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UNIQUENESS IMPLIES EXISTENCE OF SOLUTIONS FOR NONLINEAR $(k; j)$ POINT BOUNDARY VALUE PROBLEMS FOR NTH ORDER DIFFERENTIAL EQUATIONS

Jeffrey Ehme

Department of Mathematics, Spelman College
Atlanta, GA 30314 USA
e-mail: jehme@spelman.edu

Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

Given appropriate growth conditions for f and a uniqueness assumption on $y^{(n)} = 0$ with respect to certain $(k; j)$ point boundary value problems, it is shown that uniqueness of solutions to the nonlinear differential equation

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}),$$

subject to nonlinear $(k; j)$ boundary conditions of the form

$$g_{ij}(y(t_j), \dots, y^{(n-1)}(t_j)) = y_{ij},$$

implies existence of solutions.

Key words and phrases: $(k; j)$ point boundary conditions, nonlinear, continuous dependence, uniqueness, existence.
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1 Introduction

In this paper, we will consider the differential equation, $n \geq 3$,

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}), \quad a < x < b. \tag{1}$$

We begin by defining an apppropriate linear point boundary condition. Given $1 \leq j \leq n-1$ and $1 \leq k \leq n-j$, positive integers m_1, ..., m_k such that $m_1 + \cdots + m_k = n - j$, points $a < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_{k+j} < b$, real numbers y_{il}, $1 \leq i \leq m_l$, $1 \leq l \leq k$, and real numbers y_{n-j+l}, $1 \leq l \leq j$, a boundary condition of the form

$$\begin{aligned} &y^{(i-1)}(x_l) = y_{il}, \quad 1 \leq i \leq m, \quad 1 \leq l \leq k, \\ &y'(x_{k+l}) = y_{n-j+l}, \quad 1 \leq l \leq j, \end{aligned} \tag{2}$$

will be referred to as a *linear* $(k;j)$ *point boundary condition*, and a condition of the form

$$\begin{aligned} g_{il}(y(t_l),\ldots,y^{(n-1)}(t_l)) &= y_{il}, \quad 1\le i\le m, \quad 1\le l\le k, \\ g_{kl}(y(t_{k+l}),\ldots,y^{(n-1)}(t_{k+l})) &= y_{n-j+l}, \quad 1\le l\le j, \end{aligned} \tag{3}$$

will be referred to as a *nonlinear* $(k;j)$ *point boundary condition.* If

$$g_{il}(x_0,x_1,\ldots,x_{n-1}) = x_{i-1}$$

for $1\le i\le m$, $1\le l\le k$, and if

$$g_{kl}(x_0,x_1,\ldots,x_{n-1}) = x_1$$

$1\le l\le j$, then the nonlinear $(k;j)$ boundary conditions become linear $(k;j)$ boundary conditions. Our interest in these boundary conditions is stimulated by recent work by Eloe and Henderson in [6].

Moreover, it will be assumed that for $x_p\ge 0$, $x_p\le g_{pl}(x_0,x_1,\ldots,x_{n-1})$, and for $x_p<0$, $g_{pl}(x_0,x_1,\ldots,x_{n-1})\le x_p$, for all p and l. We will refer to this property by saying the nonlinear $(k;j)$ condition dominates the linear $(k;j)$ condition. Let $(x_0,\ldots,x_{n-1})$ denote a vector in R^n. If $h:R\to R$ is an odd continuous function and $k:R^n\to R$ is any continuous positive function, then $g_{il}(x_0,\ldots,x_{n-1}) = x_{i-1}+h(x_{i-1})k(x_0,\ldots,x_{n-1})$, $1\le i\le m$, $1\le l\le k$ and $g_{kl}(x_0,\ldots,x_{n-1}) = x_1+h(x_1)k(x_0,\ldots,x_{n-1})$, $1\le l\le j$, is an example of such a boundary condition satisfying the property.

Our goal will be to establish uniqueness implies existence results for (1), (2) and (1), (3). Our standing assumptions for this paper are the following.

(A1) $f:[a,b]\times R^n\to R$ is continuous.

(A2) Solutions of initial value problems for (1) are unique and extend across the interval $[a,b]$.

Later, in the statements of our main theorems, growth conditions will be placed on the function f.

Results concerning uniqueness implies existence have been considered by authors for many types of boundary conditions. For several examples of these types of arguments as applied to conjugate, focal, or Lidstone problems, see [2]-[9] and the references cited within. Eloe and Henderson obtained existence and uniqueness results for non-local boundary value problems in [5]. Some papers in which nonlinear boundary conditions have also been studied include Abadi and Thompson [1] and Thompson's studies of fully nonlinear problems in [14]-[16]. Schrader [17] and Ehme, Eloe, and Henderson [4] considered various problems with non-linear boundary conditions. In this paper, we will establish uniqueness implies existence results for nonlinear linear $(k;j)$ problems. These new results will yield as special cases the linear $(k;j)$ problems in [6].

2 Preliminary Results

By ordering the linear $(k;j)$ boundary conditions (2) in some order, we may denote these conditions by $L_i(y)$, where $1 \le i \le n$. Likewise, we can label the nonlinear $(k;j)$ boundary conditions (3) as $S_i(y)$ by using the same ordering as in the linear conditions. We obtain $|L_i(y)| \le |S_i(y)|$, for all i, because we have assumed the nonlinear condition dominates the linear condition.

Next, we define $\varphi : R^n \to R^n$ by

$$\varphi(c_0, c_1, \dots, c_{n-1}) = (S_1(y(t, c_0, \dots, c_{n-1})), , \dots, S_n(y(t, c_0, \dots, c_{n-1}))), \tag{4}$$

where $y(t, c_0, \dots, c_{n-1})$ denotes the unique solution of the initial value problem

$$\begin{aligned} &y^{(n)} = f(t, y, y', \dots, y^{(n-1)}), \\ &y^{(i)}(t_0) = c_i, \ 0 \le i \le n-1. \end{aligned}$$

and t_0 is a fixed point in (a, b).

The following representation theorem will be indispensable. For completeness, we state and prove this result here.

Theorem 2.1 (**Representation Theorem**). *Let $u(t) \in C^{(n)}[a, b]$. Assume solutions of $y^{(n)} = 0$ satisfying $L_i(y) = 0, \ i = 1, \dots, n$, are unique when they exist. Then*

$$u(t) = L_1(u)p_1(t) + L_2(u)p_2(t) + \cdots + L_n(u)p_n(t) + \int_a^b G(t, s)u^{(n)}(s) \ ds,$$

where p_j is a polynomial of degree less than or equal to $n-1$, $L_i(p_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta, and $G(t, s)$ is the Green's function for $y^{(n)} = 0$ satisfying $L_i(y) = 0$.

Proof. Let p_j denote the unique solution of

$$\begin{cases} y^{(n)} = 0, \\ L_i(y) = \delta_{ij}, \ \text{for } i = 1, \dots, n. \end{cases}$$

The existence of the Green's function implies such p_j's exist. Clearly, p_j is a polynomial with a degree at most $n-1$, and $L_i(p_j) = \delta_{ij}$. Next, let

$$w(t) = u(t) - L_1(u)p_1(t) - L_2(u)p_2(t) - \cdots - L_n(u)p_n(t) - \int_a^b G(t, s)u^{(n)}(s) \ ds.$$

Then $w^{(n)}(t) = u^{(n)}(t) - L_1(u) \cdot 0 - \cdots - L_n(u) \cdot 0 - u^{(n)}(t) = 0$. Thus, using the fact that

$$L_i\left(\int_a^b G(t, s)u^{(n)}(s)ds\right) = 0,$$

we obtain

$$\begin{aligned} L_i(w) &= L_i(u) - L_1(u)L_i(p_1) - \cdots - L_n(u)L_i(p_n) - 0 \\ &= L_i(u) - L_i(u) \\ &= 0. \end{aligned}$$

By uniqueness of of solutions of $y^{(n)} = 0, L_i(y) = 0$, it follows that $w(t) = 0$. This completes the proof.

We next establish that solutions depend continuously on the boundary values.

Theorem 2.2 (Continuous Dependence). *Assume (A1) and (A2), and suppose solutions to (1), (3) are unique when they exist. Then, given a solution y_0 of (1), (3), and $\varepsilon > 0$, there exists $\delta > 0$ such that if $|S_i(y_0) - r_i| < \delta$, $i = 1, \ldots, n$, then there exists a solution z of (1) such that $S_i(z) = r_i$, for $i = 1, \ldots, n$, and $|y_0^{(i)}(t) - z^{(i)}(t)| < \varepsilon$, $i = 0, \ldots, n-1$.*

Proof. Let φ be defined as in (4). If $\varphi(\vec{c}_1) = \varphi(\vec{c}_2)$, then this implies $S_i(y(t, \vec{c}_1)) = S_i(y(t, \vec{c}_2))$, $i = 1, 2, \ldots, n$. Our uniqueness assumption on the nonlinear boundary value problems implies that $y(t, \vec{c}_1) = y(t, \vec{c}_2)$, for all t, including t_0. But, this implies $\vec{c}_1 = \vec{c}_2$ because solutions of initial value problems are unique, and, hence, φ is one-to-one. The results now follows from the Brouwer Theorem on Invariance of Domain and the fact that solutions depend continuously on initial conditions.

We note that the previous theorem also establishes that φ is continuous. As the linear $(k; j)$ problems are special cases of the nonlinear $(k; j)$ problems, the above theorem also establishes continuous dependence for the linear $(k; j)$ problems.

Lemma 2.1. *If φ is onto, then solutions to (1), (3) exist for all choices of y_{il}.*

The proof of this lemma follows immediately from the definition of ϕ.

3 Main Results

We now in a position to state our first main theorem.

Theorem 3.1. *Assume (A1) and (A2), and solutions to (1), (3) are unique when they exist. Assume solutions to the linear problem $y^{(n)} = 0$, $L_i(y) = 0$, $i = 1, \ldots, n$, are unique when they exist. Also assume*

$$\frac{|f(t, y_1, \ldots, y_n)|}{[\max\{|y_1|, \ldots, |y_n|\}]^p} \leq M,$$

for some $M > 0$ and for all $(y_1, \ldots, y_n) \in R^n$ such that $|(y_1, \ldots, y_n)| > R$, for some $R > 0$, $0 < p < 1$. Then solutions to the nonlinear $(k; j)$ problem (1), (3) exist for all choices of boundary values.

Proof. Assume $\varphi : R^n \to R^n$ is defined as in (4). Theorem 1.1 implies the image of φ is open. The image of φ is clearly non-empty. To complete the proof we will show the image of φ is both open and closed which will imply the image is R^n, and hence the mapping φ is onto. By Lemma 1.3, we will obtain existence of solutions to (1), (3).

First, suppose there exists a sequence $\vec{c}_k \in R^n$ such that $\varphi(\vec{c}_k) \to y_0$. If $< \vec{c}_k >$ is bounded, then there exists a convergent subsequence $< \vec{c}_{k_l} >$. Suppose $\vec{c}_{k_l} \to \vec{c}_0$ for some $\vec{c}_0 \in R^n$. Then, the continuity of φ implies $\varphi(\vec{c}_{k_l}) \to \varphi(\vec{c}_0)$, which implies $y_0 = \varphi(\vec{c}_0)$. And, hence, the image of φ is closed.

We may now assume $\varphi(\vec{c}_k) \to y_0$ and $|\vec{c}_k| \to \infty$. Let $\vec{c}_k = (c_{0_k}, c_{1_k}, \ldots, c_{n-1_k})$. Since $y_n^{(i)}(t, \vec{c}_k)$ is continuous in t for $0 \le i \le n-1$, there exists $t_k \in [a,b]$ such that $|y^{(i_k)}(t_k, \vec{c}_k)| = \max\left\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\right\}$ for some $i_k \in \{0, \ldots, n-1\}$. Since $0 \le i_k \le n-1$ and each i_k is an integer, there exists a subsequence of the i_k that is constant. By choosing this subsequence, we may assume $|y^{(j)}(t_k, \vec{c}_k)| = \max \|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|$, for all k and some fixed j. Applying our Representation Theorem, we have

$$\begin{aligned} y^{(j)}(t_k, \vec{c}_k) &= L_1(y(t,\vec{c}_k))p_1^{(j)}(t_k) + \cdots + L_n(y(t,\vec{c}_k))p_n^{(j)}(t_k) \\ &+ \int_a^b \frac{\partial^{(j)} G}{\partial t^j}(t_k, s) f(s, y(s,\vec{c}_k), \ldots, y^{(n-1)}(s,\vec{c}_k))ds. \end{aligned}$$

This will then give us

$$\begin{aligned} \frac{y^{(j)}(t_k,\vec{c}_k)}{\left[\max\left\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\right\}\right]^p} &= \frac{L_1(y(t,\vec{c}_k))p_1^{(j)}(t_k)}{\left[\max\left\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\right\}\right]^p} \\ &+ \quad \vdots \\ &+ \frac{L_n(y(t,\vec{c}_k))p_n^{(j)}(t_k)}{\left[\max\left\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\right\}\right]^p} \end{aligned}$$

$$+ \int_a^b \frac{\partial^{(j)} G}{\partial t^j}(t_k, s) \frac{f(s, y(s,\vec{c}_k), \ldots, y^{(n-1)}(s,\vec{c}_k))}{\left[\max\left\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\right\}\right]^p} ds. \tag{5}$$

As

$$\begin{aligned} |y^{(j)}(t_k,\vec{c}_k)| &= \max\left\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\right\} \\ &\ge \max\{|c_{0_k}|, |c_{1_k}|, \ldots, |c_{n_k-1}|\} \to \infty \end{aligned}$$

and $p < 1$, we see that

$$\frac{|y^{(j)}(t_k,\vec{c}_k)|}{\left[\max\left\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\right\}\right]^p} = |y^{(j)}(t_k,\vec{c}_k)|^{1-p} \to \infty$$

We will obtain a contradiction by showing the right hand side of (5) is bounded. By assumption $\varphi(\vec{c}_k) \to y_0$ implies $< \varphi(\vec{c}_k) >$ is a bounded sequence. From the definition of $\varphi(\vec{c}_k)$, we obtain each of its component functions $S_i(y(t,\vec{c}_k))$ is bounded. By our dominance assumption, $|L_i(y(t,\vec{c}_k))| \le |S_i(y(t,\vec{c}_k))|$, we obtain that $L_i(y(t,\vec{c}_k))$ are all bounded.

Moreover,

$$\int_a^b \left|\frac{\partial^{(j)}G}{\partial t^j}(t_k,s)\right| \cdot \left|\frac{f(s,y(s,\vec{c}_k),\ldots,y^{(n-1)}(s,\vec{c}_k))}{[\max\{\|y(t,\vec{c}_k)\|,\ldots,\|y^{(n-1)}(t,\vec{c}_k)\|\}]^p}\right| ds \le \int_a^b \left|\frac{\partial^{(j)}G}{\partial t^j}(t_k,s)\right| M ds,$$

which is clearly bounded.This implies the right hand side of (5) is bounded, which is a contradiction. Hence, $|\vec{c}_k| \nrightarrow \infty$. This yields the image of φ is closed and the proof is complete.

If the g_{il} are actually linear $(k;j)$ boundary conditions, then we obtain the following special case of results recently proved by Eloe and Henderson in [6] using completely different techniques.

Theorem 3.2. *Assume (A1) and (A2), and solutions of (1), (2) are unique when they exist. Assume solutions of the linear problem $y^{(n)} = 0$ satisfying $L_i(y) = 0,\ i = 1,\ldots,n$, are unique when they exist. Also assume*

$$\frac{|f(t,y_1,\ldots,y_n)|}{[\max\{|y_1|,\ldots,|y_n|\}]^p} \le M,$$

for some $M > 0$ and for all $(y_1,\ldots,y_n) \in R^n$ such that $|(y_1,\ldots,y_n)| > R$, for some $R > 0$, $0 < p < 1$. Then solutions to the linear $(k;j)$ boundary problem (1), (2) exist for all choices of boundary values.

The hypotheses on f in Theorem 3.1 are satisfied by any bounded function f. In addition, any unbounded function f such that

$$|f(t,y_1,\ldots,y_n)| \le \alpha_1 y_1^{p_1} + \cdots + \alpha_n y_n^{p_n},$$

where $\alpha_i \ge 0,\ 0 < p_i < 1, i = 1,...,n$, for all $(y_1,\ldots,y_n) \in R^n$ such that $|(y_1,\ldots,y_n)| > R$, for some $R > 0$ will also satisfy the hypothesis of Theorem 3.1.

We will now consider the case when $p_i = 1,\ i = 1,...,n$. This case is satisfied by functions f that are bounded by linear functions. That is,

$$|f(t,y_1,\ldots,y_n)| \le \alpha_1|y_1| + \cdots + \alpha_n|y_n|,$$

for appropriate $\alpha_i \in R^+$ and for all $(y_1,\ldots,y_n) \in R^n$ such that $|(y_1,\ldots,y_n)| > R$, for some $R > 0$.

Theorem 3.3. *Assume (A1) and (A2), and solutions to (1), (3) are unique when they exist. Assume solutions of $y^{(n)} = 0$ satisfying $L_i(y) = 0,\ i = 1,\ldots,n$, are unique. Also assume there exists an α such that*

$$\frac{|f(t,y_1,\ldots,y_n)|}{\max\{|y_1|,\ldots,|y_n|\}} < \alpha < \frac{1}{\max_{l\in\{0,1,\ldots,n-1\}}\left\{\max_{t\in[a,b]}\left\{\int_a^b \frac{\partial^{(l)}G}{\partial t^l}(t,s)ds\right\}\right\}}$$

for all $(y_1, \ldots, y_n) \in R^n$ such that $|(y_1, \ldots, y_n)| > R$, for some $R > 0$. Then solutions to (1), (3) exist for all choices of boundary values.

Because the proof of Theorem 3.3 is similar to the proof of Theorem 3.1, we provide a sketch of the proof.

Sketch of Proof. Building on our work in the proof of Theorem 3.1, it suffices to consider the equality

$$\begin{aligned} y^{(j)}(t_k, \vec{c}_k) &= L_1(y(t,\vec{c}_k))p_1^{(j)}(t_k) + \cdots + L_n(y(t,\vec{c}_k))p_n^{(j)}(t_k) \\ &+ \int_a^b \frac{\partial^{(j)}G}{\partial t^j}(t_k, s) f(s, y(s,\vec{c}_k), \ldots, y^{(n-1)}(s,\vec{c}_k))ds, \end{aligned}$$

where $|y^{(j)}(t_k, \vec{c}_k)| = \max\left\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\right\}$, for all k, as in the proof of Theorem 3.1. As before, $|L_i(y(t,\vec{c}_k))| \le |S_i(y(t,\vec{c}_k))|$, for all i and all k, and hence the $|L_i(y(t,\vec{c}_k))|$ are bounded. Thus,

$$\begin{aligned} \frac{y^{(j)}(t_k,\vec{c}_k)}{\max\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\}} &= \frac{L_1(y(t,\vec{c}_k))p_1^{(j)}(t_k)}{\max\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\}} + \cdots \\ &+ \frac{L_n(y(t,\vec{c}_k))p_n^{(j)}(t_k)}{\max\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\}} \end{aligned}$$

$$+ \int_a^b \frac{\partial^{(j)}G}{\partial t^j}(t_k, s)\frac{f(s, y(s,\vec{c}_k), \ldots, y^{(n-1)}(s,\vec{c}_k))}{\max\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\}}ds. \tag{6}$$

Then, $\max\left\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\right\} \to \infty$, as $k \to \infty$, and as before,

$$\frac{L_i(y(t,\vec{c}_k))p_i^{(j)}(t_k)}{\max\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\}} \to 0.$$

By construction, $\left|\frac{y^{(j)}(t_k,\vec{c}_k)}{\max\left\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\right\}}\right| = 1$, for all k. However,

$$\begin{aligned} &\left|\int_a^b \frac{\partial^{(j)}G}{\partial t^j}(t_k, s)\frac{f(s, y(s,\vec{c}_k), \ldots, y^{(n-1)}(s,\vec{c}_k))}{\max\{\|y(t,\vec{c}_k)\|, \ldots, \|y^{(n-1)}(t,\vec{c}_k)\|\}}ds\right| \\ \le &\int_a^b \left|\frac{\partial^{(j)}G}{\partial t^j}(t_k, s)\right| \cdot \left|\frac{f(s, y(s,\vec{c}_k), \ldots, y^{(n-1)}(s,\vec{c}_k))}{\max\{|y(t,\vec{c}_k)|, \ldots, |y^{(n-1)}(t,\vec{c}_k)|\}}ds\right| \\ \le &\int_a^b \left|\frac{\partial^{(j)}G}{\partial t^j}(t_k, s)\right| \cdot \alpha \, ds < 1, \end{aligned}$$

from our assumption on α. We see the right hand side of (6) is eventually less than 1 in an absolute value, which is a contradiction.

Applying Theorem 3.3 to linear $(k;j)$ boundary value problems yields the following theorem.

Theorem 3.4. *Assume (A1) and (A2), and solutions to (1), (2) are unique when they exist. Assume solutions of* $y^{(n)} = 0$ *satisfying* $L_i(y) = 0,\ i = 1, \ldots, n$, *are unique. Also assume there exists an* α *such that*

$$\frac{|f(t, y_1, \ldots, y_n)|}{\max\{|y_1|, \ldots, |y_n|\}} < \alpha < \frac{1}{\max_{l \in \{0,1,\ldots,n-1\}} \left\{ \max_{t \in [a,b]} \left\{ \int_a^b \frac{\partial^{(l)} G}{\partial t^l}(t, s) ds \right\} \right\}}$$

for all $(y_1, \ldots, y_n) \in R^n$ *such that* $|(y_1, \ldots, y_n)| > R$, *for some* $R > 0$. *Then solutions to (1), (2) exist for all choices of boundary values.*

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