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ON THE CRITICAL VALUES OF PARAMETRIC RESONANCE IN MEISSNER'S EQUATION BY THE METHOD OF DIFFERENCE EQUATIONS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

In this paper the second order liner differential equation

$$\begin{cases} x'' + a^2(t)x = 0, \\ a(t) = \begin{cases} \pi + \varepsilon, & \text{if } 2nT \le t < 2nT + T_1, \\ \pi - \varepsilon, & \text{if } 2nT + T_1 \le t < 2nT + T_1 + T_2, \quad (n = 0, 1, 2, \dots), \end{cases} \end{cases}$$

is investigated, where $T_1 > 0$, $T_2 > 0$ ($T := (T_1 + T_2)/2$) and $\varepsilon \in [0, \pi)$. We say that a parametric resonance occurs in this equation if for every $\varepsilon > 0$ sufficiently small there are $T_1(\varepsilon)$, $T_2(\varepsilon)$ such that the equation has solutions with amplitudes tending to ∞, as $t \to \infty$. The period $2T_*$ of the parametric excitation is called a critical value of the parametric resonance if $T_* = T_1(\varepsilon) + T_2(\varepsilon)$ with some T_1, T_2 for all sufficiently small $\varepsilon > 0$. We give a new simple geometric proof for the fact that the critical values are the natural numbers. We apply our method also to find the most effective control destabilizing the equilibrium $x = 0$, $x' = 0$, and to give a sufficient condition for the parametric resonance in the asymmetric case $T_1 \ne T_2$.

Key words and phrases: Parametric resonance, Meissner's equation, impulsive differential equations, difference equations, problem of swinging.
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1 Introduction

Consider a system of differential equations

$$x' = f(t, x) \qquad (f(t, 0) \equiv 0), \tag{1}$$

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with a right-hand side periodic with respect to t, i.e., $f\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$, $f(t+P) \equiv f(t)$ for some $P>0$. It can happen that the zero solutions of equations $x' = f(t_0, x)$ are stable in Lyapunov's sense for every $t_0 \geq 0$, but the zero solution of equation (1) is unstable. This phenomenon is called *parametric resonance*. The conditions of this phenomenon can be studied [2, 5, 13, 15] by the method of period mapping (Poincaré mapping) $x(P; 0, \cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$, where $x(\cdot; 0, x_0)\colon [0,\infty) \to \mathbb{R}^n$ denotes the solution of (1) satisfying the initial condition $x(0; 0, x_0) = x_0$. If (1) is linear, then any fundamental matrix $X(t)$ of the system

$$x' = A(t)x \quad (A(t+P) \equiv A(t)) \tag{2}$$

has a representation of the form

$$X(t) = Y(t)e^{Rt} \qquad (Y(t+P) \equiv Y(t)),$$

where $R \in \mathbb{R}^{n\times n}$ is called the monodromy matrix to (2) (Floquet theory [5]). Stability properties of the zero solution of (2) (among them conditions of the parametric resonance) are determined by the constant monodromy matrix R. However, to have matrix R one has to generate a set of n linearly independent solutions of (2).

A special case of (2) is Hill's equation [9]

$$x'' + a^2(t)x = 0 \quad (x \in \mathbb{R};\ a(t+P) \equiv a(t)) \tag{3}$$

describing, e.g., the motion of the moon. The key step of generating two linearly independent solutions to (3) can be done by elementary functions in the case of Meissner's equation [2, 10, 12, 13], when the coefficient a in (3) is a piece-wise constant function. This type of equation plays an important role in technical applications [16]. It has been found out that parametric resonance is possible only at certain critical values of the period P.

In this paper, using a purely elementary approach independent of Floquet's theory, we deduce the critical values of parametric resonance for that general case of Meissner's equation when the lengths in the time of the two pieces in the coefficient a in (3) can be different. D. R. Merkin [13, p. 263], proved that the critical values are independent of the relation between these lengths. We show, however, that the realization of parametric resonance at these critical values does depend on the difference between the lengths, and we give a condition (in term of the difference) sufficient for the parametric resonance. We apply our approach to the problem of swinging to get an effective state-dependent control of the lengths of the pieces in the coefficient a.

2 The Method

If a is a piece-wise constant function then equation (3) has the following form. Given two sequences of positive numbers $\{a_k\}_{k=1}^{\infty}$, $\{t_k\}_{k=1}^{\infty}$, which will be denoted simply by $\{a_k\}$ and $\{t_k\}$, respectively, $(t_k \leq t_{k+1},\ k \in \mathbb{N})$, $t_0 := 0$, consider the equation

$$x'' + a^2(t)x = 0, \qquad a(t) := a_k \text{ if } t_{k-1} \leq t < t_k \quad (k \in \mathbb{N}). \tag{4}$$

Definition 2.1 *A function* $x : [0, \infty) \to \mathbb{R}$ *is called a* solution *of* (4) *if it is continuously differentiable on* $[0, \infty)$*, it is twice differentiable and solves the equation on every* $[t_{k-1}, t_k)$ *for* $k \in \mathbb{N}$.

At first we transform equation (4) into a system of impulsive differential equations, which will be replaced with a system of difference equations. The dynamics generated by this system can be described by elementary geometric transformations. This method was introduced in [6], and it was also used in [7, 8]; we reformulate it to make the present paper self-contained.

Introducing the new state variable $y := x'/a_k$ on the interval $[t_{k-1}, t_k)$, we can reexpress equation (4) in the form of a 2-dimensional system

$$x' = a_k y, \quad y' = -a_k x, \qquad \text{if } t_{k-1} \le t < t_k, \qquad (k \in \mathbb{N}). \tag{5}$$

Since we want to have a system of first order differential equations equivalent to (4), we ought to require additional "connectivity" conditions of solutions of (5) which follow from Definition 2.1: a function $t \mapsto (x(t), y(t))$ is a solution of (5) on $[0, \infty)$ if $t \mapsto (x(t), a(t)y(t))$ is continuous on $[0, \infty)$, and $t \mapsto (x(t), y(t))$ is differentiable and solves system (5) on $[t_{k-1}, t_k)$ for $k \in \mathbb{N}$.

Since the function $t \mapsto x'(t) = a(t)y(t)$ has to be continuous on $[0, \infty)$, the function $t \mapsto y(t)$ is right-continuous for all $t \ge 0$ and satisfies $a_k y(t_k - 0) = a_{k+1} y(t_k)$ for $k \in \mathbb{N}$, where $y(t_k - 0)$ denotes the left-hand side limit of y at t_k. Accordingly, the system of first order differential equations equivalent to (4) is

$$\begin{cases} x' = a_k y, \ y' = -a_k x, & \text{if } t_{k-1} \le t < t_k, \\ y(t_k) = \dfrac{a_k}{a_{k+1}} y(t_k - 0), & (k \in \mathbb{N}). \end{cases} \tag{6}$$

This is a so-called impulsive differential equation (see [3, 4, 11, 14] and the references therein): the evolution of $(x(t), y(t))$ is governed by a differential equation for $t \ne t_k$ $(k \in \mathbb{N})$, and $y(t)$ makes jumps at $t = t_k$. Due to its special form, the impulsive differential equation (6) can be represented as a discrete dynamical system on the plane in the following way. Introducing the polar coordinates r, φ by the formulae $x = r\cos\varphi$, $y = r\sin\varphi$ $(r > 0,\ -\infty < \varphi < \infty)$, we can transform system (5) into

$$r' = 0, \quad \varphi' = -a_k, \qquad (t_{k-1} \le t < t_k,\ k \in \mathbb{N}).$$

Therefore, during the evolution governed by (6) the points of the plane revolve uniformly around the origin for $t \in [t_{k-1}, t_k)$, then a contraction of size a_k/a_{k+1} along the y-axis occurs at $t = t_k$. Now introduce the following notation:

$$\begin{aligned} &\tau_k := t_k - t_{k-1}, \quad \varphi_k := a_k \tau_k, \quad d_k := \frac{a_k}{a_{k+1}}, \\ &M_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ &M_k = M_k(d_k, \varphi_k) := \begin{pmatrix} 1 & 0 \\ 0 & d_k \end{pmatrix} \begin{pmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{pmatrix}, \qquad (k \in \mathbb{N}). \end{aligned} \tag{7}$$

Then from (6) we have

$$z_k := \begin{pmatrix} x(t_k) \\ y(t_k) \end{pmatrix} = M_k M_{k-1} \cdots M_1 M_0 \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \in \mathbb{R}^2, \qquad (k = 0, 1, 2, \ldots). \tag{8}$$

Since the discrete dynamical system (8) has the same stability properties as our original system (6), in the remaining part of the paper we shall investigate (8).

3 Parametric Resonance

For given $T_1 > 0$, $T_2 > 0$ ($T := (T_1 + T_2)/2$) and $\varepsilon \in [0, \pi)$ consider the second order differential equation

$$\begin{cases} x'' + a^2(t)x = 0, \\ a(t) = \begin{cases} \pi + \varepsilon, & \text{if } 2nT \le t < 2nT + T_1, \\ \pi - \varepsilon, & \text{if } 2nT + T_1 \le t < 2nT + T_1 + T_2, \quad (n = 0, 1, 2, \ldots), \end{cases} \end{cases} \tag{9}$$

which is a Meissner's equation.

Definition 3.1 *We say that a* parametric resonance *occurs in equation* (9) *if for arbitrarily small* $\varepsilon > 0$ *there exists a non-empty open set* $S(\varepsilon)$ *such that if* $2T \in S(\varepsilon)$*, then there are* $T_1 > 0$, $T_2 > 0$ *such that* $T_1 + T_2 = 2T$ *and equation* (9) *has solutions with the property*

$$\lim_{n\to\infty} (\max\{|x(t)| : 2nT \le t \le 2(n+1)T\}) = \infty,$$

i.e., solutions with amplitudes tending to ∞*, as* $t \to \infty$.

$C \subset [0, \infty)$ *is called the set of the* critical values *of the parametric resonance if* $2T \in C$ *implies* $2T \in S(\varepsilon)$ *for all sufficiently small* $\varepsilon > 0$.

It is known [13] that *the critical values of the parametric resonance are the multiples of the half of the own period of the harmonic oscillator obtained from* (9) *by the substitution* $\varepsilon = 0$*, i.e.,*

$$C = \bigcup_{m=1}^{\infty} \left(\bigcap_{0<\varepsilon<\frac{1}{m}} S(\varepsilon) \right) = \left\{ \left(\frac{1}{2} \frac{2\pi}{\pi} \right) j : j \in \mathbb{N} \right\} = \mathbb{N}.$$

Now we prove this assertion by the method described in Section 2. Equation (9) is a special case of (4). Setting

$$a_{2n-1} = \pi + \varepsilon, \ a_{2n} = \pi - \varepsilon, \qquad d_{2n-1} = \frac{\pi + \varepsilon}{\pi - \varepsilon}, \ d_{2n} = \frac{\pi - \varepsilon}{\pi + \varepsilon},$$

$$\tau_{2n-1} = T_1, \ \tau_{2n} = T_2 \qquad (n \in \mathbb{N}),$$

and introducing the notations

$$\varphi_1 := (\pi+\varepsilon)T_1,\ \varphi_2 := (\pi-\varepsilon)T_2, \qquad \kappa := \frac{\pi+\varepsilon}{\pi-\varepsilon}, \tag{10}$$

for matrices (7) we get

$$\begin{aligned} M_+ := M_{2n-1} &= \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix} \begin{pmatrix} \cos\varphi_1 & \sin\varphi_1 \\ -\sin\varphi_1 & \cos\varphi_1 \end{pmatrix}, \\ M_- := M_{2n} &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\kappa} \end{pmatrix} \begin{pmatrix} \cos\varphi_2 & \sin\varphi_2 \\ -\sin\varphi_2 & \cos\varphi_2 \end{pmatrix}, \quad (n\in\mathbb{N}). \end{aligned}$$

Instead of (8) we may study the system of difference equations

$$w_n := \begin{pmatrix} x(2nT) \\ y(2nT) \end{pmatrix} = (M_-M_+)^n \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}, \quad (n = 0,1,2,\ldots). \tag{11}$$

A simple computation shows that

$$\begin{aligned} &M(\varepsilon) := M_-M_+ \\ &= \begin{pmatrix} \cos\varphi_1\cos\varphi_2 - \kappa\sin\varphi_1\sin\varphi_2 & \sin\varphi_1\cos\varphi_2 + \kappa\cos\varphi_1\sin\varphi_2 \\ -\frac{1}{\kappa}\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2 & -\frac{1}{\kappa}\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2 \end{pmatrix}; \end{aligned}$$

i.e., (11) can be rewritten into the form

$$w_n = \begin{pmatrix} x(2nT) \\ y(2nT) \end{pmatrix} = M(\varepsilon)^n \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}, \quad (n = 0,1,2,\ldots). \tag{12}$$

Since $\det(M(\varepsilon)) = \det(M_+)\det(M_-) = 1$, the eigenvalues λ_1, λ_2 of $M(\varepsilon)$ are determined by the equation

$$\lambda^2 - \mathrm{Trace}(M(\varepsilon))\lambda + 1 = 0.$$

If λ_1, λ_2 are not real then they are located on the unit circle of the complex plane, so the trivial solution of (11) is stable. Therefore,

$$S(\varepsilon) \subset D(\varepsilon) := \{2T > 0 : |\mathrm{Trace}(M(\varepsilon))| \geqq 2 \quad \text{for some} \quad T_1, T_2\}.$$

For the trace of $M(\varepsilon)$ we have

$$\begin{aligned} \mathrm{Trace}(M(\varepsilon)) &= 2\cos\varphi_1\cos\varphi_2 - \left(\kappa + \frac{1}{\kappa}\right)\sin\varphi_1\sin\varphi_2 \\ &= \cos(\varphi_1+\varphi_2)\left\{2 + \left(\kappa + \frac{1}{\kappa} - 2\right)\frac{1}{2}\right\} - \cos(\varphi_1-\varphi_2)\left(\kappa + \frac{1}{\kappa} - 2\right)\frac{1}{2}, \end{aligned} \tag{13}$$

whence, introducing the notation $\gamma = \gamma(\varepsilon) = (\kappa + 1/\kappa - 2)/2$, and taking into account the identities

$$\varphi_1 + \varphi_2 = 2T\pi + (T_1 - T_2)\varepsilon, \quad \varphi_1 - \varphi_2 = 2T\varepsilon + (T_1 - T_2)\pi, \tag{14}$$

we get

$$D(\varepsilon) = \{2T > 0 : |[2+\gamma]\cos(2T\pi + (T_1 - T_2)\varepsilon) - \gamma\cos(2T\varepsilon + (T_1 - T_2)\pi)| \geqq 2 \text{ for some } T_1, T_2\}. \tag{15}$$

If $2T_*$ is a critical value of the parametric resonance, then $2T_* \in D(\varepsilon)$ for all $\varepsilon > 0$, so (15) with $\varepsilon \to 0+0$ and $\lim_{\varepsilon\to 0}\gamma(\varepsilon) = 0$ yields $|2\cos(2T_*\pi)| \geqq 2$, i.e., $2T_* \in \mathbb{N}$, which means that $C \subset \mathbb{N}$.

To prove the reversed relation it is enough to show that for every $j \in \mathbb{N}$ and sufficiently small $\varepsilon > 0$ there is a $\delta > 0$ such that $j + \tau \in S(\varepsilon)$ for $|\tau| < \delta$ with the choice $T_1 = T_2$. This will be implied by the inequality $|f(\tau)| > 2$, where f is defined by

$$f(\tau) := [2+\gamma]\cos((j+\tau)\pi) - \gamma\cos((j+\tau)\varepsilon) = [2+\gamma](-1)^j\cos\tau - \gamma\cos((j+\tau)\varepsilon)$$

(see (15)). Since

$$\lim_{\tau\to 0} f(\tau) = (-1)^j 2 + \gamma\left((-1)^j - \cos j\varepsilon\right)$$

and $\gamma > 0$, for $\varepsilon < \pi/(4j)$ we have $\left|\lim_{\tau\to 0} f(\tau)\right| > 2$, which implies the existence of the desired δ, and the proof is complete.

4 The Problem of Swinging

In the problem of swinging (see §25 in [1]) the swinger changes the height of their center of gravity periodically. Small oscillations of the equivalent mathematical pendulum are described by the equation

$$\varphi'' + \frac{g}{\ell(t)}\varphi = 0,$$

where $\ell(t)$ denotes the length of the thread, which is a step function; φ denotes the angle measured anticlockwise between the axis directed vertically downward and the thread; g is the constant of gravity. After choosing an appropriate unit of length, the equation of motion is of the form (9). The swinger (perhaps a child with small $\varepsilon > 0$) would like to destabilize the equilibrium position $x = 0$, so the problem is to find the critical values of parametric resonance. As was shown in the previous section, they are the natural numbers. The most natural choice for the swinger is the smallest critical value $2T_* = 1$, i.e., $T_* = 1/2$. This means that the period of the behaviour of the swinger is approximately equal to the half of the own period of the swing, which is in accordance with the practice of swinging.

4.1 Realistic and Most Effective Control

In Arnold's model [1, §25] only the case $T_1 = T_2$ is allowed. However, this assumption is not realistic because it is not the elapsed time t but the *position* $x(t)$ *of the swing* that the swinger is observing and is using to control the swing.

To make the situation clearer, let us suppose that the swinger has chosen $T_* = 1/2$. Without loss of the generality we may assume that $y(0) = 0$, i.e., $|x(0)| \neq 0$ is the first amplitude of the motion. If the swinger keeps the rule $T_1 = T_2 = 1/2$, then $\varphi_1 = (\pi + \varepsilon)/2 > \pi/2$ (see (10)), which means that the swinger changes the height of their center of gravity (makes the pendulum longer) first *after* crossing the equilibrium position $x = 0$. When the next change (shortening the pendulum) happens, then $\varphi(1) = \varphi_1 + \varphi_2 = (\pi + \varepsilon)/2 + (\pi - \varepsilon)/2 = \pi$, i.e., $x'(1) = 0$ and $|x(t)|$ has a maximum (in other words, $|x(1)|$ is the second amplitude). It is more realistic to assume that the swinger changes the height of their gravity *exactly* at the moments when $|x(t)|$ has extreme values. Then $\varphi_1 = \varphi_2 = \pi/2$, and

$$T_1 = \frac{\pi}{2(\pi + \varepsilon)} = \frac{1}{2}\frac{1}{1 + \frac{\varepsilon}{\pi}} < \frac{1}{2}\frac{1}{1 - \frac{\varepsilon}{\pi}} = \frac{\pi}{2(\pi - \varepsilon)} = T_2.$$

From (13) we get

$$\mathrm{Trace}(M(\varepsilon)) = -\left\{2 + \left(\kappa + \frac{1}{\kappa} - 2\right)\frac{1}{2}\right\} - \left(\kappa + \frac{1}{\kappa} - 2\right)\frac{1}{2} = -\left(\kappa + \frac{1}{\kappa}\right) < -2,$$

which implies parametric resonance. Actually, this obviously follows from the geometry of our method: when contractions happen, then $y = 0$, so the points $z_{2n} = (x(2nT), 0)$ of the trajectory remain on the circles during the contractions (see Figure 1. a)).

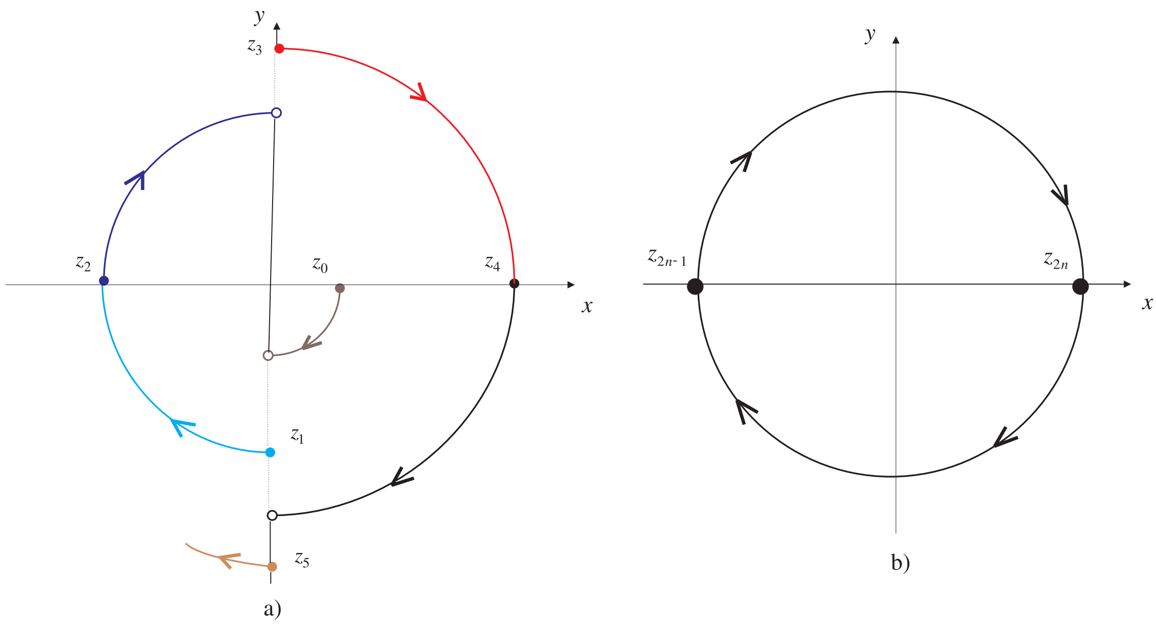

Figure 1: a) The most effective control; b) $T_* = 1$, $\varphi_1 = \varphi_2$: no parametric resonance

It is also clear that this is the most effective control. In addition, if A_n denotes the

n-th amplitude of the motion ($A_n = |x(2nT)|$), then

$$A_n = \left(\frac{\pi+\varepsilon}{\pi-\varepsilon}\right)^n, \quad (n = 0, 1, 2, \ldots),$$

because only the dilations have influence on the motion.

Now let us take the second critical value $2T_* = 2$, i.e., $T_* = 1$, and assume $\varphi_1 = \varphi_2 = \pi$. Then neither the dilations nor the contractions have influence on the motion with $y(0) = 0, x(0) \neq 0$, so the amplitudes are constant (see Figure 1. b)). This means that the choice

$$T_1 = \frac{\pi}{\pi+\varepsilon} = \frac{1}{1+\frac{\varepsilon}{\pi}} < 1, \quad T_2 = \frac{\pi}{\pi-\varepsilon} = \frac{1}{1-\frac{\varepsilon}{\pi}} > 1$$

does not result in instability.

4.2 Asymmetric Swinging

Finally suppose that the swinger can keep the rule $2T = 2T_*$ for a critical value of the parametric resonance, but T_1 and T_2 may be essentially different. We study how the occurrence of the instability depends on the difference $T_1 - T_2$.

Consider the practically most important case $T_* = 1/2$. By (14) we have

$$\varphi_1 + \varphi_2 = \pi + (T_1 - T_2)\varepsilon, \quad \varphi_1 - \varphi_2 = \varepsilon + (T_1 - T_2)\pi;$$

moreover,

$$\gamma = \left(\kappa + \frac{1}{\kappa} - 2\right)\frac{1}{2} = \frac{2\varepsilon^2}{\pi^2 - \varepsilon^2} = \frac{2}{\pi^2}\varepsilon^2\left(1 + \frac{\varepsilon^2}{\pi^2} + O(\varepsilon^4)\right), \quad (0 < \varepsilon \ll 1).$$

Consequently,

$$\begin{aligned} \mathrm{Trace}(M(\varepsilon)) &= [2+\gamma]\cos(\pi + (T_1 - T_2)\varepsilon) - \gamma\cos(\varepsilon + (T_1 - T_2)\varepsilon) \\ &= -[2+\gamma]\cos((T_1 - T_2)\varepsilon) - \gamma\left\{\cos\varepsilon\cos((T_1 - T_2)\pi) - \sin\varepsilon\sin((T_1 - T_2)\pi)\right\} \\ &= -2 + \left\{(T_1 - T_2)^2 - \frac{2}{\pi^2}(1 + \cos((T_1 - T_2)\pi))\right\}\varepsilon^2 + O(\varepsilon^3), \quad (0 < \varepsilon \ll 1). \end{aligned}$$

Since $1 + \cos 2\alpha = 2\cos^2\alpha$, the inequality $|\mathrm{Trace}(M(\varepsilon))| > 2$ will be satisfied for small $\varepsilon > 0$ if

$$\frac{\pi}{2}|T_1 - T_2| < \cos\left(\frac{\pi}{2}|T_1 - T_2|\right).$$

So we have proved the following theorem:

Theorem 4.1 *Let* $\alpha_*(\approx 0.739085)$ *denote the root of the equation* $\cos\alpha = \alpha$ $(\alpha \in \mathbb{R})$. *Then at the critical value* $2T_* = 1$, *i.e., at* $T_* = 1/2$ *there occurs a parametric resonance in equation* (9) *if*

$$2T = T_1 + T_2 = 1, \quad |T_1 - T_2| < \alpha_*. \tag{16}$$

In other words, for sufficiently small $\varepsilon > 0$, *conditions* (16) *imply the existence of solutions to equation* (9) *whose amplitudes tend to* ∞ *as* $t \to \infty$.

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