

A CANTILEVER EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We prove new results on the existence of positive solutions for some cantilever equation subject to nonlocal and nonlinear boundary conditions. Our main ingredient is the classical fixed point index.

Key words and phrases: Fixed point index, cone, positive solution, cantilever equation.

AMS (MOS) Subject Classifications: Primary 34B18, secondary 34B10, 47H10, 47H30

1 Introduction

In this paper we establish new results on the existence of positive solutions for the fourth order differential equation

$$u^{(4)}(t) = g(t)f(t, u(t)), \quad t \in (0, 1), \quad (1)$$

subject to the nonlocal boundary conditions (BCs)

$$u(0) = u'(0) = u''(1) = 0, \quad u'''(1) + k_0 + B(\alpha[u]) = 0. \quad (2)$$

Here k_0 is a non-negative constant, B is a non-negative continuous function and $\alpha[u]$ is a positive functional given by

$$\alpha[u] = \int_0^1 u(s) dA(s),$$

involving a Stieltjes integral.

Equation (1) models the stationary states of the deflection of an elastic beam. Beam equations have been studied recently by several authors under different boundary conditions. For example, the BCs

$$u(0) = 0, u(1) = 0, u'(0) = 0, u'(1) = 0,$$

correspond to both ends of the beam being clamped, these have been studied recently in [32, 37, 42, 43, 48]; the BCs

$$u(0) = 0, u(1) = 0, u''(0) = 0, u''(1) = 0,$$

model a bar with hinged ends and have been studied for example in [5, 7, 9, 12, 14, 19, 35, 36, 37, 42, 43, 46]. Also the BCs

$$u(0) = 0, u'(0) = 0, u(1) = 0, u''(1) = 0,$$

have been studied in [1, 42, 49] and correspond to the left end being clamped and the right end being hinged. The conditions

$$u(0) = 0, u'(0) = 0, u''(1) = 0, u'''(1) = 0$$

model the so-called cantilever bar, that is a bar clamped on the left end and where the right end is free to move with vanishing bending moment and shearing force. These types of BCs have been investigated in [4, 47] and, in particular, [4] provides a detailed insight on the physical motivation for this problem.

The BCs (2) describe a cantilever beam with forces acting on its right end, for example,

- $u'''(1) + k_0 = 0$ models a force acting in 1,
- $u'''(1) + k_1 u(1) = 0$ describes a spring in 1,
- $u'''(1) + B(u(1)) = 0$ models a spring with a strongly nonlinear rigidity (this could happen, for example, due to the type of material),
- $u'''(1) + B(u(\eta)) = 0$ describes a feedback mechanism, where the spring reacts to the displacement registered in a point η of the beam.

Thus the condition

$$u'''(1) + k_0 + B(\alpha[u]) = 0$$

covers a variety of cases and includes, as special cases when $k_0 = 0$ and $B(w) = w$, multi-point and integral boundary conditions, that are widely studied objects also in the case of fourth order BVPs, see for example [8, 11, 13, 15, 16, 18, 29, 30, 39]. BCs of nonlinear type are widely studied objects in the case of second and fourth order equations, see for example [3, 6, 10, 20, 21, 23, 25, 31, 45] and references therein. The

study of positive solutions of BVPs that involve Stieltjes integrals has been done, in the case of positive measures, in [27, 28]. *Signed* measures were used in [41] and in the subsequent papers [40, 42]; here, as in [23, 25], we are forced, due to some inequalities involved in our theory, to restrict our attention to positive measures only.

Our approach is to rewrite the BVP (1)-(2) as a perturbed Hammerstein integral equation of the form

$$u(t) = \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds.$$

This type of integral equation has been investigated, when $k_0 = 0$ and $B(w) \equiv w$, in [38, 41, 43, 44], when $B(w) \equiv w$ in [22, 24, 27], and when $k_0 = 0$ in [21, 23, 25, 45]. In order to utilize classical fixed point index theory, we provide here a modification of the results of [27, 41] in order to prove the existence of multiple positive solutions of the BVP (1)-(2).

In a last remark we briefly illustrate how our approach may also be applied to the nonlinear BCs

$$u(0) = k_0 + B(\alpha[u]), \quad u'(0) = 0, \quad u''(1) = 0, \quad u'''(1) = 0, \tag{3}$$

$$u(0) = 0, \quad u'(0) = k_0 + B(\alpha[u]), \quad u''(1) = 0, \quad u'''(1) = 0, \tag{4}$$

$$u(0) = 0, \quad u'(0) = 0, \quad u''(1) = k_0 + B(\alpha[u]), \quad u'''(1) = 0. \tag{5}$$

As in [42], where a different set of BCs were investigated, we point out that these nonlocal boundary conditions can be interpreted as feedback controls; in particular, the BCs (3) can be seen as a control on the displacement in the left end, the BCs (4) would be a device handling the angular attitude at the left end, whereas the BCs (5) describe a control on the bending moment at the right end.

2 Positive Solutions of the Fourth-Order BVPs

We firstly describe in details our approach for the BVP

$$u^{(4)}(t) = g(t)f(t, u(t)), \quad t \in (0, 1), \tag{6}$$

$$u(0) = u'(0) = u''(1) = 0, \quad u'''(1) + k_0 + B(\alpha[u]) = 0. \tag{7}$$

Throughout the paper we assume that:

- (C₁) $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ satisfies Carathéodory conditions, that is, for each u , $t \mapsto f(t, u)$ is measurable and for almost every t , $u \mapsto f(t, u)$ is continuous, and for every $r > 0$ there exists a L^∞ -function $\phi_r : [0, 1] \rightarrow [0, \infty)$ such that

$$f(t, u) \leq \phi_r(t) \quad \text{for almost all } t \in [0, 1] \text{ and all } u \in [0, r],$$

(C₂) $g \in L^1[0, 1]$, $g \geq 0$ for almost every $t \in [0, 1]$,

(C₃) A is a function of bounded variation and dA is a *positive* measure,

(C₄) $B : [0, \infty) \rightarrow [0, \infty)$ is continuous and there exist $\delta_1, \delta_2 \geq 0$ such that

$$\delta_1 w \leq B(w) \leq \delta_2 w \text{ for every } w \in [0, \infty).$$

The homogeneous BVP that corresponds to the BCs

$$u(0) = u'(0) = u''(1) = u'''(1) = 0,$$

has been studied in [4, 47], by means of an Hammerstein integral equation of the type

$$u(t) = \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds, \quad (8)$$

where G_0 is the Green's function associated with these BCs, that is

$$G_0(t, s) = \begin{cases} \frac{1}{6}(3t^2s - t^3), & s \geq t \\ \frac{1}{6}(3s^2t - s^3), & s \leq t. \end{cases}$$

In order to rewrite the BVP (6)-(7) as a perturbation of the integral equation (8), we observe that the function

$$\gamma(t) = \frac{1}{6}(3t^2 - t^3) \text{ for all } t \in [0, 1],$$

is the unique solution of the BVP

$$\gamma^{(4)}(t) = 0, \quad \gamma(0) = \gamma'(0) = \gamma''(1) = 0, \quad \gamma'''(1) + 1 = 0.$$

This allows us to associate with the BVP (6)-(7) the perturbed Hammerstein integral equation

$$u(t) = \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds, \quad (9)$$

where γ and G_0 are as above, and utilize the classical fixed point index for compact maps (see for example [2] or [17]).

Since the derivative of the function G_0 with respect to t is non-negative for all $t \in [0, 1]$, G_0 is a non-decreasing function of t that attains its maximum when $t = 1$. Then we set

$$\Phi(s) := \max_{0 \leq t \leq 1} G_0(t, s) = G_0(1, s) = \frac{1}{2}s^2 - \frac{1}{2}s^3.$$

We now look for a suitable interval $[a, b] \subset [0, 1]$ and for the constants $c_0, c_\gamma > 0$ that satisfy the inequalities

$$\begin{aligned} G_0(t, s) &\geq c_0\Phi(s), \text{ for every } (t, s) \in [a, b] \times [0, 1], \\ \gamma(t) &\geq c_\gamma\|\gamma\|, \text{ for every } t \in [a, b]. \end{aligned}$$

Since the derivative of the function $G_0(t, s)/\Phi(s)$ with respect to s is non-negative for all $s \in [0, 1]$, the function $G_0(t, s)/\Phi(s)$ is a non-decreasing function of s .

If we set

$$c_0(t) = \min\left\{\frac{2t}{3-t}, \frac{1}{2}t^2(3-t)\right\} = \frac{1}{2}t^2(3-t),$$

we may take, for $[a, b] \subset (0, 1]$,

$$c_0 := \min_{a \leq t \leq b} c_0(t) = \frac{1}{2}a^2(3-a).$$

We observe that $\|\gamma\| = \frac{1}{3}$, $\min_{t \in [a, b]} \gamma(t) = \gamma(a)$ and therefore we have

$$\gamma(t) \geq 3\gamma(a)\|\gamma\|, \text{ for every } t \in [a, b],$$

so that we can take

$$c_\gamma := 3\gamma(a).$$

Thus, for an arbitrary $[a, b] \subset (0, 1]$, we may set

$$c := \min\{c_0, c_\gamma\} = \frac{1}{2}a^2(3-a). \tag{10}$$

By a solution of the BVP (6)-(7) we mean a solution of the corresponding integral equation (9) and we work in the Banach space $C[0, 1]$ endowed with the usual supremum norm $\|u\| := \sup\{|u(t)| : t \in [0, 1]\}$ and the above hypotheses enable us to utilize the cone

$$K = \{u \in C[0, 1] : \min_{t \in [a, b]} u(t) \geq c\|u\|\},$$

where $[a, b] \subset (0, 1]$ and c is as in (10).

If Ω is a bounded open subset of K (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K . We write

$$K_r = \{u \in K : \|u\| < r\} \text{ and } \overline{K}_r = \{u \in K : \|u\| \leq r\}.$$

We consider now the map $T : C[0, 1] \rightarrow C[0, 1]$ defined by

$$Tu(t) := \gamma(t)(k_0 + B(\alpha[u])) + Fu(t),$$

where

$$Fu(t) := \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds.$$

Theorem 2.1 *If the hypotheses (C_1) - (C_4) hold for some $r > 0$, then T maps \overline{K}_r into K . When these hypotheses hold for each $r > 0$, T maps K into K . Moreover, T is a compact map.*

Proof. Take $u \in \overline{K}_r$. Then we have, for $t \in [0, 1]$,

$$\begin{aligned} Tu(t) &= \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds \\ &\leq \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 \Phi(s)g(s)f(s, u(s)) ds, \end{aligned}$$

therefore

$$\|Tu\| \leq \|\gamma\|(k_0 + B(\alpha[u])) + \int_0^1 \Phi(s)g(s)f(s, u(s)) ds.$$

Then we have

$$\min_{t \in [a, b]} Tu(t) \geq c \left[\|\gamma\|(k_0 + B(\alpha[u])) + \int_0^1 \Phi(s)g(s)f(s, u(s)) ds \right] \geq c\|Tu\|.$$

Hence we have $Tu \in K$ for every $u \in \overline{K}_r$. Moreover, the map T is compact, since it is a sum of two compact maps. In fact, the compactness of F is well-known and the perturbation $\gamma(t)(k_0 + B(\alpha[u]))$ is compact since γ and B are continuous and it maps a bounded set into a bounded subset of a 1-dimensional space. \square

We make use of the following numbers

$$\begin{aligned} f^{0, \rho} &:= \sup_{0 \leq u \leq \rho, 0 \leq t \leq 1} \frac{f(t, u)}{\rho}, & f_{\rho, \rho/c} &:= \inf_{\rho \leq u \leq \rho/c, a \leq t \leq b} \frac{f(t, u)}{\rho}, \\ \frac{1}{m} &:= \sup_{t \in [0, 1]} \int_0^1 G_0(t, s)g(s) ds, & \frac{1}{M(a, b)} &:= \inf_{t \in [a, b]} \int_a^b G_0(t, s)g(s) ds, \end{aligned}$$

use the notation

$$\mathcal{G}(s) := \int_0^1 G_0(t, s) dA(t),$$

and assume

$$(C_5) \quad 1 - \delta_2 \alpha[\gamma] > 0.$$

First, we give a condition that implies the index is 1 on the set K_ρ .

Lemma 2.1 *Assume that there exists $\rho > 0$ such that*

(I_ρ^1) *the following inequality holds:*

$$\frac{k_0}{3\rho(1 - \delta_2 \alpha[\gamma])} + f^{0, \rho} \left(\frac{\delta_2}{3(1 - \delta_2 \alpha[\gamma])} \int_0^1 \mathcal{G}(s)g(s) ds + \frac{1}{m} \right) < 1. \quad (11)$$

Then the fixed point index, $i_K(T, K_\rho)$, is 1.

Proof. We show that

$$\lambda u \neq Tu \text{ for every } u \in \partial K_\rho \text{ and for every } \lambda \geq 1,$$

which implies that the index is 1 on the set K_ρ . In fact, if there exists $\lambda \geq 1$ and $u \in \partial K_\rho$ such that

$$\lambda u(t) = Tu(t) = \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds$$

then we have

$$\lambda u(t) \leq \gamma(t)(k_0 + \delta_2\alpha[u]) + \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds \tag{12}$$

and

$$\lambda \alpha[u] \leq \alpha[\gamma](k_0 + \delta_2\alpha[u]) + \int_0^1 \mathcal{G}(s)g(s)f(s, u(s)) ds.$$

Hence we obtain

$$(\lambda - \delta_2\alpha[\gamma])\alpha[u] \leq k_0\alpha[\gamma] + \int_0^1 \mathcal{G}(s)g(s)f(s, u(s)) ds.$$

Substituting into (12) gives

$$\lambda u(t) \leq \frac{\lambda k_0 \gamma(t)}{\lambda - \delta_2\alpha[\gamma]} + \frac{\gamma(t)\delta_2}{\lambda - \delta_2\alpha[\gamma]} \int_0^1 \mathcal{G}(s)g(s)f(s, u(s)) ds + \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds.$$

Taking the supremum for $t \in [0, 1]$ gives

$$\begin{aligned} \lambda \rho &\leq \frac{\lambda k_0 \|\gamma\|}{\lambda - \delta_2\alpha[\gamma]} + \frac{\|\gamma\|\delta_2}{\lambda - \delta_2\alpha[\gamma]} \int_0^1 \mathcal{G}(s)g(s)\rho f^{0,\rho} ds + \sup_{t \in [0,1]} \int_0^1 G_0(t, s)g(s)\rho f^{0,\rho} ds \\ &\leq \frac{k_0 \|\gamma\|}{1 - \delta_2\alpha[\gamma]} + \frac{\|\gamma\|\delta_2}{1 - \delta_2\alpha[\gamma]} \int_0^1 \mathcal{G}(s)g(s)\rho f^{0,\rho} ds + \sup_{t \in [0,1]} \int_0^1 G_0(t, s)g(s)\rho f^{0,\rho} ds. \end{aligned}$$

Thus we have,

$$\lambda \leq \frac{k_0}{3\rho(1 - \delta_2\alpha[\gamma])} + f^{0,\rho} \left(\frac{\delta_2}{3(1 - \delta_2\alpha[\gamma])} \int_0^1 \mathcal{G}(s)g(s) ds + \frac{1}{m} \right) < 1.$$

This contradicts the fact that $\lambda \geq 1$ and proves the result. □

In order to give a condition that implies the fixed point index is equal to 0, we make use of the open set

$$V_\rho = \{u \in K : \min_{t \in [a,b]} u(t) < \rho\}.$$

V_ρ is equal to the set called $\Omega_{\rho/c}$ in [33]. Note that $K_\rho \subset V_\rho \subset K_{\rho/c}$.

Lemma 2.2 Assume that there exists $\rho > 0$ such that

(I $_{\rho}^0$) the following inequality holds:

$$\frac{k_0 c}{3\rho(1 - \delta_1 \alpha[\gamma])} + \left(\frac{c\delta_1}{3(1 - \delta_1 \alpha[\gamma])} \int_a^b \mathcal{G}(s)g(s) ds + \frac{1}{M(a, b)} \right) f_{\rho, \rho/c} > 1. \quad (13)$$

Then we have $i_K(T, V_{\rho}) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in [0, 1]$. Then $e \in K$. We prove that

$$u \neq T(u) + \lambda e \quad \text{for all } u \in \partial V_{\rho} \quad \text{and } \lambda \geq 0,$$

which ensures that the index is 0 on the set V_{ρ} . In fact, if this does not happen, there exist $u \in \partial V_{\rho}$ and $\lambda \geq 0$ such that $u = Tu + \lambda e$, that is

$$u(t) = \gamma(t)(k_0 + B(\alpha[u])) + \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds + \lambda.$$

Then we have

$$u(t) \geq \gamma(t)(k_0 + \delta_1 \alpha[u]) + \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds + \lambda, \quad (14)$$

and

$$\begin{aligned} \alpha[u] &\geq \alpha[\gamma](k_0 + \delta_1 \alpha[u]) + \int_0^1 \mathcal{G}(s)g(s)f(s, u(s)) ds + \alpha[\lambda] \\ &\geq \alpha[\gamma](k_0 + \delta_1 \alpha[u]) + \int_0^1 \mathcal{G}(s)g(s)f(s, u(s)) ds. \end{aligned}$$

Hence we obtain

$$(1 - \delta_1 \alpha[\gamma])\alpha[u] \geq k_0 \alpha[\gamma] + \int_0^1 \mathcal{G}(s)g(s)f(s, u(s)) ds.$$

Substituting into (14) gives

$$u(t) \geq \frac{k_0 \gamma(t)}{1 - \delta_1 \alpha[\gamma]} + \frac{\gamma(t)\delta_1}{1 - \delta_1 \alpha[\gamma]} \int_0^1 \mathcal{G}(s)g(s)f(s, u(s)) ds + \int_0^1 G_0(t, s)g(s)f(s, u(s)) ds + \lambda.$$

Then we have, for $t \in [a, b]$,

$$u(t) \geq \frac{k_0 c_{\gamma} \|\gamma\|}{1 - \delta_1 \alpha[\gamma]} + \frac{c_{\gamma} \|\gamma\| \delta_1}{1 - \delta_1 \alpha[\gamma]} \int_a^b \mathcal{G}(s)g(s)\rho f_{\rho, \rho/c} ds + \int_a^b G_0(t, s)g(s)\rho f_{\rho, \rho/c} ds + \lambda,$$

and therefore

$$\min_{t \in [a,b]} u(t) \geq \frac{k_0 c}{3\rho(1 - \delta_1 \alpha[\gamma])} + f_{\rho, \rho/c} \left(\frac{c\delta_1}{3(1 - \delta_1 \alpha[\gamma])} \int_a^b \mathcal{G}(s)g(s) ds + \frac{1}{M(a, b)} \right) + \lambda.$$

By (I_ρ^0) we have that $\min_{t \in [a,b]} u(t) > \rho + \lambda \geq \rho$. This contradicts the fact that $u \in \partial V_\rho$ and proves the result. \square

We can now state a result for the existence of one or two positive solutions for the integral equation (9).

Theorem 2.2 Equation (9) has a positive solution in K if either of the following conditions hold.

(S_1) There exist $\rho_1, \rho_2 \in (0, \infty)$, with $\rho_1 < \rho_2$, such that $(I_{\rho_1}^1)$ and $(I_{\rho_2}^0)$ hold.

(S_2) There exist $\rho_1, \rho_2 \in (0, \infty)$, with $\rho_1 < c\rho_2$, such that $(I_{\rho_1}^0)$ and $(I_{\rho_2}^1)$ hold.

Equation (9) has two positive solutions in K if either of the following conditions hold.

(D_1) There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, with $\rho_1 < \rho_2 < c\rho_3$, such that $(I_{\rho_1}^1)$, $(I_{\rho_2}^0)$ and $(I_{\rho_3}^1)$ hold.

(D_2) There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, with $\rho_1 < c\rho_2$ and $\rho_2 < \rho_3$, such that $(I_{\rho_1}^0)$, $(I_{\rho_2}^1)$ and $(I_{\rho_3}^0)$ hold.

We omit the proof which follows simply from properties of fixed point index, for details of similar proofs see [26, 34]. Note that, if the nonlinearity f has a suitable oscillatory behavior, one may establish, by the same method, the existence of more than two positive solutions (we refer the reader to [27, 33] to see the type of results that may be stated).

Example 2.1 We now assume that $g \equiv 1$, $\alpha[u] = u(\xi)$, $\xi \in (0, 1)$ and we make, as in [25], the choice

$$B(w) = \begin{cases} \frac{1}{4}w, & 0 \leq w \leq 1, \\ \frac{1}{8}w + \frac{1}{8}, & w \geq 1. \end{cases}$$

In this case we have $\delta_1 = \frac{1}{8}$ and $\delta_2 = \frac{1}{4}$,

$$\alpha[\gamma] = \int_0^1 \gamma(t) dA(t) = \frac{1}{6}(3\xi^2 - \xi^3)$$

and

$$\int_0^1 \mathcal{G}(s) ds = \xi^2 \left(-\frac{1}{24}\xi^2 - \frac{1}{6}\xi + \frac{1}{4} \right).$$

By direct calculation one gets $m = 8$. The ‘optimal’ $[a, b]$, the interval for which

$$M := M(a, b) = \frac{12}{3a^2b^2 - 2a^3b - a^4}$$

is a minimum, is given by the interval

$$\left[\frac{-3 + \sqrt{33}}{4}, 1 \right], \quad (15)$$

this gives $M = 22.032$ and $c = 0.545$.

Remark 2.1 *Until now we have discussed in details the case of the BCs (2). We point out that the same approach may be applied to the BCs (3), (4), (5).*

In all these cases the growth assumptions on the nonlinearities, that are the conditions equivalent to (11) and (13), can be written in the form

$$\frac{k_0 \|\gamma\|}{(1 - \delta_2 \alpha[\gamma]) \rho} + f^{0,\rho} \left(\frac{\delta_2 \|\gamma\|}{(1 - \delta_2 \alpha[\gamma])} \int_0^1 \mathcal{G}(s) g(s) ds + \frac{1}{m} \right) < 1,$$

and

$$\frac{k_0 c_\gamma \|\gamma\|}{(1 - \delta_1 \alpha[\gamma]) \rho} + \left(\frac{c_\gamma \delta_1 \|\gamma\|}{(1 - \delta_1 \alpha[\gamma])} \int_a^b \mathcal{G}(s) g(s) ds + \frac{1}{M(a, b)} \right) f_{\rho, \rho/c} > 1,$$

where

	BCs (2)	BCs (3)	BCs (4)	BCs (5)
$\gamma(t)$	$(3t^2 - t^3)/6$	1	t	$t^2/2$
c_γ	$a^2(3 - a)/2$	1	a	$a^2/2$
c	$a^2(3 - a)/2$	$a^2(3 - a)/2$	$a^2(3 - a)/2$	$a^2/2$

This also illustrates that the cone K , given by the constant c , varies according to the non-homogeneous BCs considered.

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