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On similarity solutions for non-Newtonian boundary layer flows*

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Abstract

The boundary layer equations for non-Newtonian fluids are examined under the classical conditions of uniform flow past a semi infinite flat plate. Using the similarity transformation, the system of partial differential equations are transformed into a similarity non-linear ordinary differential equation. The existence and uniqueness of some classes of similarity solutions are studied and their properties are investigated.

1 Introduction

The aim of this paper is to obtain existence result for nonlinear boundary value problem of the equation

$$[\phi(f'')]' + ff'' = 0, \tag{1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $\phi(0) = 0$. Such homeomorphisms ϕ are in particular motivated by the one-dimensional p-Laplacian, for which $\phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-1} s$ for $s \neq 0$, with $p > 0$ and $\phi_p(0) = 0$. Several papers have been recently devoted to the existence of solutions to differential equations $[\phi_p(f')]' + g(\eta, f, f\prime) = 0$, see, for example ([6], [10], [11], [12], [13]) and the references therein. Various two-point boundary value problems containing the operator $[\phi(f')]'$ have received a lot of attention with respect to existence of solutions (see [4], [9]).

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the following conditions:

(H1) (Strict monotonicity) For any $x_1, x_2 \in \mathbb{R}, x_1 \neq x_2$

$$\langle \phi(x_1) - \phi(x_2), x_1 - x_2 \rangle > 0.$$

(H2) (Coercivity) There exists a function $\sigma : [0, \infty) \to [0, \infty)$, $\sigma(s) \to \infty$ as $s \to \infty$, such that

$$\langle \phi(x), x \rangle \geq \sigma(|x|)\, |x|, \quad \text{for all} \quad x \in \mathbb{R}.$$

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(H3) (Homogenity of degree δ) For any $\delta, \lambda \in \mathbb{R}^+$,

$$\phi(\lambda x) = \lambda^\delta \phi(x).$$

In [7] it has been shown that under conditions (H1) and (H2) ϕ is a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and $\left|\phi^{-1}(y)\right| \to +\infty$ as $|y| \to +\infty$.

We shall understand the solution of (1) in the following sense: $f : I \to \mathbb{R}$ is of class C^2, $\phi(f'')$ is absolutely continuous and f satisfies (1) a.e. on $(0, \infty)$.

This paper is organized as follows. In Section 2 we begin by the derivation of the equation, show how the boundary layer approximation leads to the two point boundary value problem and the similarity solution. This new model, written in terms of stream function, allows to introduce similarity variables to reduce the partial differential equation into ordinary differential equation of the third order with appropriate boundary values. Then, this two-point boundary value problem is studied using shooting method. We show the existence and uniqueness of its solution.

2 Mathematical model

We investigate a one layer model of laminar non-Newtonian fluid with constant speed V_∞ past a semi infinite vertical plate. In the absence of body force and external pressure gradients, the laminar boundary layer equations expressing conservation of mass and momentum are governed by ([1], [14], [15]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y}, \tag{3}$$

where $y = 0$ is the plate, the x and y axes are taken along and perpendicular to the plate, the functions u and v are the velocity components of the fluid along the x and y axes and ρ is the density. Here we study non-Newtonian fluids, when the rheological behaviour of the fluid in between parallel plates is described by the shear stress-shear rate relationship

$$\tau = \kappa\, \phi\left(\frac{\partial u}{\partial y}\right), \tag{4}$$

where the τ is the shear stress, κ is an empirical constant and $\partial u/\partial y$ is the velocity gradient perpendicular to the flow direction. For $\phi(s) = s\ h(s)$ the quantity $\kappa h(\partial u/\partial y)$ is called the "effective viscosity".

As an example, we mention the so-called one-dimensional p-Laplacian operator $\phi_p(s) = |s|^{p-1} s$ corresponding to the Ostwald-de Waele power-law model, where $p = 1$ is the Newtonian case and for $0 < p < 1$ has been proposed as being descriptive of pseudo-plastic non-Newtonian fluid and $p > 1$ describes the dilatant fluid [15].

The model (2)-(4) describes the steady plane flow of a fluid past a thin plate, provided the boundary layer assumptions are verified ($u \gg v$ and the existence of a very thin layer attached to the plate). The boundary conditions to be applied are given by

$$\begin{aligned} u(x,0) &= 0, && (5)\\ v(x,0) &= 0, && (6)\\ u(x,y) &\to V_\infty \quad \text{as} \quad y \to \infty, && (7) \end{aligned}$$

where $V_\infty > 0$. The boundary condition at $y \to \infty$ means that the velocity v tends to the main-stream velocity V_∞ asymptotically.

Equation (2) ensures that $u\ dy - v\ dx$ is an exact differential equal to $d\psi$, then

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

The lines in the fluid whose tangent is parallel to $(u,\ v)$ are given by $\psi = const$. The non-dimensional function $\psi(x,y)$ is called stream function.

Introducing function ψ, the boundary value problem $(2)-(7)$ can be rewritten as follows

$$\begin{gathered} \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x\,\partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \upsilon \frac{\partial}{\partial y}\phi\left(\frac{\partial^2 \psi}{\partial y^2}\right), \quad \upsilon = \kappa/\rho, \\ \frac{\partial \psi}{\partial x}(x,0) = 0, \quad \frac{\partial \psi}{\partial y}(x,0) = 0, \\ \frac{\partial \psi}{\partial y}(x,y) \to V_\infty \quad \text{as} \quad y \to \infty. \end{gathered} \tag{8}$$

Blasius, when $\phi(s) = s$, or ϕ_p with $p = 1$, obtained the family of particular solutions to $(2)-(7)$ such that the velocity profile $u(x,y)$ depends only on the variable $\eta(x,y) = yx^{-1/2}$ and $\psi(x,y) = x^{1/2}f(\eta)$ (see [5]). Consequently, the two-point boundary value problem $(2)-(7)$ is reduced to the so-called Blasius problem

$$\begin{gathered} f''' + \frac{1}{2}ff'' = 0, \\ f(0) = 0, \quad f'(0) = 0, \\ \lim_{\eta \to \infty} f'(\eta) = 1. \end{gathered} \tag{9}$$

Here we look for similarity solutions using the linear transformation (see [2])

$$x \to \lambda x, \quad y \to \lambda^\beta y, \quad \psi \to \lambda^\alpha \psi$$

where λ is a positive parameter, and we introduce the stream function ψ and the similarity variable η with

$$\psi(x,y) = bx^{-\alpha} f(\eta), \quad \eta = a\frac{y}{x^{\beta}},$$

where according to (H3) the exponents α and β satisfy the scaling relation

$$\alpha(\delta - 2) + \beta\,(2\delta - 1) = 1,$$

and real numbers a and $b > 0$ are such that

$$\gamma a^{2\delta-1} b^{\delta-2} = -\alpha \;\text{ and }\; ab = V_{\infty}.$$

That means

$$a = [(\delta+1)\,\upsilon]^{-\frac{1}{\delta+1}}\, V_{\infty}^{\frac{2-\delta}{\delta+1}}, \quad b = [(\delta+1)\,\upsilon]^{\frac{1}{\delta+1}}\, V_{\infty}^{\frac{2\delta-1}{\delta+1}}.$$

In terms of similarity solutions we obtain the following nonlinear ordinary differential equation

$$[\phi(f'')]' + ff'' = -\,(\alpha+\beta)\, f'^{2}.$$

Condition $\frac{\partial\psi}{\partial y}(x,y) \to V_{\infty}$ as $y \to \infty$ implies $\alpha = -\beta$, i.e., $\alpha = -1/\,(\delta+1)\,,$ $\beta = 1/\,(\delta+1)\,.$ Therefore we arrive to the two-point boundary value problem

$$[\phi(f'')]' + ff'' = 0, \quad \text{for} \quad \eta > 0, \tag{10}$$

$$f(0) = 0, \quad f'(0) = 0, \tag{11}$$

$$\lim_{\eta\to\infty} f'(\eta) = 1. \tag{12}$$

The existence and uniqueness of the solution to this problem will be investigated. For the one-dimensional p-Laplacian it has been proved for $0 < p < 1$ by Nachman and Callegari in [8] and for $p > 1$ by Benlahsen et al. in [3].

3 Existence of solutions to the boundary value problem

We use the shooting method and replace the condition at infinity by one at $\eta = 0$. Therefore, (10)-(12) is converted into an initial value problem of (10) with initial conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = \gamma. \tag{13}$$

We determine the value γ such that the corresponding solution satisfies condition (12). We will denote by f_{γ} the solution of the initial value problem (10), (13). Taking the integral of (10)

$$\phi\,(f''(\eta)) + f(\eta)f'(\eta) = \phi\,(f''(0)) + \int_0^{\eta} f'^{2}(s)\; ds \tag{14}$$

holds for all $\eta > 0$.

Let us denote the interval of existence of f by $(0, \eta_\gamma)$ with $\eta_\gamma \leq \infty$. If $\gamma \neq 0$ then there exists $\varepsilon \in (0, \eta_\gamma)$, $\eta_\gamma \leq \infty$, such that $f''(\eta) \neq 0$ for all $\eta \in (0, \varepsilon)$.

Lemma 1 *For $\eta \in (0, \eta_\gamma)$ as long as $f''(\eta) \neq 0$,*

$$f''(\eta) = \gamma \exp\left(-\int_0^\eta \frac{f(s)}{\phi'(f''(s))} ds \right) \tag{15}$$

holds.

Proof. A local in η nontrivial solution exists for any $\gamma \in \mathbb{R}$ and is unique. From (10)

$$\frac{f'''}{f''} + \frac{f}{\phi'(f'')} = 0$$

holds in $(0, \varepsilon)$ where $f'' \neq 0$. Hence, one gets (15). ■

Note that $\gamma < 0$ would imply that $f''(\eta) < 0$ and the solution cannot satisfy boundary conditions (11), (12). Therefore we shall suppose that $\gamma > 0$.

Theorem 2 *Solution f_γ exists on $\mathbb{R}^+$ and $\lim_{\eta \to \infty} f'_\gamma(\eta) = C$, where constant $C > 0$ may depend on γ.*

Proof. Assume for contradiction that $\eta_\gamma < \infty$. Function f_γ satisfies identity (14) for all $\eta < \eta_\gamma$, that is

$$\phi\left(f''_\gamma(\eta)\right) + f_\gamma(\eta) f'_\gamma(\eta) = \phi(\gamma) + \int_0^\eta f'^2_\gamma(s)\, ds. \tag{16}$$

It follows from (16) that f_γ cannot have a local maximum and $f'_\gamma(\eta) > 0$ on $(0, \eta_\gamma)$ and $f_\gamma(\eta) > 0$. We shall vary γ such that f_γ is global ($\eta_\gamma < \infty$) and satisfies the desired condition at infinity.

Consider $E = \phi\left(f''_\gamma(\eta)\right)$ which satisfies

$$E' = -f''_\gamma f_\gamma \leq 0,$$

see (10). Thus f''_γ is bounded, $f''_\gamma(\eta) \leq \gamma$ for all $\eta > 0$. This implies that f_γ and f'_γ are also bounded in $(0, \eta_\gamma)$, i.e., $f_\gamma(\eta) \leq \frac{1}{2}\gamma\eta_\gamma^2$, $f'_\gamma(\eta) \leq \gamma\eta_\gamma$. Therefore $[0, \eta_\gamma)$ cannot be the maximal interval of existence. Therefore f_γ exists on all of $0 \leq \eta < \infty$, hence, $\eta_\gamma = \infty$ and f_γ is global. Furthermore, there exists a real number $l \in [0, \gamma]$ such that $\lim_{\eta \to \infty} f''_\gamma(\eta) = l$. Since f_γ is convex, monotonically increasing, then $\lim_{\eta \to \infty} f_\gamma(\eta) = \infty$.

Next we show that f'_γ tends to a positive limit as $\eta \to \infty$. Let us consider $H' = f'^2_\gamma$, where $H = \phi(f''_\gamma) + f'_\gamma f_\gamma$. Then H is monotonically increasing and has a limit at infinity. This shows that there exists the limit

$$\lim_{\eta \to \infty} f_\gamma(\eta) f'_\gamma(\eta) = C_2, \quad C_2 \in (0, \infty].$$

First, assume that C_2 is finite. This implies that $f'_\gamma(\eta) \approx \eta^{-1/2}$ at infinity, and with (16) it leads to $\lim_{\eta\to\infty} f_\gamma(\eta)\ f'_\gamma(\eta) = \infty$, a contradiction. Then, $C_2 = \infty$. Applying (10) one can get

$$\frac{[\phi(f'')]'}{f\ f'} = -\frac{f''}{f'},$$

then the following limits exist

$$\lim_{\eta\to\infty}\int_1^\eta \frac{[\phi(f''(s))]'}{f(s)f'(s)}\ ds = K \quad \text{and} \quad \lim_{\eta\to\infty}\int_1^\eta \frac{f''(s)}{f'(s)}\ ds = -K.$$

Hence, there exists constant $C > 0$ depending on γ such that $\lim_{\eta\to\infty} f'_\gamma(\eta) = C$ and $\lim_{\eta\to\infty} f''_\gamma(\eta) = l = 0$. ■

To sum up we have

$$\lim_{\eta\to\infty} f_\gamma(\eta) = \infty, \qquad \lim_{\eta\to\infty} f'_\gamma(\eta) = C, \qquad \lim_{\eta\to\infty} f''_\gamma(\eta) = 0. \tag{17}$$

Applying the second limit in (17) and (H3) one can get

$$f(\eta) = C^{\frac{1-2\delta}{\delta+1}} f(C^{\frac{\delta-2}{\delta+1}}\eta) \quad \text{and} \quad f''(0) = \gamma C^{-\frac{3}{\delta+1}}.$$

Theorem 3 *Equation (10) has a unique solution satisfying (11) and (12).*

Proof. In order to show the uniqueness of the solution to (10) we employ the following Crocco-like transformation $s = f'$ and $G = f''$ for (10)-(12) (see [8]), and we arrive at the following nonlinear boundary value problem

$$[\phi'(G)G']'\, G + s = 0, \tag{18}$$

$$G'(0) = 0, \quad G(1) = 0, \tag{19}$$

with $G' = f'''/f''$ under the assumptions that $f'' > 0$ on $[0,1)$ and $0 \le f' \le 1$ for $0 \le \eta \le 1$. It is remarkable that for the one-dimensional p-Laplacian the Crocco transformation has been applied for $0 < p < 1$ by Nachman and Callegari in [8]. Here we use the same method.
For $s \in [0,1)$ we show that there exist at most one positive solution to (18), (19). Taking the integral of (18) from 0 to s one can obtain

$$\phi'(G(s))G'(s) = -\int_0^s \frac{\xi}{G(\xi)}\ d\xi \tag{20}$$

for any $s < 1$. Let us denote by G_1 and G_2 two positive solutions of (18)-(19) for which $G_1(0) > G_2(0)$. Then there exists an $\varepsilon \in (0,1)$ such that $G_1(s) > G_2(s)$ for in $(0,\varepsilon)$. We assume that $G_1(s_0) = G_2(s_0)$ and $G_1(s) > G_2(s)$ for a $s \in (0,s_0)$, which means that $G'_1(s_0) \le G'_2(s_0)$. But from (20) one gets

$$\phi'(G_1(s_0))G'_1(s_0) = -\int_0^{s_0} \frac{\xi}{G_1(\xi)}\ d\xi > -\int_0^{s_0} \frac{\xi}{G_2(\xi)}\ d\xi = \phi'(G_2(s_0))G'_2(s_0).$$

Then $G_1'(s_0) > G_2'(s_0)$ leads to contradiction. Hence, $s_0 = 1$, and $\phi'(G_1(s))G_1'(s) > \phi'(G_2(s))G_2'(s)$ for $s \in (0,1)$. Consequently, $[\phi(G_1(s)) - \phi(G_2(s))]' > 0$, it implies that $\phi(G_1(s)) - \phi(G_2(s)) > 0$ is monotonically increasing and $\phi(G_1(1)) - \phi(G_2(1)) = 0$, which contradicts to our previous assumptions. We deduce that there exists a unique solution to (18), (19). ∎

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