 0$ for all $x > 0$. Suppose now that there is an $x_0 > 0$ such that $\varphi_0(x_0) = 0$. Then, by the continuity of φ_0, we have

$$0 = \varphi_0(x_0) = \inf_{\|u\|=x_0} \langle u, \phi(x_0 u)\rangle = \min_{\|u\|=1} \langle u, \phi(x_0 u)\rangle = \langle u^*, \phi(x_0 u^*)\rangle$$

for some $u^* \in \mathbb{R}^d$ such that $\|u^*\| = 1$. But then, with $x^* = x_0 u^* \neq 0$, we have $\langle x^*, \phi(x^*)\rangle = 0$, contradicting (5.3).

13.2. **Proof of Proposition 3.** It is easy to see that $\langle x, \phi(x)\rangle > 0$ for $x \neq 0$ ensures that $x = 0$ is the only equilibrium of the unperturbed equation (5.2). Suppose that $x_0 \neq 0$ is another equilibrium. Then $\phi(x_0) = 0$. But $0 = \langle x_0, \phi(x_0)\rangle > 0$ by (5.3), a contradiction. The global asymptotic stability of solutions is achieved by taking $u(t) = \|z(t)\|^2$ for $t \geq 0$.

If $z(0) = 0$, then $z(t) = 0$, for all $t \geq 0$. Otherwise, suppose $\dot{z}(0) \neq 0$, then $||z(0)|| > 0$ and $u'(t) = -2\langle z(t), \phi(z(t))\rangle \leq 0$, so $t \to u(t)$ is non-increasing. Either $u(t) \to 0$, as $t \to \infty$ or $u(t) \to L^2 > 0$ as $t \to \infty$. Suppose that the latter holds. We establish that

$$\liminf_{t\to\infty} \langle z(t), \phi(z(t))\rangle =: \lambda > 0$$

from which a contradiction will result.

Since $\|z(t)\| \to L > 0$ as $t \to \infty$, $\|z(t)\| > 0$ for all $t \geq 0$ and

$$\frac{\langle z(t), \phi(z(t))\rangle}{\|z(t)\|} \geq \inf_{\|u\|=\|z(t)\|} \frac{\langle u, \phi(u)\rangle}{\|u\|} = \varphi_0(\|z(t)\|).$$

By Lemma 1, φ_0 is locally Lipschitz continuous, so since $\|z(t)\| \to L$ as $t \to \infty$,

$$\liminf_{t\to\infty} \frac{\langle z(t), \phi(z(t))\rangle}{\|z(t)\|} \geq \liminf_{t\to\infty} \varphi_0(\|z(t)\|) = \varphi_0(L).$$

Also, as $L > 0$, Lemma 1 ensures that $\varphi_0(L) > 0$. Thus $\liminf_{t\to\infty}\langle z(t), \phi z(t))\rangle \geq L\varphi_0(L) > 0$. Recalling that $u'(t) \leq -2\langle z(t), \phi(z(t))\rangle$, we get

$$\limsup_{t\to\infty} u'(t) \leq \limsup_{t\to\infty} -2\langle z(t), \phi(z(t))\rangle \leq -2L\varphi_0(L) < 0.$$

Therefore $u(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $u(t) \geq 0$ for all $t \geq 0$.

13.3. **Proof of Proposition 4.** The local Lipschitz continuity of ϕ ensures that there is a unique continuous solution of (5.1), defined up to a maximal time $T > 0$ for which $\lim_{t\uparrow T} \|x(t)\| = +\infty$, or $x(t)$ is uniquely defined for all $t \geq 0$. Suppose the former and let $y(t) = \|x(t)\|^2$ for $t \in [0, T)$. Then for $t \in (0, T)$ we get

$$y'(t) = 2\langle -\phi(x(t)) + \gamma(t), x(t))\rangle \leq 2\langle \gamma(t), x(t)\rangle \leq 2\|\gamma(t)\|\|x(t)\|,$$

using (5.3) and the Cauchy Schwarz inequality. Since γ is continuous on $[0, T]$, we have that $\|\gamma(t)\| \leq \Gamma$ for all $t \in [0, T]$ and some $\Gamma > 0$. Hence

$$y'(t) \leq 2\Gamma\sqrt{y(t)}, \quad t \in (0, T), \quad y(0) = \|\xi\|^2 \geq 0.$$

Since $y(t) \to \infty$ as $t \uparrow T$ and y is continuous, there exists $T_1 \in (0, T)$ such that $y(t) \geq 1$ for all $t \in [T_1, T)$. Dividing by $\sqrt{y(t)}$ on both sides of this differential inequality for $t \in [T_1, T)$ and then integrating yields

$$y(t)^{1/2} - y(T_1)^{1/2} \leq 2\Gamma(t - T_1), \quad t \in [T_1, T).$$

Letting $t \uparrow T$ on both sides of the inequality now leads to the desired contradiction.

13.4. **Proof of Theorem 18.** By hypothesis (5.6), φ is locally Lipschitz continuous and obeys $\varphi(0) = 0$. Therefore (5.7) has a unique continuous solution. Moreover, we see that $x_\epsilon(t) > 0$ for all $t \geq 0$, by considering $t_0 = \inf\{t > 0 : x_{\eta,\epsilon}(t) = 0\}$ and showing that such a t_0 cannot be finite. Clearly, we must have $x'_{\eta,\epsilon}(t_0) \leq 0$, so that

$$0 \geq x'_{\eta,\epsilon}(t_0) = -\varphi(x_{\eta,\epsilon}(t_0)) + \|\gamma(t_0)\| + \frac{\epsilon}{2}e^{-t_0} = \|\gamma(t_0)\| + \frac{\epsilon}{2}e^{-t_0} \geq \epsilon e^{-t_0} > 0,$$

a contradiction. Thus $x_\epsilon(t) > 0$ for all $t \geq 0$.

Let $y(t) = \|x(t)\|^2$ and $y_\epsilon(t) = x_{\eta,\epsilon}(t)^2$ for $t \geq 0$. We show that $y(t) \leq y_\epsilon(t)$ and this proves the result. Now as $y(t) = \langle x(t), x(t)\rangle$ and $x \in C^1([0,\infty); \mathbb{R}^d)$, we have that $y \in C^1((0,\infty); \mathbb{R})$ and moreover

$$y'(t) = -2\langle \phi(x(t)), x(t)\rangle + 2\langle \gamma(t), x(t)\rangle.$$

By the Cauchy–Schwarz inequality and (5.6), we get

$$\frac{2\langle \phi(x(t)), x(t)\rangle}{||x(t)||} \geq 2\varphi(\|x(t)\|)$$

when $\|x(t)\| \neq 0$, so $-2\langle \phi(x(t)), x(t)\rangle \leq -2\varphi(\|x(t)\|)\|x(t)\|$. In the case that $\|x(t)\| = 0$, $2\langle \phi(x(t)), x(t)\rangle = 0$. Therefore, for all $t \geq 0$, $-2\langle \phi(x(t)), x(t)\rangle \leq -2\varphi(\|x(t)\|)\|x(t)\|$. Thus $y'(t) \leq -2\varphi(\|x(t)\|)\|x(t)\| + 2\|\gamma(t)\|\|x(t)\|$ for $t > 0$ or

$$y'(t) \leq -2\varphi(\sqrt{y(t)})\sqrt{y(t)} + 2\|\gamma(t)\|\sqrt{y(t)}, \quad t > 0.$$

Moreover $y_\epsilon(0) = (\|x(0)\| + \epsilon/2)^2 > \|x(0)\|^2 = y(0)$.

Suppose there is $t_2 > 0$ such that $y(t_2) = y_\epsilon(t_2)$ but $y(t) < y_\epsilon(t)$ for $t \in [0, t_2)$. Then as y_ϵ is in $C^1((0,\infty), \mathbb{R})$, we have that $y'(t_2) \geq y'_\epsilon(t_2)$. By construction

$$\begin{aligned} y'_\epsilon(t) &= 2x_{\eta,\epsilon}(t)\{-\varphi(x_{\eta,\epsilon}(t)) + \|\gamma(t)\| + \epsilon/2e^{-t}\} \\ &= -2\sqrt{y_\epsilon(t)}\varphi(\sqrt{y_\epsilon(t)}) + 2\sqrt{y_\epsilon(t)}\|\gamma(t)\| + \epsilon\sqrt{y_\epsilon(t)}e^{-t}. \end{aligned}$$

Thus

$$\begin{aligned} y'_\epsilon(t_2) &= -2\sqrt{y_\epsilon(t_2)}\varphi(\sqrt{y_\epsilon(t_2)}) + 2\sqrt{y_\epsilon(t_2)}\|\gamma(t_2)\| + \epsilon\sqrt{y_\epsilon(t_2)}e^{-t_2} \\ &= -2\sqrt{y(t_2)}\varphi(\sqrt{y(t_2)}) + 2\sqrt{y(t_2)}\|\gamma(t_2)\| + \epsilon\sqrt{y_\epsilon(t_2)}e^{-t_2} \\ &\geq y'(t_2) + \epsilon\sqrt{y_\epsilon(t_2)}e^{-t_2} \\ &\geq y'_\epsilon(t_2) + \epsilon\sqrt{y_\epsilon(t_2)}e^{-t_2}. \end{aligned}$$

or $\sqrt{y_\epsilon(t_2)} \leq 0$. This implies $x_{\eta,\epsilon}(t_2) = 0$. But this is impossible as $x_{\eta,\epsilon}(t) > 0$ for all $t \geq 0$. Therefore $y(t) < y_\epsilon(t)$ for all $t \geq 0$, or $\|x(t)\|^2 < x_{\eta,\epsilon}(t)^2$ for all $t \geq 0$, which proves the result.

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