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ASYMPTOTIC PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH BOUNDED Φ-LAPLACIAN

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

In this paper we deal with the asymptotic problem

$$\big(a(t)\Phi(x')\big)' + b(t)F(x) = 0\,, \quad \lim_{t\to\infty} x'(t) = 0\,, \quad x(t) > 0 \ \text{ for large } \ t\,. \qquad (*)$$

Motivated by searching for positive radially symmetric solutions in a fixed exterior domain in $\mathbb{R}^N$ for partial differential equations involving the curvature operator, the global positiveness and uniqueness of (*) is also considered.

Several examples illustrate the discrepancies between the bounded and unbounded Φ. The results for the curvature operator and the classical Φ-Laplacian are compared, too.

Key words and phrases: Ordinary differential equations, nonlinear boundary value problems, bounded Φ-Laplacian, nonoscillation.
AMS (MOS) Subject Classifications: 34B10, 34C10

1 Introduction

In this paper we deal with the second order nonlinear differential equation

$$\big((a(t)\Phi(x')\big)' + b(t)F(x) = 0, \quad (t \geq t_0), \tag{1}$$

where:

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(i) $\Phi\colon \mathbb{R} \to (-\sigma,\sigma)$, $0 < \sigma \le \infty$, is an increasing odd homeomorphism, such that $\Phi(u)u > 0$ for $u \neq 0$;

(ii) $F\colon \mathbb{R} \to \mathbb{R}$ is a continuous increasing function such that $F(u)u > 0$ for $u \neq 0$;

(iii) $a, b\colon [t_0,\infty) \to (0,\infty)$ are continuous functions and

$$\int_{t_0}^{\infty} b(t)\,dt < \infty\,.$$

Our aim is to study the existence of positive solutions x of (1) satisfying the asymptotic boundary conditions

$$\lim_{t\to\infty} x(t) = \ell_x\,, \quad \lim_{t\to\infty} x'(t) = 0\,, \quad 0 < \ell_x < \infty\,, \tag{2}$$

$$\lim_{t\to\infty} x(t) = \infty\,, \quad \lim_{t\to\infty} x'(t) = 0\,. \tag{3}$$

The prototype of (1) is the equation

$$\big(a(t)\Phi_C(x')\big)' + b(t)F(x) = 0\,, \quad (t \ge t_0)\,, \tag{4}$$

where the map $\Phi_C\colon \mathbb{R} \to (-1,1)$ is given by

$$\Phi_C(u) = \frac{u}{\sqrt{1+|u|^2}}\,. \tag{5}$$

This equation arises in the study of the radially symmetric solutions of partial differential equation with the curvature operator

$$\operatorname{div}\left(g(|x|)\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + B(|x|)F(u) = 0\,, \tag{6}$$

where $x = (x_1,\dots x_n) \in \mathbb{R}^n$, $n \ge 2$, $\nabla u = (D_1u,\dots,D_nu)$, $D_i = \partial/\partial x_i$, $i = 1,\dots n$, $|x| = \sqrt{\sum_{i=1}^n x_i^2}$, $E = \{x \in R^n : |x| \ge d\}$, $d > 0$ and $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a weight function.

Denote $r = |x|$ and $\frac{du}{dr} = u_r$ the radial derivative of u. Since $\nabla u = \frac{x}{r}u_r$, we have

$$g(r)\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = x\frac{g(r)}{r}\Phi_C(u_r)$$

and, by a direct computation, we get that the function u is a radially symmetric solution of (6) if and only if the function $y = y(r) = u(|x|)$ is a solution of

$$\big(r^{n-1}g(r)\Phi_C(y')\big)' + r^{n-1}B(r)F(y) = 0\,, \quad (r \ge d)\,,$$

which is a special case of (1).

Boundary value problems on a compact interval associated to the partial differential equations with the mean curvature operator have been investigated in [4, 5, 6]; see also the references therein.

When Φ is the classical Φ-Laplacian, i.e.

$$\Phi(u) = |u|^{p-2}u\,, \quad p > 1\,,$$

various types of asymptotic problems for equation (1) have been deeply investigated. We refer to [2, 3, 8, 13, 14] and the monographs [1, 10] for further references.

In a recent paper [9], the authors studied all possible types of nonoscillatory solutions of (1) and their mutual coexistence under the assumption that there exists $\lambda > 0$ such that

$$\lambda a^{-1}(t) \in \operatorname{Im}\Phi \quad \text{for any} \quad t \geq t_0\,. \tag{7}$$

This classification depends on the asymptotic behavior of the vector $(x, x^{[1]})$, where x is a solution of (1) and $x^{[1]}$ denotes its quasiderivative

$$x^{[1]}(t) = a(t)\Phi(x'(t))\,.$$

If $\sigma = \infty$, i.e. $\operatorname{Im}\Phi$ is unbounded, then (7) is satisfied for any $\lambda > 0$. So, condition (7) plays a role only when $\operatorname{Im}\Phi$ is bounded and requires

$$\liminf_{t\to\infty} \quad a(t) > 0\,. \tag{8}$$

Moreover, when (8) holds, then

$$\lim_{t\to\infty} x^{[1]}(t) = 0 \quad \Longrightarrow \quad \lim_{t\to\infty} x'(t) = 0$$

for any solution x of (1). If $\liminf_{t\to\infty} a(t) = 0$, then this is not in general true.

For these reasons, particular attention will be devoted to the equation (1) with Φ bounded and its special case (4). Consequently, throughout this paper, we assume

$$\operatorname{Im}\Phi \quad \text{is bounded,} \quad \liminf_{t\to\infty} a(t) = 0\,. \tag{Hp}$$

It is easy to show (see below) that, when (Hp) holds, any nonoscillatory solution x of (1) satisfies $\lim_{t\to\infty} x^{[1]}(t) = 0$.

Moreover, the global positiveness and uniqueness of solutions of (1)–(2) will be also considered. This problem is motivated by searching for positive radially symmetric solutions in a fixed exterior domain in $\mathbb{R}^N$ for (6).

We will show that the lack of the homogeneity property of Φ can produce several new phenomena, which are illustrated by some examples. With minor changes, our results can be applied also when $\sigma = \infty$ and so they complement the previous ones stated in [7, 9] for a general Φ and in [8] for the classical Φ-Laplacian. Similarities and discrepancies with these cases complete the paper jointly with a discussion on the meaning of the assumption (Hp).

Finally, we introduce the integral

$$J_\mu = \int^\infty \Phi^* \left(\mu \frac{1}{a(t)} \int_t^\infty b(s)\, ds \right) dt \,,$$

where Φ^* denotes the inverse function to Φ and μ is a positive constant. This integral plays a crucial role for the asymptotic behavior of solutions, similarly as in case when Φ is unbounded ([7, 8, 9]).

2 Necessary Conditions

Throughout this paper we shall consider only the solutions of (1) which exist on some ray $[t_x, \infty)$, where $t_x \geq t_0$ may depend on the particular solution. As usual, a solution x of (1) defined in some neighborhood of infinity is said to be *nonoscillatory* if $x(t) \neq 0$ for large t, and *oscillatory* otherwise.

If x is eventually positive [negative], then its quasiderivative $x^{[1]}$ is decreasing [increasing] for large t. The following holds.

Lemma 2.1. *Assume* (Hp). *Then any nonoscillatory solution x of* (1) *satisfies*

$$x(t)x^{[1]}(t) > 0 \quad \textit{for large } t \textit{ and} \quad \lim_{t\to\infty} x^{[1]}(t) = 0 \,.$$

Proof. Let x be a nonoscillatory solution of (1) and, without loss of generality, assume $x(t) > 0$ for $t \geq T \geq t_0$. From (Hp), there exists $\{t_k\}$, $t_k \to \infty$ such that $\lim_k x^{[1]}(t_k) = 0$ and, because $x^{[1]}$ is eventually decreasing, the assertion follows.

In virtue of Lemma 2.1, nonoscillatory solutions of (1) are eventually monotone. Necessary conditions for the solvability of (1)–(2), or (1)–(3), are given by the following.

Proposition 2.1. *Assume* (Hp).

i$_1$) *If*

$$\limsup_{t\to\infty} \frac{1}{a(t)} \int_t^\infty b(s)\, ds = \infty \,,$$

then any continuable solution of (1) *is oscillatory.*

i$_2$) *Assume* $\lim_{|u|\to\infty} |F(u)| = \infty$. *If*

$$\limsup_{t\to\infty} \frac{1}{a(t)} \int_t^\infty b(s)\, ds > 0 \,, \tag{9}$$

then (1) *does not have unbounded nonoscillatory solutions.*

i$_3$) *If*

$$\liminf_{t\to\infty} \frac{1}{a(t)} \int_t^\infty b(s)\, ds > 0 \,,$$

then (1) *does not have bounded nonoscillatory solutions. In addition, if* $\lim_{|u|\to\infty} |F(u)| = \infty$*, then any continuable solution of* (1) *is oscillatory.*

i$_4$) *If*

$$\lim_{t\to\infty} \frac{1}{a(t)} \int_t^\infty b(s)\, ds = 0\,, \tag{10}$$

then any bounded nonoscillatory solution x *of* (1) *satisfies*

$$\lim_{t\to\infty} x'(t) = 0\,. \tag{11}$$

Conversely, if (1) *has nonoscillatory solutions* x *satisfying* (11)*, then* (10) *holds.*

Proof. Let x be a nonoscillatory solution of (1). In view of Lemma 2.1 we can suppose, without loss of generality, $x(t) > 0$, $x'(t) > 0$ for $t \ge T$.

Claim i$_1$). Integrating (1), we obtain for $t \ge T$

$$x^{[1]}(t) = \int_t^\infty b(s) F\big(x(s)\big)\, ds \ge F\big(x(t)\big) \int_t^\infty b(s)\, ds \tag{12}$$

or

$$\frac{\sigma}{F(x(t))} > \frac{1}{a(t)} \int_t^\infty b(s)\, ds$$

which yields a contradiction as $t \to \infty$.

Claim i$_2$). Now assume $\lim_{t\to\infty} x(t) = \infty$. Using the same argument, we obtain for $t \ge T$

$$\sigma > \Phi\big(x'(t)\big) \ge F\big(x(t)\big) \frac{1}{a(t)} \int_t^\infty b(s)\, ds$$

which contradicts (9) as $t \to \infty$.

Claim i$_3$). From (12) it follows that

$$\Phi(x'(t)) \ge F(x(T)) \frac{1}{a(t)} \int_t^\infty b(s)\, ds\,, \tag{13}$$

so $\liminf_{t\to\infty} x'(t) > 0$, i.e. x is unbounded. The second assertion follows from claim i_2).

Claim i$_4$). If $\lim_{t\to\infty} x(t) = \ell_x < \infty$, then for $t \ge T$ we have

$$\Phi^* \left(F(\ell_x) \frac{1}{a(t)} \int_t^\infty b(s)\, ds \right) \ge x'(t) \ge \Phi^* \left(F(x(T)) \frac{1}{a(t)} \int_t^\infty b(s) ds \right),$$

which yields (11). Conversely, from (13), the condition (10) immediately follows.

From (13) the following result follows.

Proposition 2.2. *Assume* (Hp)*. If* $J_\mu = \infty$ *for any sufficiently small* $\mu > 0$*, then bounded nonoscillatory solutions of* (1) *do not exist.*

Remark 2.1. *Proposition 2.1*-i$_2$) *remains to hold if the unboundedness of F is replaced by*

$$\lim_{|u|\to\infty} |F(u)| = M_F < \infty\,, \quad \limsup_{t\to\infty} \frac{1}{a(t)} \int_t^\infty b(s)\,ds > \frac{\sigma}{M_F}\,. \tag{14}$$

The following example shows that the condition (14) is optimal.

Example 2.1. *The equation*

$$\Big(t^{-1}\Phi_C(x')\Big)' + \frac{\sqrt{t^2+4}}{\sqrt{5}t^3}\Phi_C(x) = 0\,, \quad (t \geq 1)\,,$$

has the unbounded solution $x(t) = t/2$, i.e., the statement of Proposition 2.1-i$_2$) *does not hold. In this case*

$$\limsup_{t\to\infty} \frac{1}{a(t)} \int_t^\infty b(s)\,ds = \limsup_{t\to\infty} t \int_t^\infty \frac{1}{s^2}\,ds = 1\,,$$

so (14) *is not verified. However, Proposition 2.1*-i$_3$) *is applicable and any nonoscillatory solutions is unbounded.*

3 Nonoscillatory Bounded Solutions

In this section we deal with solutions of (1) satisfying the asymptotic conditions (2) and with their global positiveness and uniqueness.

Theorem 3.1. *Assume* (Hp),

$$\limsup_{t\to\infty} \frac{1}{a(t)} \int_t^\infty b(s)\,ds < \infty \tag{15}$$

and there exists a positive constant $\overline{\mu}$ such that

$$J_{\overline{\mu}} = \int^\infty \Phi^*\left(\frac{\overline{\mu}}{a(t)} \int_t^\infty b(s)\,ds\right) dt < \infty\,.$$

Then, for each $L > 0$, L sufficiently small, (1) *has nonoscillatory solutions, x, such that $\lim_{t\to\infty} x(t) = L$.*

In addition, if (10) *holds, then $\lim_{t\to\infty} x'(t) = 0$, i.e. the asymptotic problem* (1)–(2) *is solvable.*

Proof. In view of (15), there exists $L > 0$ such that $F(L) < \overline{\mu}$ and

$$\sup_{t\geq t_0} \frac{F(L)}{a(t)} \int_t^\infty b(s)\,ds < \sigma\,.$$

Choose $t_1 \geq t_0$ large so that

$$\int_{t_1}^{\infty} \Phi^* \left(\frac{F(L)}{a(t)} \int_t^{\infty} b(s)\, ds \right) dt \leq \frac{L}{2}\,. \tag{16}$$

Denote with $C[t_1, \infty)$ the Fréchet space of all continuous functions on $[t_1, \infty)$ endowed with the topology of uniform convergence on compact subintervals of $[t_1, \infty)$ and consider the set $\Omega \subset C[t_1, \infty)$ given by

$$\Omega = \{u \in C[t_1, \infty) : L/2 \leq u(t) \leq L\}\ .$$

Define on Ω the operator T as follows

$$T(u)(t) = L - \int_t^{\infty} \Phi^* \left(\frac{1}{a(s)} \int_s^{\infty} b(\tau) F(u(\tau))\, d\tau \right) ds\,.$$

Obviously, $T(u)(t) \leq L$. From (16), because it results for $s \geq t_1$

$$\int_s^{\infty} b(\tau) F(u(\tau))\, d\tau \leq F(L) \int_s^{\infty} b(\tau)\, d\tau\,,$$

we obtain $T(u)(t) \geq L/2$, that is T maps Ω into itself. Let us show that $T(\Omega)$ is relatively compact, i.e. $T(\Omega)$ consists of functions equibounded and equicontinuous on every compact interval of $[t_1, \infty)$. Because $T(\Omega) \subset \Omega$, the equiboundedness follows. Moreover, in view of the above estimates, for any $u \in \Omega$ we have

$$0 < \frac{d}{dt} T(u)(t) \leq \Phi^* \left(\frac{F(L)}{a(s)} \int_s^{\infty} b(\tau)\, d\tau \right)\,,$$

which proves the equicontinuity of the elements of $T(\Omega)$. The continuity of T in Ω follows by using the Lebesgue dominated convergence theorem and taking into account (16). Thus, by the Tychonov fixed point theorem, there exists a fixed point x of T. Clearly, x is a solution of (1) such that $\lim_{t\to\infty} x(t) = L$ and the solvability of the BVP (1)-(2) follows from Proposition 2.1–i_4).

Theorem 3.1 answers the existence problem of bounded eventually positive solutions of (1). For the equation (4) this result can be improved by obtaining sufficient conditions for their global positivity and uniqueness. To this end, the following Gronwall type lemma is needed.

Lemma 3.1 ([11, Lemma 4.1]). *Let w and ψ be two nonnegative continuous functions such that $\psi \in L^1[T, \infty)$ and $w\,\psi \in L^1[T, \infty)$. If*

$$w(t) \leq A + \int_t^{\infty} \psi(s) w(s)\, ds\,, \quad (t \geq T)\,,$$

for some nonnegative constant A, then

$$w(t) \leq A \exp\left(\int_t^{\infty} \psi(s)\, ds \right), \quad (t \geq T)\,.$$

If, in particular, $A = 0$, then $w(t) = 0$ identically on $[T, \infty)$.

Theorem 3.2. *Assume* (Hp) *and suppose that* F *is continuously differentiable in a neighborhood of zero such that*

$$\lim_{u\to 0}\frac{F(u)}{u}=0\,. \tag{17}$$

If conditions (15) *and*

$$I=\int_{t_0}^{\infty}\frac{1}{a(t)}\int_t^{\infty}b(s)\,ds\,dt<\infty \tag{18}$$

are verified, then for any positive and sufficiently small L*, there exists a unique solution* x *of* (4) *such that*

$$x(t)>0 \quad \textit{for} \quad t\ge t_0\,, \qquad \lim_{t\to\infty}x(t)=L\,. \tag{19}$$

Proof. In view of (15), there exists $\overline{\lambda}>0$ such that for any $t\ge t_0$

$$\sup_{t\ge t_0}\frac{\overline{\lambda}}{a(t)}\int_t^{\infty}b(s)\,ds<\frac{1}{2}\,. \tag{20}$$

Thus

$$\Phi_C^*\left(\frac{\overline{\lambda}}{a(t)}\int_t^{\infty}b(s)\,ds\right)<\frac{2}{\sqrt{3}}\frac{\overline{\lambda}}{a(t)}\int_t^{\infty}b(s)\,ds \tag{21}$$

and so, from (18), we have $J_{\overline{\lambda}}<\infty$. In virtue of (17), choose $L>0$ such that F is continuously differentiable on $(0,L]$ and

$$F(L)<\min\left\{\overline{\lambda},\frac{\sqrt{3}L}{4\,I}\right\}\,.$$

From (21) we get

$$\int_{t_0}^{\infty}\Phi_C^*\left(\frac{F(L)}{a(t)}\int_t^{\infty}b(s)\,ds\right)\,dt<\frac{2}{\sqrt{3}}I\;F(L)<\frac{L}{2}$$

and so (16) is satisfied with $t_1=t_0$. Reasoning as in the proof of Theorem 3.1-i$_1$), there exists at least one solution x of (4) satisfying the boundary conditions (19). It remains to show the uniqueness of this solution. Let z,y be two solutions of (4) satisfying (19). Since z and y are increasing, we have $0<y(t)<L$, $0<z(t)<L$ on $[t_0,\infty)$. Setting

$$h_w(t)=\frac{1}{a(t)}\int_t^{\infty}b(s)F(w(s))\,ds,$$

in view of (20) we have $0<h_y(t)<2^{-1}$, $0<h_z(t)<2^{-1}$. Integrating (4), we obtain

$$|y(t)-z(t)|\le\int_t^{\infty}\left|\Phi_C^*(h_y(s))-\Phi_C^*(h_z(s))\right|\,ds\,. \tag{22}$$

A direct calculation gives

$$\frac{d}{du}\Phi_C^*(u) \leq \frac{8}{\sqrt{27}} \quad \text{on} \quad [0, \frac{1}{2}],$$

and so, the mean value theorem implies

$$\begin{aligned}|\Phi_C^*(h_y(t)) - \Phi_C^*(h_z(t))| &\leq \frac{8}{\sqrt{27}}\,|h_y(t) - h_z(t)| \\ &\leq \frac{8}{\sqrt{27}a(t)}\int_t^\infty b(r)\,|F(y(r)) - F(z(r))|\,dr\,.\end{aligned}$$

Using again the mean value theorem, we get

$$|\Phi_C^*(h_y(t)) - \Phi_C^*(h_z(t))| \leq \frac{8M_L}{\sqrt{27}a(t)}\int_t^\infty b(r)\,|y(r) - z(r)|\,dr\,, \tag{23}$$

where

$$M_L = \max_{\xi\in[0,L]} \frac{dF}{du}_{|u=\xi}\,.$$

Putting

$$w(t) = \sup_{\xi\geq t}|y(\xi) - z(\xi)|\,,$$

from (22) and (23) we get

$$w(t) \leq \frac{8}{\sqrt{27}}M_L\int_t^\infty \left(\frac{1}{a(s)}\int_s^\infty b(r)\,dr\right) w(s)\,ds$$

and applying Lemma 3.1 the assertion follows.

Remark 3.1. *As already claimed, Theorem 3.2 plays an important role in searching for positive radially symmetric solutions in a fixed exterior domain in* $\mathbb{R}^N$ *for the partial differential equation* (6)*. Moreover, Theorem 3.2 can be easily extended to an equation involving a more general* Φ*, by assuming that* Φ *is continuously differentiable in a neighborhood of zero. The details are left to the reader.*

A closer examination of proofs of Theorems 3.1 and 3.2 shows that these results hold also when $\operatorname{Im}\Phi$ is unbounded. So, in particular, they can be applied to the equation associated to the Sturm-Liouville operator

$$(a(t)z')' + b(t)F(z) = 0\,. \tag{24}$$

The boundedness of nonoscillatory solutions of (24) is strongly related to the boundedness of nonoscillatory solutions of (4). We make this observation precise in Corollary 3.1. To show this fact, the following lemma concerning the map Φ_C is needed.

Lemma 3.2. *Assume* (15) *and let*

$$H = \sup_{t \ge t_0} \frac{1}{a(t)} \int_t^\infty b(s)\, ds\,.$$

For any $\mu > 0$ such that

$$\mu H < 1 \tag{25}$$

we have

$$I = \int_{t_0}^\infty \frac{1}{a(t)} \int_t^\infty b(s)\, ds\, dt < \infty \iff J_\mu^C = \int_{t_0}^\infty \Phi_C^* \left(\frac{\mu}{a(t)} \int_t^\infty b(s)\, ds \right) dt < \infty\,.$$

Then, in particular, the convergence of the integral J_μ^C does not depend on the values of the parameter μ, i.e. either $J_\mu^C < \infty$ or $J_\mu^C = \infty$ for any $\mu > 0$ satisfying (25)*.*

Proof. From (25) we have

$$\sup_{t \ge t_0} \frac{\mu}{a(t)} \int_t^\infty b(s)\, ds < 1\,.$$

Then

$$\frac{\mu}{a(t)} \int_t^\infty b(s) ds \le \Phi_C^* \left(\mu \frac{1}{a(t)} \int_t^\infty b(s)\, ds \right) \le \frac{\mu}{\sqrt{1 - H^2}} \frac{1}{a(t)} \int_t^\infty b(s)\, ds$$

and the assertion follows.

Corollary 3.1. *Assume* (15) *and*

$$\liminf_{t \to \infty} a(t) = 0\,. \tag{26}$$

Then the following statements are equivalent.

i_1) *Equation* (24) *has bounded nonoscillatory solutions.*

i_2) *Equation* (4) *has bounded nonoscillatory solutions.*

i_3) *$I < \infty$.*

Proof. $\mathrm{i}_1) \Longrightarrow \mathrm{i}_2)$. If $\int_{t_0}^\infty \frac{1}{a(t)}\, dt < \infty$, then $I < \infty$ and Lemma 3.2 yields $J_\mu^C < \infty$ for any $\mu > 0$ satisfying (25). So, the assertion follows from Theorem 3.1. If $\int_{t_0}^\infty \frac{1}{a(t)}\, dt = \infty$, then $I < \infty$, as it follows by applying to the equation (24), for instance, [7, Theorem 4.2-i_1)] or [13, Theorem 2.2]. Hence, using the same argument, the assertion again follows.

$\mathrm{i}_2) \Longrightarrow \mathrm{i}_3)$. From Proposition 2.2 we have $J_\mu^C < \infty$ for a sufficiently small constant $\mu > 0$. So, in view of Lemma 3.2, the assertion follows.

$\mathrm{i}_3) \Longrightarrow \mathrm{i}_1)$. The assertion follows by applying Theorem 3.1, which, as claimed, holds also when $\sigma = \infty$.

4 Asymptotic Estimates

When Φ is the classical Φ-Laplacian, for two bounded nonoscillatory solutions x, y, such that $\lim_{t\to\infty} x(t) = \ell_x \neq 0$, $\lim_{t\to\infty} y(t) = \ell_y \neq 0$ we have that the limit

$$\lim_{t\to\infty} \frac{x(t)-\ell_x}{y(t)-\ell_y} = \lim_{t\to\infty} \frac{x'(t)}{y'(t)}$$

is finite and different from zero. Roughly speaking, all bounded nonoscillatory solutions of (1) with the classical Φ-Laplacian have an equivalent growth at infinity. This fact can fail for a general Φ, with Im Φ bounded, as the following example illustrates.

Example 4.1. *Consider the equation*

$$\left(\frac{1}{t}\Phi(x')\right)' + \frac{\log t - 1}{(t\log t)^2}x = 0 \quad (t \geq 3)\,, \tag{27}$$

where $\Phi\colon \mathbb{R} \to (-1,1)$ *is a continuous odd function such that*

$$\Phi(u) = -(\log u)^{-1} \quad \mathit{if} \quad 0 < u < 1/e\,. \tag{28}$$

Let us show that (27) *has two bounded solutions such that*

$$\lim_{t\to\infty} \frac{x'(t)}{y'(t)} = \infty\,. \tag{29}$$

We have

$$\Phi^*(w) = e^{-1/w} \quad \mathit{if} \quad 0 < w < 1\,.$$

Because

$$\frac{1}{a(t)}\int_t^\infty b(s)\,ds = -t\int_t^\infty \frac{d}{ds}\left(\frac{1}{s\log s}\right)ds = \frac{1}{\log t}\,,$$

we get for $\lambda \in (0,1]$

$$\Phi^*\left(\lambda\,\frac{1}{a(t)}\int_t^\infty b(s)\,ds\right)dt = \frac{1}{t^{1/\lambda}}\,. \tag{30}$$

Taking into account that

$$\sup_{t\geq 3} \frac{1}{a(t)}\int_t^\infty b(s)\,ds < 1\,,$$

in virtue of Theorem 3.1 and its proof, there exist two bounded nonoscillatory solutions x, y *of* (27) *such that* $\lim_{t\to\infty} x(t) = 2^{-1}$, $\lim_{t\to\infty} y(t) = 8^{-1}$. *The l'Hopital rule yields*

$$\lim_{t\to\infty} \frac{\Phi(x'(t))}{(\log t)^{-1}} = \frac{1}{2}, \quad \lim_{t\to\infty} \frac{\Phi(y'(t))}{(\log t)^{-1}} = \frac{1}{8}\,.$$

Hence there exists $T \geq t_0$ *such* $x'(t) > t^{-3}, y'(t) < t^{-4}$ *for* $t > T$. *Then* $x'(t) > ty'(t)$ *and* (29) *follows.*

A sufficient condition, in order that bounded nonoscillatory solutions have an equivalent growth at infinity, is given by the following.

Corollary 4.1. *Assume* (Hp), (10) *and that*

$$\lim_{u\to 0+}\frac{\Phi(u)}{u^{\alpha}}=d\,,\quad 0<d<\infty\,, \tag{31}$$

for some $\alpha>0$. *If* x, y *are two bounded nonoscillatory solutions of* (1) *such that*

$$\lim_{t\to\infty}x(t)=c_x\,,\quad \lim_{t\to\infty}y(t)=c_y\,,\quad 0<c_x,\ c_y<\infty\,, \tag{32}$$

then the limit

$$\lim_{t\to\infty}\frac{x(t)-c_x}{y(t)-c_y}$$

is finite and different from zero. Moreover, any bounded nonoscillatory solution x *of* (1) *satisfies*

$$x'(t)=O\left(\left(\frac{1}{a(t)}\int_t^{\infty}b(s)\,ds\right)^{1/\alpha}\right)\quad as\quad t\to\infty\,. \tag{33}$$

Proof. Without loss of generality, let x, $x^{[1]}$, y and $y^{[1]}$ be positive for $t\ge T\ge t_0$. Hence, the l'Hopital rule gives

$$\lim_{t\to\infty}\frac{\Phi(x'(t))}{\Phi(y'(t))}=\lim_{t\to\infty}\frac{x^{[1]}(t)}{y^{[1]}(t)}=\frac{F(c_x)}{F(c_y)}\,.$$

In virtue of Theorem 3.1 we have $\lim_{t\to\infty}x'(t)=0,\quad \lim_{t\to\infty}y'(t)=0$. Thus

$$\frac{\Phi(x'(t))}{\Phi(y'(t))}=\frac{\Phi(x'(t))}{(x'(t))^{\alpha}}\frac{(y'(t))^{\alpha}}{\Phi(y'(t))}\left(\frac{x'(t)}{y'(t)}\right)^{\alpha}$$

which implies that the limit

$$\lim_{t\to\infty}\frac{x'(t)}{y'(t)}$$

is finite and different from zero and the first assertion follows.

Finally, let x be a solution of (1) such that $\lim_{t\to\infty}x(t)=\ell_x, 0<\ell_x<\infty$. In view of Proposition 2.1-i$_4$), we have $\lim_{t\to\infty}x'(t)=0$ and, from (31),

$$\lim_{t\to\infty}\frac{\Phi(x'(t))}{(x'(t))^{\alpha}}=d\,.$$

Since

$$\frac{a(t)(x'(t))^{\alpha}}{\int_t^{\infty}b(s)ds}=\frac{(x'(t))^{\alpha}}{\Phi(x'(t))}\frac{x^{[1]}(t)}{\int_t^{\infty}b(s)\,ds}\,,$$

by using Lemma 2.1 and the l'Hopital rule, we obtain (33).

It follows from the proof of Corollary 4.1 that bounded solutions for equations with the map Φ_C have the same growth as that ones with Sturm-Liouville operator.

Corollary 4.2. *Assume* (10) *and* (26)*. If x is a bounded nonoscillatory solution of* (4) *and z is a bounded nonoscillatory solution of* (24) *such that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t)$, then*

$$x'(t) - z'(t) = o\left(\frac{1}{a(t)}\int_t^\infty b(s)ds\right) \quad as \quad t \to \infty\,.$$

Proof. Without loss of generality, suppose x, x', z, z' are positive for $t \geq T \geq t_0$. In virtue of Proposition 2.1-i$_4$), we have $\lim_{t\to\infty} x'(t) = 0$. Because the same argument holds for (24), we have also $\lim_{t\to\infty} z'(t) = 0$. Moreover, since $a(t)z'(t)$ is positive decreasing for $t \geq T$ and (26) holds, we get

$$\lim_{t\to\infty} a(t)z'(t) = 0\,.$$

From the equality

$$\frac{x'(t) - z'(t)}{a^{-1}(t)\int_t^\infty b(s)\,ds} = \frac{x'(t)}{\Phi_C(x'(t))}\frac{x^{[1]}(t)}{\int_t^\infty b(s)ds} - \frac{a(t)z'(t)}{\int_t^\infty b(s)\,ds}\,,$$

taking into account Lemma 2.1, by using the l'Hopital rule, the assertion follows.

5 Unbounded Solutions

In this section we study the existence of solutions of (1) satisfying the asymptotic conditions (3).

Theorem 5.1. *Let* (Hp) *be satisfied. Assume there exists k, $0 < \Phi(k) < \sigma$, such that*

$$\int_{t_0}^\infty b(s)F(ks)\,ds < \infty \tag{34}$$

and

$$\lim_{t\to\infty} \frac{1}{a(t)}\int_t^\infty b(s)F(ks)\,ds = 0\,. \tag{35}$$

If there exists a positive constant $\overline{\mu} \in \operatorname{Im} F$ such that

$$J_{\overline{\mu}} = \int_{t_0}^\infty \Phi^*\left(\overline{\mu}\,\frac{1}{a(t)}\int_t^\infty b(s)\,ds\right)\,dt = \infty\,, \tag{36}$$

then the asymptotic problem (1)–(3) *is solvable.*

Proof. Let L be such that

$$F(L) = \overline{\mu}\,. \tag{37}$$

In virtue of (35), we can choose $t_1 > 0$ so large that

$$\sup_{t\geq t_1} \frac{1}{a(t)}\int_t^\infty b(s)F(ks)\,ds \leq \Phi(k)\,, \quad kt_1 > L\,. \tag{38}$$

Now, as in the proof of Theorem 3.1, denote by $C[t_1,\infty)$ the Fréchet space of all continuous functions on $[t_1,\infty)$ endowed with the topology of uniform convergence on compact subintervals of $[t_1,\infty)$ and consider the set $\Omega \subset C[t_1,\infty)$ given by

$$\Omega = \{u \in C[t_1,\infty) : L \le u(t) \le kt\}\ .$$

Define in Ω the operator T as follows

$$T(u)(t) = L + \int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\int_s^\infty b(\tau)F(u(\tau))\,d\tau\right) ds\,.$$

In view of (38), we have

$$T(u)(t) \le L + \int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\int_s^\infty b(\tau)F(k\tau)\,d\tau\right) \le L + k(t-t_1) \le kt.$$

Obviously, $T(u)(t) \ge L$ and so T maps Ω into itself. Reasoning as in the proof of Theorem 3.1 and applying the Tychonov fixed point theorem, there exists a solution x of the integral equation

$$x(t) = L + \int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\int_s^\infty b(\tau)F(x(\tau))\,d\tau\right) ds.$$

Clearly, x is a solution of (1). Because

$$\Phi(x'(t)) = \frac{1}{a(t)}\int_t^\infty b(\tau)F(x(\tau))\,d\tau \le \frac{1}{a(t)}\int_t^\infty b(\tau)F(k\tau)\,d\tau\,,$$

in virtue of (35), we obtain $\lim\limits_{t\to\infty} x'(t) = 0$. Moreover, from (37) we have

$$\int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\int_s^\infty b(\tau)F(x(\tau))\,d\tau\right) ds \ge \int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\int_s^\infty b(\tau)F(L)\,d\tau\right) ds = \int_{t_1}^{t} \Phi^*\left(\frac{\overline{\mu}}{a(s)}\int_s^\infty b(\tau)\,d\tau\right)$$

and so (36) yields $\lim\limits_{t\to\infty} x(t) = \infty$.

Remark 5.1. *Theorem 5.1 holds for a general map Φ, bounded or unbounded, and complements similar results stated in [7, Theorem 3.1], [8, Theorem 3.3], and [9, Theorem 1].*

The following example illustrates Theorem 5.1.

Example 5.1. *Consider the equation*

$$\left(t^{-3}\Phi_C(x')\right)' + t^{-5}\sqrt{|x|}\ \operatorname{sgn} x = 0\,, \quad (t \ge 1)\,.$$

Because all the assumptions of Theorem 5.1 are satisfied for any $k > 0$ and $\overline{\mu} > 0$, this equation has unbounded nonoscillatory solutions x such that $\lim\limits_{t\to\infty} x'(t) = 0$.

Remark 5.2. *When* (8) *holds and* $\operatorname{Im}\Phi$ *is bounded, the existence of nonoscillatory solutions* x *of* (1) *such that* $\lim_{t\to\infty} x^{[1]}(t) = 0$ *has been obtained in [9] as limit of a sequence* $\{z_n\}$*, where* z_n *are solutions of* (1) *such that* $\lim_{t\to\infty} z_n^{[1]}(t) > 0$*. Hence, in virtue of Lemma 2.1, the argument used in [9, Theorem 1] cannot be adapted to the case here considered.*

In the next theorem, we give an asymptotic estimate for unbounded solutions of (1).

Theorem 5.2. *Let* (Hp) *be satisfied. Assume that for some* $\alpha > 0$ *the function* Φ *satisfies* (31) *and the function* F *satisfies* (35) *and*

$$\lim_{u\to\infty} \frac{F(u)}{u^\alpha} = k\,, \quad 0 < k < \infty\,. \tag{39}$$

If x *is a solution of the BVP* (1)–(3)*, then*

$$x'(t) = o\left(\left(\frac{1}{a(t)}\int_t^\infty b(s)F(ks)\,ds\right)^{1/\alpha}\right) \quad as \quad t\to\infty\,.$$

Proof. We have by the l'Hopital rule

$$\lim_{t\to\infty} \frac{x(t)}{kt} = 0\,.$$

Proceeding as in the proof of Corollary 4.1, we have

$$\frac{a(t)(x'(t))^\alpha}{\int_t^\infty b(s)F(ks)\,ds} = \frac{(x'(t))^\alpha}{\Phi(x'(t))}\frac{x^{[1]}(t)}{\int_t^\infty b(s)F(ks)\,ds}$$

and by (39)

$$\lim_{t\to\infty} \frac{F(x(t))}{F(kt)} = \lim_{t\to\infty} \frac{F(x(t)}{x^\alpha(t)}\frac{(kt)^\alpha}{F(kt)}\frac{(x(t)^\alpha}{(kt)^\alpha} = 0\,.$$

The assertion follows by the l'Hopital rule.

6 Coexistence Result

From Theorems 3.1 and 5.1 we have the following coexistence result.

Corollary 6.1. *Let* (Hp) *be satisfied. Assume there exists* k*,* $0 < \Phi(k) < \sigma$*, such that* (34) *and* (35) *hold. If there exist two positive constants* λ *and* μ*,* $\lambda < \mu \in \operatorname{Im} F$ *such that* $J_\lambda < \infty$ *and* $J_\mu = \infty$*, then* (1) *has both bounded and unbounded nonoscillatory solutions* x *such that* $\lim_{t\to\infty} x'(t) = 0$*.*

The following example illustrates Corollary 6.1.

Example 6.1. *Consider the equation*

$$\left(\frac{1}{t}\Phi(x')\right)' + \frac{\log t - 1}{(t\log t)^2}F(x) = 0, \quad (t \geq 3)\,, \tag{40}$$

where $\Phi\colon \mathbb{R} \to (-1,1)$ *is, as in Example 4.1, a continuous odd function defined by* (28) *and* F *is a continuous odd function such that*

$$F(u) = \frac{\log u}{\log\log u} \quad on \quad [9,\infty)\,.$$

We have for $t \geq 9$

$$\begin{aligned} b(t)F(t) &\leq \frac{\log t - 1}{(t\log t)^2}\left(\frac{\log t}{\log(\log t)} + \frac{1}{(\log(\log t))^2}\right) \\ &\leq \frac{1}{t^2\log(\log t)} + \frac{1}{t^2\log t(\log(\log t))^2} = -\frac{d}{dt}\frac{1}{t(\log(\log t))}\,. \end{aligned}$$

Thus (34), (35) *are verified with* $k = 1$. *Reasoning as in Example4.1, condition* (30) *holds for* $\lambda \in (0,1]$. *Hence* $J_{1/2} < \infty$ *and* $J_1 = \infty$ *and from Corollary 6.1, equation* (40) *has both bounded and unbounded solutions* x *such that* $\lim_{t\to\infty} x'(t) = 0$

Example 6.1 also shows that the convergence of the integral J_μ can depend on the values of the parameter μ. In view of Lemma 3.2, for the map Φ_C^* this fact does not occur when (15) holds. Because (35) implies (15), Corollary 6.1 cannot be applied to equation (4).

7 Open Problems and Suggestions

(1) Asymptotic estimations for bounded solutions. Does (33) hold for any bounded nonoscillatory solution, x, by assuming, instead of (31), that Φ is asymptotically homogeneous near zero, i.e.

$$\lim_{u\to 0}\frac{\Phi(\lambda u)}{\Phi(u)} = \lambda^\alpha \quad \text{for} \quad \lambda \in (0,1] \quad \text{and some} \quad \alpha > 0\ ? \tag{41}$$

Condition (41) means that Φ is a regularly varying function at zero. This notion, and the analogous one at infinity, are often used both in searching for radial solutions of elliptic problems and in asymptotic theory of ordinary differential equations, see, e.g., [6, 12] and references therein.

(2) The growth of solutions. When (Hp) and $\limsup_{t\to\infty} a(t) > 0$ hold, then, in virtue of Lemma 2.1, equation (1) does not have unbounded solutions x such that $\lim_{t\to\infty} x'(t) = \ell_x$, $0 < \ell_x \leq \infty$. Nevertheless, when

$$\lim_{t\to\infty} a(t) = 0\,,$$

equation (1) can have unbounded solutions x such that $\lim_{t\to\infty} x'(t) = \ell_x$ with $0 < \ell_x \le \infty$, as the following example shows.

Example 7.1. *Consider the equation*

$$\left(\frac{\sqrt{1+t^2}}{t^2}\Phi_C(x')\right)' + \frac{8}{t^8}x^3 = 0 \quad (t \ge 1)\,, \tag{42}$$

A direct calculation shows that $x(t) = 2^{-1}t^2$ *is a solution of* (42). *Observe that the conditions* (34) *and* (35) *are verified, while* $J^C_\mu < \infty$ *for any small* μ. *In addition, from Theorem 3.1, equation* (42) *has also nonoscillatory bounded solutions.*

It should be interesting to give criteria for the existence of unbounded solutions x of (1) satisfying the boundary condition $\lim_{t\to\infty} x'(t) = \ell_x$, $0 < \ell_x \le \infty$.

(3) Coexistence result. When the convergence of the integral J_μ does not depend on μ, the coexistence result stated in Corollary 6.1 cannot be applied. If J_μ diverges for μ in a neighboorhod of zero, then, by Proposition 2.2, bounded nonoscillatory solutions of (1) do not exist.

When J_μ converges for any $\mu > 0$, Example 7.1 illustrates that bounded and unbounded nonoscillatory solutions of (1) can coexist. It is an open problem if in this case always bounded and unbounded solutions satisfying (11) coexist.

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