Electronic Journal of Qualitative Theory of Differential Equations
Proc. 9th Coll. QTDE, 2012, No. **13**, 1–17
http://www.math.u-szeged.hu/ejqtde/

ON FINDING SOLUTIONS OF TWO-POINT BOUNDARY VALUE PROBLEMS FOR A CLASS OF NON-LINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. We consider the two-point boundary value problems for a certain class of non-linear functional differential equations. To study the problem, we use a method based upon a special type of successive approximations that are constructed analytically and, under suitable conditions, converge uniformly on the given interval.

Our techniques lead one to a certain finite-dimensional system of numerical determining equations that "detect" all the solutions of the problem. Based on properties of these equations, we give efficient conditions ensuring the solvability of the original problem. The conditions are formulated in terms of functions that are potential candidates for approximate solutions and, being such, are constructed explicitly.

1. Introduction

The purpose of this paper is to extend the numerical-analytic techniques, which had been used in [8, 9] in relation to a two-point boundary value problem for some systems of linear differential equations with argument deviations, to study similar problems for a class of functional differential systems of the form

$$x'(t) = (fx)(t), \qquad t \in [a, b], \tag{1.1}$$

determined by a (generally speaking, non-linear) operator $f : C \to L_1$.

Equation (1.1) is considered under the two-point linear boundary conditions of a non-separated type

$$Ax(a) + Bx(b) = d, \tag{1.2}$$

where B is a non-singular matrix.

System (1.1) is a very general object and comprises, in particular, various equations of the form

$$x_i'(t) = g_i\left(t, x_1(t), x_2(t), \ldots, x_n(t), x_1(\tau_{i1}(t)), x_2(\tau_{i2}(t)), \ldots, x_n(\tau_{in}(t))\right),$$

where $t \in [a, b]$, $g_i : [a, b] \times \mathbb{R}^{2n} \to \mathbb{R}^n$, $i = 1, 2, \ldots, n$, and τ_{ij}, $i, j = 1, 2, \ldots, n$, which represent the argument deviations, are Lebesgue measurable functions transforming the given interval $[a, b]$ into itself. It is important to note that the latter condition imposed on the argument deviations,

1991 *Mathematics Subject Classification.* 34B15.

Key words and phrases. Non-linear functional differential equation, two-point conditions, successive approximations, convergence, existence.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

in fact, does not bring about any loss of generality. For more details on this subject, we refer the reader to the book [1].

2. Notation

The following notation is used in the sequel:

(1) $C := C([a, b], \mathbb{R}^n)$ is the Banach space of the continuous functions $[0, T] \to \mathbb{R}^n$ with the standard uniform norm.
(2) $L_1 := L_1([a, b], \mathbb{R}^n)$ is the usual Banach space of the vector functions $[a, b] \to \mathbb{R}^n$ with Lebesgue integrable components.
(3) $\mathscr{L}(\mathbb{R}^n)$ is the algebra of all the square matrices of dimension n with real elements.
(4) $r(Q)$ is the maximal in module eigenvalue of the matrix $Q \in \mathscr{L}(\mathbb{R}^n)$.
(5) $\mathbf{1}_n$ is the unit matrix of dimension n.
(6) $\mathbf{0}_k$ is the zero square matrix of dimension k.
(7) For any x_i, $i = 1, 2, \ldots, n$, we use the notation $\operatorname{col}(x_1, x_2, \ldots, x_n)$ and $x = (x_i)_{i=1}^n$ for the column vector constituted by $x_1, x_2, \ldots, x_n$.
(8) $\partial\Omega$ is the boundary of a set $\Omega \subset \mathbb{R}^n$.
(9) For any vectors v_i, $i = 1, 2, \ldots, n$, we denote by $[v_1, v_2, \ldots, v_n]$ the $n \times n$ matrix with the columns $v_1, v_2, \ldots, v_n$.
(10) By e_i, $i = 1, 2, \ldots, n$, we denote the n-dimensional unit vectors

$$e_i := \operatorname{col}(\underbrace{0, 0, \ldots, 0,}_{i-1} 1, 0, \ldots, 0). \tag{2.1}$$

(11) For u and v from $\mathbb{R}^n$, we put $\langle u, v\rangle := \{x \in \mathbb{R}^n \mid u \leq x \leq v\}$.
(12) For any $x \in \mathbb{R}$, $[x]_- := -\min\{x, 0\}$ and $[x]_+ := \max\{x, 0\}$.
(13) $\deg F$ is the Brower degree of a vector field F.

The inequalities and the absolute value sign for vectors and matrices, as well as the operations $\max_{t\in[a,b]}$, $\sup_{z\in\langle z_0,z_1\rangle}$, etc., applied to vector and matrix-valued functions, are understood elementwise.

3. Problem setting

We consider the system of $n \geq 1$ non-linear functional differential equations (1.1), where $f : C \to L_1$ is a continuous operator. By a solution of (1.1), as usual, one understands an absolutely continuous function $x : [a, b] \to \mathbb{R}^n$ satisfying (1.1) at almost every point of the interval $[a, b]$.

Equation (1.1) is studied under the two-point boundary conditions (1.2) where $d \in \mathbb{R}^n$, the matrix $A \in \mathscr{L}(\mathbb{R}^n)$ is arbitrary, and $\det B \neq 0$. Note at once that, without loss of generality, one may restrict oneself to the boundary condition of the particular form

$$Ax(a) + x(b) = 0. \tag{3.1}$$

For the latter purpose, it is sufficient to carry out, e. g., the change of variable

$$y(t) = Bx(t) - \frac{t-a}{b-a}d, \qquad t \in [a, b],$$

and make use of the fact that B is non-singular. In what follows, skipping the explicit change of variable, we replace condition (1.2) by (3.1) and deal with problem (1.1), (3.1) directly.

We shall show that the question of finding a solution of the problem under consideration can be efficiently approached by using certain techniques based on successive approximations (cf. [3, 5–7, 11–18]).

4. Main assumptions

We look for a solution of problem (1.1), (3.1) among functions having initial value in a certain set $\langle z_0, z_1 \rangle$. It is convenient to define $\langle z_0, z_1 \rangle$ as

$$\langle z_0, z_1 \rangle := \{ z \in \mathbb{R}^n \mid z_0 \leq z \leq z_1 \}, \tag{4.1}$$

where z_0 and z_1 are fixed vectors. Recall that here and below the inequalities for vectors and matrices are understood in the componentwise sense.

Definition 4.1. An operator $l : C \to L_1$ is said to be positive if $(lu)(t) \geq 0$ for a. e. $t \in [a, b]$ whenever $u(t) \geq 0$ for all $t \in [a, b]$.

Definition 4.2. An operator $f : C \to L_1$ is said to satisfy the Lipschitz condition on a set $\mathscr{B} \subset C$ if there exists a positive linear operator $l : C \to L_1$ such that

$$|(fu)(t) - (fv)(t)| \leq (l|u - v|)(t), \qquad t \in [a, b], \tag{4.2}$$

for all u and v from $\mathscr{B}$.

Given any vectors y_0 and y_1 from $\mathbb{R}^n$, we define the set $\mathscr{B}(y_0, y_1)$ by putting

$$\mathscr{B}(y_0, y_1) := \{ x \in C : y_0 \leq x(t) \leq y_1 \text{ for all } t \in [a, b] \}. \tag{4.3}$$

5. Construction of the successive approximations and convergence conditions

Prior to formulation of the theorem, we introduce some notation. Let us put

$$(Py)(t) := \int_a^t y(s)ds - \frac{t-a}{b-a} \int_a^b y(s)ds, \qquad t \in [a, b], \tag{5.1}$$

for any y from L_1.

Our study of solutions of the boundary value problem (1.1), (1.2) is based upon the use of the function sequence determined by the recurrence relation

$$x_{m+1}(\cdot, z) := Pfx_m(\cdot, z) + \varphi_z, \qquad m = 0, 1, 2, \ldots, \; z \in \langle z_0, z_1 \rangle, \tag{5.2}$$

with $x_0(\cdot, z) := \varphi_z$, where

$$\varphi_z(t) := z - \frac{t-a}{b-a} (A + \mathbf{1}_n) z, \qquad t \in [a, b]. \tag{5.3}$$

It can be easily verified that, for every $m = 0, 1, 2, \ldots$ function (5.2) satisfy the boundary condition (1.2) for arbitrary $z \in \mathbb{R}^n$.

Let us introduce into consideration the $n \times n$ matrices $\bar{A}_- = (\bar{a}_{-;\,i,j})_{i,j=1}^n$ and $\bar{\bar{A}}_- = (\bar{\bar{a}}_{-;\,i,j})_{i,j=1}^n$ with the elements defined by the equalities

$$\bar{a}_{-;\,i,j} := \begin{cases} 0 & \text{if } i \neq j, \\ \min\{1, [a_{ii}]_-\} & \text{if } i = j, \end{cases} \tag{5.4}$$

and

$$\bar{\bar{a}}_{-;\,i,j} := \begin{cases} [a_{ij}]_- & \text{if } i \neq j, \\ \max\{1, [a_{ii}]_-\} & \text{if } i = j. \end{cases} \tag{5.5}$$

With any given positive linear operator $l : C \to L_1$, we associate the matrix function $K_l : [a,b] \to \mathscr{L}(\mathbb{R})$ of the form

$$K_l := [le_1, le_2, \dots, le_n], \tag{5.6}$$

with e_i, $i = 1, 2, \dots, n$, given by (2.1), and set

$$Q_l := \max_{t \in [a,b]} \left(\left(1 - \frac{t-a}{b-a}\right) \int_a^t K_l(s)ds + \frac{t-a}{b-a} \int_t^b K_l(s)ds \right). \tag{5.7}$$

We emphasize that the maximum in (5.7) is taken elementwise, and it is, in general, not attained at a point from $[a,b]$ unless $n = 1$.

Remark 5.1. The expression le_i, $i = 1, 2, \dots, n$, appearing in (5.6) is understood in the sense that l is applied to a constant vector function equal identically to e_i. In other words, the columns of K_l are constituted by the values of l on unit vectors. For instance, if $l = (l_i)_{i=1}^n : C \to L_1$ is defined as

$$(l_i x)(t) := \sum_{j=1}^n p_{ij}(t) x_j(\tau_{ij}(t)), \qquad t \in [a,b],\ i = 1, 2, \dots, n,$$

where τ_{ij} are measurable and p_{ij} are Lebesgue integrable, then the corresponding matrix (5.6) has the form

$$K_l(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2n}(t) \\ \dots & \dots & \dots & \dots \\ p_{n1}(t) & p_{n2}(t) & \dots & p_{nn}(t) \end{pmatrix}, \qquad t \in [a,b].$$

Finally, we put

$$\omega(z) := \operatorname*{ess\,sup}_{t \in [a,b]} (f\varphi_z)(t) - \operatorname*{ess\,inf}_{t \in [a,b]} (f\varphi_z)(t) \tag{5.8}$$

for all $z \in \langle z_0, z_1 \rangle$, where φ_z is the function defined by (5.3).

The following statement establishes the convergence of sequence (5.2) and the relation of its limit function to problem (1.1), (3.1).

Theorem 5.1. *Assume that f satisfies the Lipschitz condition* (4.2) *on the set $\mathscr{B}(-\varrho^* + \bar{A}_- z_0 - A_+ z_1,\ \bar{\bar{A}}_- z_1 + \varrho^*)$, where*

$$\varrho^* := \frac{b-a}{4}(1 - Q_l)^{-1} \sup_{z \in \langle z_0, z_1 \rangle} \omega(z) \tag{5.9}$$

and $l : C \to L_1$ is a certain positive linear operator such that the corresponding matrix Q_l has the property

$$r(Q_l) < 1. \tag{5.10}$$

Then:

(1) *For any fixed $z \in \langle z_0, z_1 \rangle$, the sequence of functions* (5.2) *converges uniformly on $[a, b]$ to a function*

$$x_\infty(\cdot, z) := \lim_{m\to\infty} x_m(\cdot, z) \tag{5.11}$$

possessing the properties

$$x_\infty(a, z) = z, \tag{5.12}$$

$$Ax_\infty(a, z) + x_\infty(b, z) = 0. \tag{5.13}$$

(2) *The formula*

$$\Delta(z) := (A + \mathbf{1}_n)\, z + \int_a^b (f x_\infty(\cdot, z))(s)\, ds, \qquad z \in \langle z_0, z_1 \rangle, \tag{5.14}$$

introduces a well defined single-valued function $\Delta : \langle z_0, z_1 \rangle \to \mathbb{R}^n$.

(3) *The limit function* (5.11) *for all fixed $z \in \langle z_0, z_1 \rangle$ is a solution of the Cauchy problem*

$$x'(t) = (fx)(t) - \Delta(z), \qquad t \in [a, b], \tag{5.15}$$

$$x(a) = z, \tag{5.16}$$

where the vector function $\Delta : \langle z_0, z_1 \rangle \to \mathbb{R}^n$ is given by (5.14)*.*

(4) *For all fixed $z \in \langle z_0, z_1 \rangle$,*

$$\max_{t\in[a,b]} |x_\infty(t, z) - x_m(t, z)| \le \frac{b-a}{4} Q_l^m\, (1 - Q_l)^{-1}\, \omega(z). \tag{5.17}$$

We note that the Lipschitz condition (4.2) in Theorem 5.1 is assumed on the bounded set $\mathscr{B}(-\varrho^* + \bar{A}_- z_0 - A_+ z_1,\ \bar{\bar{A}}_- z_1 + \varrho^*)$ only and, in general, may not be satisfied globally.

6. Lemmata and proof of Theorem 5.1

Lemma 6.1 ([10, Lemma 3.2])**.** *For any non-negative function $u \in C$, the estimate*

$$(lu)(t) \le K_l(t) \max_{\xi\in[a,b]} u(\xi), \qquad t \in [a, b], \tag{6.1}$$

holds, where $K_l : [a, b] \to \mathscr{L}(\mathbb{R})$ is given by (5.6)*.*

Let us put

$$(Hy)(t) := \left(1 - \frac{t-a}{b-a}\right) \int_a^t y(s)ds + \frac{t-a}{b-a} \int_t^b y(s)ds, \qquad t \in [a, b], \tag{6.2}$$

for any y from L_1.

Lemma 6.2. *The estimate*

$$|(Pu)(t)| \le (H|u|)(t), \qquad t \in [a,b], \tag{6.3}$$

holds for any u from C.

Lemma 6.3. *For any non-negative constant vector $c \in \mathbb{R}^n$, the estimate*

$$(Hlc)(t) \le Q_l c, \qquad t \in [a,b], \tag{6.4}$$

holds, where Q_l is given by (5.7)*.*

Proof. Let $c \ge 0$. By Lemma 6.1,

$$(lc)(t) \le K_l(t)c, \quad t \in [a,b]. \tag{6.5}$$

Using (6.5) and taking the positivity of H into account, we easily arrive at (6.4). □

For any $z \in \langle z_0, z_1 \rangle$ and any vector $\varrho \in \mathbb{R}^n$ with positive components, we put

$$\mathscr{A}_z(\varrho) := \{x \in C : |x(t) - \varphi_z(t)| \le \varrho \text{ for all } t \in [a,b]\}. \tag{6.6}$$

It is obvious from (6.6) that

Lemma 6.4. $\mathscr{A}_z(\varrho_1) \subset \mathscr{A}_z(\varrho_2)$ *whenever* $\varrho_1 \le \varrho_2$*.*

For the given matrix A from the boundary condition (3.1), we define its positive and negative parts $A_+ = (a_{+;i,j})_{i,j=1}^n$ and $A_- = (a_{-;i,j})_{i,j=1}^n$ by putting

$$a_{+;i,j} := [a_{i,j}]_+, \qquad\qquad a_{-;i,j} := [a_{i,j}]_- \tag{6.7}$$

for all i and j from 1 to n. Then, obviously, A_+ and A_- are non-negative matrices and

$$A = A_+ - A_-. \tag{6.8}$$

Lemma 6.5. *For any $z \in \langle z_0, z_1 \rangle$ and non-negative ϱ, the inclusion*

$$\mathscr{A}_\varrho(z) \subset \mathscr{B}(-\varrho + \bar{A}_- z_0 - A_+ z_1,\ \bar{\bar{A}}_- z_1 + \varrho) \tag{6.9}$$

holds, where $\bar{A}_- = (\bar{a}_{-;i,j})_{i,j=1}^n$ and $\bar{\bar{A}}_- = (\bar{\bar{a}}_{-;i,j})_{i,j=1}^n$ are the matrices with the elements given by formulae (5.4), (5.5)*.*

Proof. It follows from (5.3) and (6.8) that, for any z, the function φ_z can be represented in the form

$$\varphi_z(t) = \frac{1}{b-a}\left[(b-t)\mathbf{1}_n + (t-a)A_-\right] z - \frac{t-a}{b-a} A_+ z, \qquad t \in [a,b]. \tag{6.10}$$

Therefore, taking into account the positivity of the matrices A_+ and A_-, we find that, for $z \in \langle z_0, z_1 \rangle$, the inequalities

$$\varphi_z(t) \le \frac{1}{b-a}\left[(b-t)\mathbf{1}_n + (t-a)A_-\right] z_1 - \frac{t-a}{b-a} A_+ z_0, \tag{6.11}$$

$$\varphi_z(t) \ge \frac{1}{b-a}\left[(b-t)\mathbf{1}_n + (t-a)A_-\right] z_0 - \frac{t-a}{b-a} A_+ z_1, \tag{6.12}$$

hold at every point $t \in [a,b]$.

Let us define the matrix function $M = (m_{i,j})_{i,j=1}^n : [a,b] \to \mathscr{L}(\mathbb{R})$ by setting

$$M(t) := \frac{b-t}{b-a}\mathbf{1}_n + \frac{t-a}{b-a}A_-, \qquad t \in [a,b]. \tag{6.13}$$

Then it is not difficult to see that

$$\max_{t\in[a,b]} m_{i,j}(t) = \bar{\bar{a}}_{-;i,j} \tag{6.14}$$

and

$$\min_{t\in[a,b]} m_{i,j}(t) = \bar{a}_{-;i,j}, \tag{6.15}$$

where $\bar{a}_{-;i,j}$ and $\bar{\bar{a}}_{-;i,j}$ are given by formulae (5.4) and (5.5) for all i and j.

Using (6.14) and (6.15) in (6.11), (6.12), we obtain the componentwise estimate

$$\bar{A}_- z_0 - A_+ z_1 \leq \varphi_z(t) \leq \bar{\bar{A}}_- z_1, \qquad t \in [a,b]. \tag{6.16}$$

Let now x be an arbitrary function from $\mathscr{A}_z(\varrho)$. According to (6.6), this means that

$$-\varrho + \varphi_z(t) \leq x(t) \leq \varrho + \varphi_z(t) \tag{6.17}$$

for any $t \in [a,b]$. By virtue of inequality (6.16), it follows from (6.17) that x admits the estimate

$$-\varrho + \bar{A}_- z_0 - A_+ z_1 \leq x(t) \leq \bar{\bar{A}}_- z_1 + \varrho, \qquad t \in [a,b]. \tag{6.18}$$

Since the function $x \in \mathscr{A}_z(\varrho)$ is chosen arbitrarily, estimate (6.18) proves that inclusion (6.9) holds. □

Lemma 6.6 ([8, Lemma 2]). *For an arbitrary essentially bounded function $u : [a,b] \to \mathbb{R}$, the estimate*

$$\left| \int_a^t \left(u(s) - \frac{1}{b-a}\int_a^b u(\xi)d\xi \right) ds \right| \leq \alpha(t)\Big(\operatorname*{ess\,sup}_{s\in[a,b]} u(s) - \operatorname*{ess\,inf}_{s\in[a,b]} u(s)\Big) \tag{6.19}$$

is true, where

$$\alpha(t) := (t-a)\left(1 - \frac{t-a}{b-a}\right), \qquad t \in [a,b]. \tag{6.20}$$

Let us now turn to the proof of Theorem 5.1.

Proof of Theorem 5.1. We shall show that, under the conditions assumed, (5.2) is a Cauchy sequence in the Banach space C.

Let z be an arbitrary vector from $\langle z_0, z_1 \rangle$. By Lemma 6.6, it follows from (5.1) that

$$|x_1(t,z) - \varphi_z(t)| = |(Pf\varphi_z)(t)| \leq \alpha(t)\omega(z), \qquad t \in [a,b], \tag{6.21}$$

with $\alpha : [a,b] \to [0, \frac{b-a}{4}]$ and $\omega : \langle z_0, z_1 \rangle \to \mathbb{R}^n$ defined, respectively, by (6.20) and (5.8).

It is clear from (6.20) that

$$\max_{t\in[a,b]} \alpha(t) = \frac{b-a}{4}$$

and, therefore, (6.21) yields

$$|x_1(t,z)-\varphi_z(t)| \le \frac{b-a}{4}\omega(z), \qquad t\in[a,b], \tag{6.22}$$

Hence, according to (6.6),

$$x_1(\cdot,z)\in \mathscr{A}_z\left(\tfrac{b-a}{4}\omega(z)\right). \tag{6.23}$$

In view of assumption (5.10), equality (5.9) can be represented alternatively as

$$\varrho^* = \frac{b-a}{4}\sum_{k=0}^{+\infty} Q_l^k \sup_{z\in\langle z_0,z_1\rangle}\omega(z), \tag{6.24}$$

whence it is clear that

$$\frac{b-a}{4}\omega(z)\le \varrho^*. \tag{6.25}$$

It follows from (6.23) and (6.25) that $x_1(\cdot,z)\in\mathscr{A}_z(\varrho^*)$, and therefore, by Lemma 6.5,

$$x_1(\cdot,z)\in\mathscr{B}(-\varrho^*+\bar{A}_-z_0-A_+z_1,\ \bar{\bar{A}}_-z_1+\varrho^*). \tag{6.26}$$

Since, obviously, $\mathscr{A}_z(0)=\{\varphi_z\}$, it is clear from Lemmata 6.4 and 6.5 that

$$\varphi_z\in\mathscr{B}(-\varrho^*+\bar{A}_-z_0-A_+z_1,\ \bar{\bar{A}}_-z_1+\varrho^*). \tag{6.27}$$

It follows from (6.26) and (6.27) that both functions $x_1(\cdot,z)$ and φ_z belong to the set where the operator f is assumed to satisfy the Lipschitz condition. Using this and applying Lemma 6.2, we get

$$\begin{aligned}|x_2(t,z)-\varphi_z(t)| &= |(Pfx_1(\cdot,z)(t)|\\ &\le |(Pf\varphi_z)(t)|+|(P[fx_1(\cdot,z)-f\varphi_z])(t)|\\ &\le \alpha(t)\omega(z)+Hl(\alpha\omega(z))(t), \qquad t\in[a,b].\end{aligned} \tag{6.28}$$

It follows from (6.28) that

$$|x_2(t,z)-\varphi_z(t)|\le\frac{b-a}{4}\left(\omega(z)+(Hl)(\omega(z))(t)\right), \qquad t\in[a,b]. \tag{6.29}$$

It is obvious from (5.8) that $\omega(z)\ge 0$ for all z and, hence, by Lemma 6.3,

$$\begin{aligned}|x_2(t,z)-\varphi_z(t)| \le \frac{b-a}{4}\Bigg(&\mathbf{1}_n\\ &+\left(1-\frac{t-a}{b-a}\right)\int_a^t K_l(s)ds+\frac{t-a}{b-a}\int_t^b K_l(s)ds\Bigg)\omega(z)\\ \le \frac{b-a}{4}&\left(\mathbf{1}_n+Q_l\right)\omega(z), \qquad t\in[a,b],\end{aligned} \tag{6.30}$$

where Q_l is the constant matrix given by (5.7). Consequently,

$$x_2(\cdot,z)\in\mathscr{A}_z\left(\tfrac{b-a}{4}\left(\mathbf{1}_n+Q_l\right)\omega(z)\right). \tag{6.31}$$

On the other hand, (6.24) implies that

$$\varrho^*\ge\frac{b-a}{4}\left(\mathbf{1}_n+Q_l\right)\omega(z)$$

and, therefore, due to (6.31), we have $x_2(\cdot, z) \in \mathscr{A}_z(\varrho^*)$. By Lemma 6.5,

$$x_2(\cdot, z) \in \mathscr{B}(-\varrho^* + \bar{A}_- z_0 - A_+ z_1,\ \bar{\bar{A}}_- z_1 + \varrho^*), \tag{6.32}$$

that is, $x_2(\cdot, z)$ lies in the set where f satisfies the Lipschitz condition (4.2).

Using (4.2) for the functions $x_2(\cdot, z)$ and φ_z, similarly to (6.28), (6.29), we obtain

$$\begin{aligned} |x_3(t, z) - \varphi_z(t)| &= |(Pf x_2(\cdot, z)(t)| \\ &\le |(Pf\varphi_z)(t)| + |(P[f x_2(\cdot, z) - f\varphi_z])(t)| \\ &\le \alpha(t)\omega(z) + (Hl)(\alpha\omega(z) + (Hl)(\alpha\omega(z)))(t) \end{aligned}$$

and, therefore, by (6.24),

$$\begin{aligned} |x_3(t, z) - \varphi_z(t)| &\le \frac{b-a}{4}\left(1_n + Q_l + Q_l^2\right)\omega(z) \\ &\le \varrho^*, \qquad t \in [a, b], \end{aligned} \tag{6.33}$$

whence it follows that

$$x_3(\cdot, z) \in \mathscr{B}(-\varrho^* + \bar{A}_- z_0 - A_+ z_1,\ \bar{\bar{A}}_- z_1 + \varrho^*).$$

Proceeding analogously, we find that the estimates

$$|x_m(t, z) - \varphi_z(t)| \le \frac{b-a}{4}\sum_{k=0}^{m-1} Q_l^k \omega(z) \tag{6.34}$$

$$\le \varrho^*, \qquad t \in [a, b], \tag{6.35}$$

hold for any $m \ge 1$. By virtue of Lemma 6.5, this implies that

$$\{x_m(\cdot, z) : m \ge 1\} \subset \mathscr{B}(-\varrho^* + \bar{A}_- z_0 - A_+ z_1,\ \bar{\bar{A}}_- z_1 + \varrho^*). \tag{6.36}$$

Recalling (5.2) and using Lemma 6.2, we get

$$\begin{aligned} |x_{m+1}(t, z) - x_m(t, z)| &= |(P[f x_m(\cdot, z) - f x_{m-1}(\cdot, z)])(t)| \\ &\le H\,|f x_m(\cdot, z) - f x_{m-1}(\cdot, z)|\,(t) \end{aligned} \tag{6.37}$$

for all $t \in [a, b]$ and $m \ge 1$. In view of (6.36), the Lipschitz condition for f holds at all the members of sequence (5.2) and, therefore, estimate (6.37) yields

$$\begin{aligned} |x_{m+1}(t, z) - x_m(t, z)| &\le (Hl\,|x_m(\cdot, z) - x_{m-1}(\cdot, z)|)(t) \\ &\le ((Hl)^m\,|x_1(\cdot, z) - \varphi_z|)(t) \end{aligned} \tag{6.38}$$

for all $t \in [a, b]$ and $m \ge 1$. In view of estimate (6.22) and Lemma 6.3, inequality (6.38) yields

$$\begin{aligned} |x_{m+1}(t, z) - x_m(t, z)| &\le \frac{b-a}{4}((Hl)^m\omega(z))(t) \\ &\le \frac{b-a}{4} Q_l^m \omega(z), \qquad t \in [a, b], \end{aligned} \tag{6.39}$$

for all $m \ge 1$.

Due to assumption (5.10), it follows immediately from (6.39) that

$$\begin{aligned}|x_{m+k}(t,z) - x_m(t,z)| &\le \sum_{j=0}^{k-1} |x_{m+j+1}(t,z) - x_{m+j}(t,z)| \\ &\le \frac{b-a}{4} \sum_{j=0}^{k-1} Q_l^{m+j} \omega(z) \\ &\le \frac{b-a}{4} Q_l^m \sum_{j=0}^{+\infty} Q_l^j \omega(z) \\ &= \frac{b-a}{4} Q_l^m (1 - Q_l)^{-1} \omega(z), \qquad t \in [a,b], \qquad (6.40)\end{aligned}$$

for any $m \ge 0$ and $k \ge 1$. Since, by (5.10), $\lim_{m\to+\infty} Q_l^m = 0$, estimate (6.40) proves that (5.2) is a Cauchy sequence in C.

The form of the operator P and function φ_z appearing in (5.2), (5.3) ensure that, for any $z \in \langle z_0, z_1 \rangle$ and $m \ge 1$, the function $x_m(\cdot, z)$ satisfies the two-point boundary condition

$$Ax_m(a,z) + x_m(b,z) = 0 \qquad (6.41)$$

and the initial condition

$$x_m(a,z) = z. \qquad (6.42)$$

Passing to the limit as $m \to \infty$ in (6.41), (6.42), we arrive at (5.12), (5.13).

Passing to the limit as $m \to \infty$ in equality (5.2), we show that the function $x_\infty(\cdot, z)$ given by (5.11) is the unique solution the integro-functional equation

$$x(t) = \varphi_z(t) + (Pfx)(t), \qquad t \in [a,b]. \qquad (6.43)$$

In particular, the function $\Delta : \langle z_0, z_1 \rangle \to \mathbb{R}^n$ is well defined by formula (5.14).

Differentiating both sides of (6.43) and recalling (5.1) and (5.3), we find that, for an arbitrary $z \in \langle z_0, z_1 \rangle$, the function $x = x_\infty(\cdot, z)$ is a unique solution of the Cauchy problem (5.15), (5.16).

Finally, passing to the limit as $k \to \infty$ in (6.40), we arrive at estimate (5.17). □

Let us find the relation of the function $x_\infty(\cdot, z)$ to the solution of the original boundary value problem (1.1), (3.1). For this purpose, consider the following Cauchy problem for equation (1.1) with a constant forcing term,

$$x'(t) = (fx)(t) + \mu, \qquad t \in [a,b], \qquad (6.44)$$

$$x(a) = z, \qquad (6.45)$$

where $\mu \in \mathbb{R}^n$ and $z \in \langle z_0, z_1 \rangle$ are parameters.

Theorem 6.1. *Under the conditions of Theorem 5.1, a solution $x(\cdot)$ of the initial value problem* (6.44), (6.45) *satisfies the two-point boundary condition* (3.1) *if and only if*

$$\mu = -\Delta(z), \qquad (6.46)$$

where $\Delta : \langle z_0, z_1\rangle \to \mathbb{R}^n$ *is the function given by* (5.14). *In that case,* $x(\cdot) = x_\infty(\cdot, z)$.

Proof. The assertion of Theorem 6.1 is obtained by analogy to the proof of Theorem 4.2 from [14]. □

Theorem 6.2. *Let the conditions of Theorem 5.1 be satisfied. Then the limit function* $x_\infty(\cdot, z)$ *of the recurrence sequence* (5.2) *is a solution of the boundary value problem* (1.1), (3.1) *if, and only if the value of the vector parameter* $z \in \langle z_0, z_1\rangle$ *satisfies the system of equations*

$$\Delta(z) = 0, \tag{6.47}$$

where $\Delta : \langle z_0, z_1\rangle \to \mathbb{R}^n$ *is given by* (5.14).

Proof. It is sufficient to apply Theorem 6.1 and notice that the equation (5.15) coincides with equation (1.1) if and only if relation (6.47) holds. □

Remark 6.1. Equations of type (6.47) are sometimes called "determining equations" because it is from there one has to determine the actual values of the parameters $z \in \langle z_0, z_1\rangle$ involved in the iteration process (5.2). Likewise, $\Delta : \langle z_0, z_1\rangle \to \mathbb{R}^n$ given by (5.14) is often referred to as a "determining function" for problem (1.1), (3.1).

In practice, it is natural to fix some $m \geq 1$, introduce the mth "approximate determining function" $\Delta_m : \langle z_0, z_1\rangle \to \mathbb{R}^n$ by setting

$$\Delta_m(z) := (A + \mathbf{1}_n)\, z + \int_a^b (f x_m(\cdot, z))(s)\, ds, \qquad z \in \langle z_0, z_1\rangle, \tag{6.48}$$

and, instead of the inconvenient (6.47), consider the mth "approximate determining equation"

$$\Delta_m(z) = 0. \tag{6.49}$$

It is important to point out that equation (6.49), in contrast to (6.47), is constructed directly based on the function $x_m(\cdot, z)$ and, thus, does not contain any unknown terms.

We shall see below that if equation (6.49) has an isolated solution $z = z_m$ in $\langle z_0, z_1\rangle$, then, under suitable additional assumptions, the corresponding exact system of determining equations (6.47) is also solvable and, therefore, by virtue of Theorem 6.2, the boundary value problem (1.1), (3.1) has a solution. In that case, due to estimate (5.17), the function

$$X_m(t) := x_m(t, z_m), \qquad t \in [a, b], \tag{6.50}$$

can be regarded as an mth approximation to a solution of problem (1.1), (3.1).

7. An existence theorem

To investigate the solvability of the given boundary value problem (1.1), (3.1), we need the following

Lemma 7.1. *Under the assumptions of Theorem 5.1,*

$$|\Delta(z) - \Delta_k(z)| \leq \frac{b-a}{4} \int_a^b K_l(s) ds\, Q_l^k \,(1 - Q_l)^{-1}\, \omega(z) \tag{7.1}$$

for arbitrary $z \in \langle z_0, z_1 \rangle$ *and* $k \geq 1$.

Proof. Let $z \in \langle z_0, z_1 \rangle$ and $k \geq 1$ be arbitrary. By virtue of (5.14) and (6.49), we have

$$\begin{aligned} |\Delta(z) - \Delta_k(z)| &= \left| \int_a^b [f x_\infty(\cdot, z)(t) - f x_k(\cdot, z)(t)] dt \right| \\ &\leq \int_a^b |f x_\infty(\cdot, z)(t) - f x_k(\cdot, z)(t)|\, dt. \end{aligned} \tag{7.2}$$

Since condition (5.10) is assumed, it follows that estimate (6.34) is satisfied for any $m \geq 1$. Passing to the limit as $m \to \infty$ in (6.34) and taking (6.24) into account, we obtain

$$|x_\infty(t, z) - \varphi_z(t)| \leq \frac{b-a}{4} \sum_{j=0}^{\infty} Q_l^j \omega(z) = \varrho^* \tag{7.3}$$

for all $t \in [a, b]$. Thus, $x_\infty(\cdot, z) \in \mathscr{A}_z(\varrho^*)$ and, hence, by Lemma 6.5,

$$x_\infty(\cdot, z) \in \mathscr{B}(-\varrho^* + \bar{A}_- z_0 - A_+ z_1,\ \bar{\bar{A}}_- z_1 + \varrho^*). \tag{7.4}$$

It follows from (6.36) and (7.4) that the Lipschitz condition (4.2) imposed on f can be applied for the functions $x_\infty(\cdot, z)$ and $x_k(\cdot, z)$. By doing so in (7.2), taking estimate (5.17) into account, and using Lemma 6.3, we obtain

$$\begin{aligned} |\Delta(z) - \Delta_k(z)| &\leq \int_a^b l\, |x_\infty(\cdot, z)(t) - x_k(\cdot, z)(t)|\, dt \\ &\leq \frac{b-a}{4} \int_a^b (l\, Q_l^k \,(1 - Q_l)^{-1}\, \omega(z))(t)\, dt \\ &\leq \frac{b-a}{4} \int_a^b K_l(s) ds\, Q_l^k \,(1 - Q_l)^{-1}\, \omega(z), \end{aligned}$$

which coincides with (7.1). □

Let us formulate a statement that gives conditions sufficient for the solvability of the boundary value problem (1.1), (3.1).

Definition 7.1. Let $S \subset \mathbb{R}^n$ be an arbitrary non-empty set. For any pair of functions $g_j = \operatorname{col}(g_{j,1}, \dots, g_{j,n})$, $j = 1, 2$, we write

$$g_1 \vartriangleright_S g_2 \tag{7.5}$$

if and only if there exists a function $\nu : S \to \{1, 2, \dots, n\}$ such that the strict inequality

$$g_{1,\nu(x)} > g_{2,\nu(x)} \tag{7.6}$$

holds for all $x \in S$.

In other words, relation (7.6) means that, at every single point x from S, at least one of the components of the vector g_1 is greater then the corresponding component of the vector g_2, and the number of the component may vary with x.

This relation inherits many properties of the usual strict inequality sign and, in particular, is transitive in the sense that $f \geq g$ and $g \rhd_S h$ imply the relation $f \rhd_S h$. This fact will be used below in the proof of the following

Theorem 7.1. *Let us suppose that, in addition to assumptions of Theorem 5.1, there exist a closed domain $\Omega \subset \langle z_0, z_1 \rangle$ and an integer $m \geq 1$ such that, on the boundary of Ω, the approximate determining function Δ_m given by formula* (6.48) *satisfies the condition*

$$|\Delta_m| \rhd_{\partial\Omega} \frac{b-a}{4} \int_a^b K_l(s) ds \, Q_l^k \left(1 - Q_l\right)^{-1} \omega, \tag{7.7}$$

where $\omega : \langle z_0, z_1 \rangle \to \mathbb{R}^n$ is the function given by (5.8).

Let, moreover,

$$\deg\left(\Delta_m, \Omega, 0\right) \neq 0. \tag{7.8}$$

Then there exists a certain $z^* \in \Omega$ such that the function $x_\infty(\cdot, z^*)$ is a solution of the boundary value problem (1.1), (3.1).

As is seen from equality (5.12) of Theorem 5.1, the vector z^* appearing in the last formulation, in fact, coincides with the value of the solution at the point a.

Proof of Theorem 7.1. Let us define the family of mappings $\Gamma_\theta : \langle z_0, z_1 \rangle \to \mathbb{R}^n$, $\theta \in [0, 1]$, by putting

$$\Gamma_\theta(z) := \Delta_m(z) + \theta \left[\Delta(z) - \Delta_m(z)\right] \tag{7.9}$$

for any $z \in \partial\Omega$ and $\theta \in [0, 1]$. Being a subset of a bounded set $\langle z_0, z_1 \rangle$, the set Ω is, of course, bouded itself.

Obviously, Γ_θ is a completely continuous mapping on $\partial\Omega$ for every $\theta \in [0, 1]$ and, furthermore,

$$\Gamma_0 = \Delta_m, \qquad \Gamma_1 = \Delta. \tag{7.10}$$

It follows from (7.9) and Lemma 7.1 that

$$\begin{aligned} |\Gamma_\theta(z)| &= |\Delta_m(z) + \theta \left[\Delta(z) - \Delta_m(z)\right]| \\ &\geq |\Delta_m(z)| - |\Delta(z) - \Delta_m(z)| \\ &\geq |\Delta_m(z)| - \frac{b-a}{4} \int_a^b K_l(s) ds \, Q_l^k \left(1 - Q_l\right)^{-1} \omega(z) \end{aligned}$$

for all $z \in \partial\Omega$. Therefore, by virtue of condition (7.7), we have

$$|\Gamma_\theta| \rhd_{\partial\Omega} 0. \tag{7.11}$$

Relation (7.11), in particluar, implies that Γ_θ does not vanish on $\partial\Omega$. Thus, the family $\{\Gamma_\theta : \theta \in [0, 1]\}$ is a non-degenerate homotopy connecting