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QUASILINEARIZATION METHOD AND NONLOCAL SINGULAR THREE POINT BOUNDARY VALUE PROBLEMS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

The method of upper and lower solutions and quasilinearization for nonlinear singular equations of the type

$$-x''(t) + \lambda x'(t) = f(t, x(t)),\ t \in (0, 1),$$

subject to nonlocal three-point boundary conditions

$$x(0) = \delta x(\eta), \quad x(1) = 0, \quad 0 < \eta < 1,$$

are developed. Existence of a C^1 positive solution is established. A monotone sequence of solutions of linear problems converging uniformly and rapidly to a solution of the nonlinear problem is obtained.

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1 Introduction

Nonlocal singular boundary value problems (BVPs) have various applications in chemical engineering, underground water flow and population dynamics. These problems arise in many areas of applied mathematics such as gas dynamics, Newtonian fluid mechanics, the theory of shallow membrane caps, the theory of boundary layer and so on; see for example, [2, 7, 12, 13, 16] and the references therein. An excellent resource with an extensive bibliography was produced by Agarwal and O'Regan [1]. Existence theory for nonlinear multi-point singular boundary value problems has attracted the attention of many researchers; see for example, [3, 4, 5, 14, 15, 17, 18] and the references therein.

In this paper, we study existence and approximation of C^1-positive solution of a nonlinear forced Duffing equation with three-point boundary conditions of the type

$$\begin{aligned} &-x''(t) + \lambda x'(t) = f(t, x(t)),\ t \in (0,1), \\ &x(0) = \delta x(\eta), \quad x(1) = 0, \quad 0 < \eta < 1,\ 0 < \delta < \frac{e^{\lambda} - 1}{e^{\lambda} - e^{\lambda\eta}}, \end{aligned} \tag{1}$$

where the nonlinearity $f : (0,1) \times \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous and may be singular at $x = 0$, $t = 0$ and/or $t = 1$. By singularity we mean the function $f(t, x)$ is allowed to be unbounded at $x = 0$, $t = 0$ and/or $t = 1$ and by a C^1-positive solution x we mean that $x \in C[0,1] \cap C^2(0,1)$ satisfies (1), $x(t) > 0$ for $t \in (0,1)$ and both $x'(0+)$ and $x'(1-)$ exist.

For the existence theory, we develop the method of upper and lower solutions and to approximate the solution of the BVP (1), we develop the quasilinearization technique [5, 8, 9, 10, 11]. We obtain a monotone sequence of solutions of *linear* problems and show that, under suitable conditions on f, the sequence converges uniformly and quadratically to a solution of the original nonlinear problem (1).

2 Some basic results

For $u \in C[0,1]$ we write $\|u\| = \max\{|u(t)| : t \in [0,1]\}$. For any $\lambda \in \mathbb{R} \setminus \{0\}$, consider the singular boundary value problem

$$\begin{aligned} &- x''(t) + \lambda x'(t) = f(t, x(t)),\ t \in (0,1), \\ &x(0) = \delta x(\eta), \quad x(1) = 0, \quad 0 < \eta < 1,\ 0 < \delta < \frac{e^{\lambda} - 1}{e^{\lambda} - e^{\lambda\eta}}. \end{aligned} \tag{2}$$

We seek a solution x via the singular integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s))ds + \frac{(e^{\lambda} - e^{\lambda t})\delta}{(e^{\lambda} - 1) - \delta(e^{\lambda} - e^{\lambda\eta})} \int_0^1 G(\eta, s) f(s, x(s))ds, \tag{3}$$

where

$$G(t,s) = \frac{1}{\lambda e^{\lambda s}(e^{\lambda} - 1)} \begin{cases} (e^{\lambda t} - 1)(e^{\lambda} - e^{\lambda s}), & 0 < t < s < 1, \\ (e^{\lambda s} - 1)(e^{\lambda} - e^{\lambda t}), & 0 < s < t < 1, \end{cases}$$

is the Green's function corresponding to the homogeneous two-point BVP

$$\begin{aligned} &- x''(t) + \lambda x'(t) = 0,\ t \in (0,1), \\ &x(0) = 0, \quad x(1) = 0. \end{aligned}$$

Clearly, $G(t,s) > 0$ on $(0,1)\times(0,1)$. From (3), $x \ge 0$ on $[0,1]$ provided $f \ge 0$. Hence for a positive solution we assume $f \ge 0$ on $[0,1]\times\mathbb{R}$.

We recall the concept of upper and lower solutions for the BVP (2).

Definition 2.1. *A function α is called a lower solution of the BVP (2) if $\alpha \in C[0,1]\cap C^2(0,1)$ and satisfies*

$$\begin{aligned}&-\alpha''(t)+\lambda\alpha'(t)\le f(t,\alpha(t)), \quad t\in(0,1),\\ &\alpha(0)\le\delta\alpha(\eta),\ \alpha(1)\le 0.\end{aligned}$$

An upper solution $\beta \in C[0,1]\cap C^2(0,1)$ of the BVP (2) is defined similarly by reversing the inequalities.

Choose $b>\eta$, a finite positive number, such that $\delta<\frac{e^{\lambda b}-1}{e^{\lambda b}-e^{\lambda\eta}}$. Since the homogeneous linear problem

$$\begin{aligned}&-x''(t)+\lambda x'(t)=0,\ t\in[0,b],\\ &x(0)=0,\quad x(b)=0,\end{aligned}$$

has only the trivial solution, hence, for any $\sigma\in C[0,b]$ and $\rho,\ \tau\in\mathbb{R}$, the corresponding nonhomogeneous linear three point problem

$$\begin{aligned}&-x''(t)+\lambda x'(t)=\sigma(t),\ t\in[0,b],\\ &x(0)-\delta x(\eta)=\tau,\quad x(b)=\rho,\quad 0<\eta<b,\ 0<\delta<\frac{e^{\lambda b}-1}{e^{\lambda b}-e^{\lambda\eta}},\end{aligned}\tag{4}$$

has a unique solution

$$x(t)=\int_0^b G_b(t,s)\sigma(s)ds+\frac{(e^{\lambda b}-e^{\lambda t})}{D}\{\delta\int_0^b G_b(\eta,s)\sigma(s)ds+\tau\}+\frac{\rho\psi(t)}{D},\tag{5}$$

where $\psi(t)=(e^{\lambda t}-1)+\delta(e^{\lambda\eta}-e^{\lambda t})$, $D=(e^{\lambda b}-1)-\delta(e^{\lambda b}-e^{\lambda\eta})$ and

$$G_b(t,s)=\frac{1}{\lambda e^{\lambda s}(e^{\lambda b}-1)}\begin{cases}(e^{\lambda t}-1)(e^{\lambda b}-e^{\lambda s}), & 0\le t\le s\le b,\\ (e^{\lambda s}-1)(e^{\lambda b}-e^{\lambda t}), & 0\le s\le t\le b.\end{cases}$$

We note that $\psi(t)\ge 0$ on $[0,b]$ and if $\tau\ge 0$, $\rho\ge 0$ and $\sigma\ge 0$ on $[0,b]$, then $x\ge 0$ on $[0,b]$. Thus, we have the following comparison result (maximum principle):

Maximum Principle: Let $\delta,\ \eta\in\mathbb{R}$ such that $0<\delta<\frac{e^{\lambda b}-1}{e^{\lambda b}-e^{\lambda\eta}}$ and $0<\eta<b$. For any $x\in C^1[0,b]$ such that

$$-x''(t)+\lambda x'(t)\ge 0,\ t\in(0,b),\ x(0)-\delta x(\eta)\ge 0 \text{ and } x(b)\ge 0,$$

we have $x(t)\ge 0$, $t\in[0,b]$.

In the following theorem, we prove existence of a $C^1[0,1]$ positive solution of the singular BVP (2). We generate a sequence of $C^1[0,1]$ positive solutions of nonsingular problems that has a convergent subsequence converging to a solution of the original problem.

Theorem 2.1. *Assume that there exist lower and upper solutions* $\alpha, \beta \in C[0,1] \cap C^2(0,1)$ *of the BVP* (2) *such that* $\alpha(1) = \beta(1)$, *and* $0 < \alpha \leq \beta$ *on* $[0,1)$, *and* $\alpha(0) - \delta\alpha(\eta) < \beta(0) - \delta\beta(\eta)$. *Assume that* $f : (0,1) \times \mathbb{R} \setminus \{0\} \to (0,\infty)$ *is continuous and there exists* $h(t)$ *such that* $e^{-\lambda t}h(t) \in L^1[0,1]$ *and*

$$|f(t,x)| \leq h(t) \text{ if } x \in [\bar{\alpha}, \bar{\beta}], \tag{6}$$

where $\bar{\alpha} = \min\{\alpha(t) : t \in [0,1]\} = 0$ *and* $\bar{\beta} = \max\{\beta(t) : t \in [0,1]\}$. *Then the BVP* (2) *has a* $C^1[0,1]$ *positive solution* x *such that* $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in [0,1]$.

Proof. Let $\{a_n\}$, $\{b_n\}$ be two monotone sequences satisfying

$$0 < \cdots < a_n < \cdots < a_1 < \eta < b_1 < \cdots < b_n < \cdots < 1$$

and are such that $\{a_n\}$ converges to 0, $\{b_n\}$ converges to 1. Clearly, $\cup_{n=1}^{\infty}[a_n, b_n] = (0,1)$. Let $\alpha(a_n) - \delta\alpha(\eta) \leq \beta(a_n) - \delta\beta(\eta)$ for sufficiently large n, and choose two null sequences $\{\tau_n\}$ and $\{\rho_n\}$ [that is, $\{\tau_n\}$ and $\{\rho_n\}$ both converge to 0] such that

$$\begin{aligned} \alpha(a_n) - \delta\alpha(\eta) \leq \tau_n \leq \beta(a_n) - \delta\beta(\eta), \\ \alpha(b_n) \leq \rho_n \leq \beta(b_n),\ n = 1,2,3,\dots. \end{aligned} \tag{7}$$

Define a partial order in $C[0,1] \cap C^2(0,1)$ by $x \leq y$ if and only if $x(t) \leq y(t)$, $t \in [0,1]$. Define a modification F of f with respect to α, β as follows:

$$F(t,x) = \begin{cases} f(t,\beta(t)) + \frac{x-\beta(t)}{1+|x-\beta(t)|}, & \text{if } x \geq \beta(t), \\ f(t,x(t)), & \text{if } \alpha(t) \leq x \leq \beta(t), \\ f(t,\alpha(t)) + \frac{\alpha(t)-x}{1+|\alpha(t)-x|}, & \text{if } \ x \leq \alpha(t). \end{cases} \tag{8}$$

Clearly, F is continuous and bounded on $(0,1) \times C[0,1]$. For each $n \in \mathbb{N}$, consider the nonsingular modified problems

$$\begin{aligned} -x''(t) + \lambda x'(t) = F(t,x),\ t \in [a_n, b_n], \\ x(a_n) - \delta x(\eta) = \tau_n, \quad x(b_n) = \rho_n. \end{aligned} \tag{9}$$

We write the BVP (9) as an equivalent integral equation

$$\begin{aligned} x(t) = \int_{a_n}^{b_n} G_n(t,s)F(s,x)ds + \frac{(e^{\lambda b_n} - e^{\lambda t})}{D_n}\{\delta\int_{a_n}^{b_n} G_n(\eta,s)F(s,x)ds + \tau_n\} \\ + \frac{\rho_n\psi_n(t)}{D_n},\ t \in [a_n, b_n], \end{aligned} \tag{10}$$

where $D_n = (e^{\lambda b_n} - e^{\lambda a_n}) - \delta(e^{\lambda b_n} - e^{\lambda\eta})$, $\psi_n(t) = (e^{\lambda t} - e^{\lambda a_n}) + \delta(e^{\lambda\eta} - e^{\lambda t})$ and

$$G_n(t,s) = \frac{1}{\lambda e^{\lambda s}(e^{\lambda b_n} - e^{\lambda a_n})} \begin{cases} (e^{\lambda t} - e^{\lambda a_n})(e^{\lambda b_n} - e^{\lambda s}), & a_n \leq t \leq s \leq b_n, \\ (e^{\lambda s} - e^{\lambda a_n})(e^{\lambda b_n} - e^{\lambda t}), & a_n \leq s \leq t \leq b_n. \end{cases}$$

Clearly, $G_n(t,s) \to G(t,s)$ as $n \to \infty$. By a solution of (10), we mean a solution of the operator equation

$$(I - T_n)x = 0, \text{ that is, a fixed point of } T_n,$$

where I is the identity and for each $x \in C[a_n, b_n]$, the operator $T_n : C[a_n, b_n] \to C[a_n, b_n]$ is defined by

$$\begin{aligned} T_n(x)(t) = \int_{a_n}^{b_n} G_n(t,s)F(s,x)ds + \frac{(e^{\lambda b_n} - e^{\lambda t})}{D_n}\{\delta \int_{a_n}^{b_n} G_n(\eta,s)F(s,x)ds + \tau_n\} \\ + \frac{\rho_n \psi_n(t)}{D_n},\ t \in [a_n, b_n]. \end{aligned} \tag{11}$$

Since F is continuous and bounded on $[a_n, b_n] \times C[a_n, b_n]$ for each $n \in \mathbb{N}$, hence T_n is compact for each $n \in \mathbb{N}$. By Schauder's fixed point theorem, T_n has a fixed point (say) $x_n \in C[a_n, b_n]$ for each $n \in \mathbb{N}$.

Now, we show that

$$\alpha \le x_n \le \beta \text{ on } [a_n, b_n],\ n \in \mathbb{N}$$

and consequently, x_n is a solution of the BVP

$$\begin{aligned} -x''(t) + \lambda x'(t) = f(t, x(t)),\ t \in [a_n, b_n], \\ x(a_n) - \delta x(\eta) = \tau_n, \quad x(b_n) = \rho_n. \end{aligned} \tag{12}$$

Firstly, we show that $\alpha \le x_n$ on $[a_n, b_n]$, $n \in \mathbb{N}$.
Assume that $\alpha \not\le x_n$ on $[a_n, b_n]$. Set $z(t) = x_n(t) - \alpha(t)$, $t \in [a_n, b_n]$, then

$$z \in C^1[a_n, b_n] \text{ and } z \not\ge 0 \text{ on } [a_n, b_n]. \tag{13}$$

Hence, z has a negative minimum at some point $t_0 \in [a_n, b_n]$. From the boundary conditions, it follows that

$$\begin{aligned} z(a_n) - \delta z(\eta) = [x_n(a_n) - \delta x_n(\eta)] - [\alpha(a_n) - \delta\alpha(\eta)] \ge \tau_n - \tau_n = 0, \\ z(b_n) = x_n(b_n) - \alpha(b_n) \ge \rho_n - \rho_n \ge 0. \end{aligned} \tag{14}$$

Hence, $t_0 \ne b_n$. If $t_0 \ne a_n$, then

$$z(t_0) < 0,\ z'(t_0) = 0,\ z''(t_0) \ge 0.$$

However, in view of the definition of F and that of lower solution, we obtain

$$-z''(t_0) = -z''(t_0) + \lambda z'(t_0) \ge -\frac{z(t_0)}{1 + |z(t_0)|} > 0,$$

a contradiction. Hence z has no negative local minimum.

If $t_0 = a_n$, then $z(a_n) < 0$ and $z'(a_n) \geq 0$. From the boundary condition (14), we have $z(\eta) \leq \frac{1}{\delta} z(a_n) < 0$. Let $[t_1, t_2]$ be the maximal interval containing η such that $z(t) \leq 0$, $t \in [t_1, t_2]$. Clearly, $t_1 \geq a_n$, $t_2 \leq b_n$ and $z(t_1) \geq z(a_n) \geq \delta z(\eta)$. Further, for $t \in [t_1, t_2]$, we have

$$-z''(t) + \lambda z'(t) \geq f(t, \alpha(t)) - \frac{z(t)}{1 + |z(t)|} - f(t, \alpha(t)) > 0.$$

Hence, by comparison result, $z > 0$ on $[t_1, t_2]$, again a contradiction. Thus, $\alpha \leq x_n$ on $[a_n, b_n]$.

Similarly, we can show that $x_n \leq \beta$ on $[a_n, b_n]$.

Now, define

$$u_n(t) = \begin{cases} \delta x_n(\eta) + \tau_n, \text{ if } 0 \leq t \leq a_n \\ x_n(t), \text{ if } a_n \leq t \leq b_n \\ \rho_n, \text{ if } b_n \leq t \leq 1. \end{cases}$$

Clearly, u_n is continuous extension of x_n to $[0, 1]$ and $\alpha \leq u_n \leq \beta$ on $[a_n, b_n]$. Since,

$$u_n(t) = \delta x_n(\eta) + \tau_n = x_n(a_n), \, t \in [0, a_n],$$
$$u_n(t) = \rho_n = x_n(b_n), \, t \in [b_n, 1].$$

Hence,

$$\alpha \leq u_n \leq \beta \text{ on } [0, 1], \, n \in \mathbb{N}.$$

Since $[a_1, b_1] \subset [a_n, b_n]$, for each n there must exist $t_n \in (a_1, b_1)$ such that

$$|u_n(t_n)| \leq M; |u_n'(t_n)| = |\frac{u_n(b_1) - u_n(a_1)}{b_1 - a_1}| \leq N,$$

where $M = \max_{t \in [a_1, b_1]}\{|\alpha(t)|, |\beta(t)|$, $N = \frac{2M}{b_1 - a_1}$. We can assume that

$$t_n \to t_0 \in [a_1, b_1],$$
$$u_n(t_n) \to x_0 \in [\alpha(t_0), \beta(t_0)],$$
$$u_n'(t_n) \to x_0' \in [-N, N], \text{ as } n \to \infty$$

By standard arguments [6], (also see [1, 3, 14]), there is a $C[0, 1]$ positive solution $x(t)$ of (2) such that $\alpha \leq x \leq \beta$ on $[0, 1]$, $x(t_0) = x_0$, $x'(t_0) = x_0'$ and a subsequence $\{u_{nj}(t)\}$ of $\{u_n(t)\}$ such that $u_{nj}(t)$, $u_{nj}'(t)$ converges uniformly to $x(t)$, $x'(t)$ respectively, on any compact subinterval of $(0, 1)$.

Now, using (6), we obtain

$$| - (x'(t)e^{-\lambda t})'| = e^{-\lambda t}|f(t, x(t))| \leq e^{-\lambda t} h(t) \in L^1[0, 1],$$

which implies that $x \in C^1[0, 1]$. □

3 Approximation of solution

We develop the approximation technique (quasilinearization) and show that under suitable conditions on f, there exists a bounded monotone sequence of solutions of linear problems that converges uniformly and quadratically to a solution of the nonlinear original problem. Choose a function $\Phi(t,x)$ such that Φ, Φ_x, $\Phi_{xx} \in C([0,1]\times\mathbb{R})$,

$$\Phi_{xx}(t,x) \geq 0 \text{ for every } t \in [0,1] \text{ and } x \in [0,\bar{\beta}]$$

and

$$\frac{\partial^2}{\partial x^2}[f(t,x) + \Phi(t,x)] \geq 0 \text{ on } (0,1)\times(0,\bar{\beta}]. \tag{15}$$

Here, we do not require the condition that $\frac{\partial^2}{\partial x^2} f(t,x) \geq 0$ on $(0,1)\times(0,\bar{\beta}]$.
Define $F : (0,1)\times\mathbb{R} \to \mathbb{R}$ by $F(t,x) = f(t,x) + \Phi(t,x)$. Note that $F \in C((0,1)\times\mathbb{R})$ and

$$\frac{\partial^2}{\partial x^2} F(t,x) \geq 0 \text{ on } (0,1)\times(0,\bar{\beta}], \tag{16}$$

where $\bar{\beta} = \max\{\beta(t) : t \in [0,1]\}$.

Theorem 3.1. *Assume that*

($\mathbf{A_1}$) α, β *are lower and upper solutions of the BVP* (1) *satisfying the hypotheses of Theorem 2.1.*

($\mathbf{A_2}$) f, f_x, $f_{xx} \in C((0,1)\times\mathbb{R})$ *and there exist* h_1, h_2, h_3 *such that* $e^{-\lambda t}h_i \in L^1[0,1]$ *and*

$$\left|\frac{\partial^i}{\partial x^i} f(t,x)\right| \leq h_i(t) \text{ for } |x| \leq \bar{\beta},\ t \in (0,1),\ i = 0,1,2.$$

Moreover, f *is non-increasing in* x *for each* $t \in (0,1)$.

Then, there exists a monotone sequence $\{w_n\}$ *of solutions of* linear problems *converging uniformly and quadratically to a unique solution of the BVP* (2).

Proof. The conditions (A_1) and (A_2) ensure the existence of a C^1 positive solution x of the BVP (2) such that

$$\alpha(t) \leq x(t) \leq \beta(t),\ t \in [0,1].$$

For $t \in (0,1)$, using (16), we obtain

$$f(t,x) \geq f(t,y) + F_x(t,y)(x-y) - [\Phi(t,x) - \Phi(t,y)], \tag{17}$$

where x, $y \in (0,\bar{\beta}]$. The mean value theorem and the fact that Φ_x is increasing in x on $[0,\bar{\beta}]$ for each $t \in [0,1]$, yields

$$\Phi(t,x) - \Phi(t,y) = \Phi_x(t,c)(x-y) \leq \Phi_x(t,\bar{\beta})(x-y) \text{ for } x \geq y, \tag{18}$$

where $x,\, y \in [0, \bar{\beta}]$ such that $y \le c \le x$. Substituting in (17), we have

$$f(t,x) \ge f(t,y) + [F_x(t,y) - \Phi_x(t,\bar{\beta})](x-y), \text{ for } x \ge y \tag{19}$$

on $(0,1) \times (0,\bar{\beta}]$. Define $g : (0,1) \times \mathbb{R} \times \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$g(t,x,y) = f(t,y) + [F_x(t,y) - \Phi_x(t,\bar{\beta})](x-y). \tag{20}$$

We note that $g(t,x,y)$ is continuous on $(0,1) \times \mathbb{R} \times \mathbb{R} \setminus \{0\}$. Moreover, for every $t \in (0,1)$ and $x,\, y \in (0,\bar{\beta}]$, g satisfies the following relations

$$g_x(t,x,y) = F_x(t,y) - \Phi_x(t,\bar{\beta}) \le F_x(t,y) - \Phi_x(t,y) = f_x(t,y) \le 0 \text{ and}$$

$$\begin{cases} f(t,x) \ge g(t,x,y), \text{for } x \ge y, \\ f(t,x) = g(t,x,x). \end{cases} \tag{21}$$

Moreover, for every $t \in (0,1)$ and $x, y \in (0,\bar{\beta}]$, using mean value theorem, we have

$$g(t,x,y) = f(t,y) + f_x(t,y)(x-y) - \Phi_{xx}(t,c)(\bar{\beta}-y)(x-y),$$

where $y < c < \bar{\beta}$. Consequently, in view of (A_2), we obtain

$$\begin{aligned} |g(t,x,y)| &\le |f(t,y)| + |f_x(t,y)||(x-y)| + |\Phi_{xx}(t,c)||\bar{\beta}-y||x-y| \\ &\le h_1(t) + h_2(t)\bar{\beta} + M = H(t) \text{ (say)}, \text{ for every } t \in (0,1) \text{ and } x, y \in (0,\bar{\beta}], \end{aligned} \tag{22}$$

where $M = \max\{|\Phi_{xx}(t,c)||\bar{\beta}-y||x-y| : t \in [0,1],\, x,\, y \in [0,\bar{\beta}]\}$. Hence

$$e^{-\lambda t}H(t) = e^{-\lambda t}h_1(t) + e^{-\lambda t}h_2(t)\bar{\beta} + e^{-\lambda t}M \in L^1[0,1].$$

Now, we develop the iterative scheme to approximate the solution. As an initial approximation, we choose $w_0 = \alpha$ and consider the linear problem

$$\begin{aligned} -x''(t) + \lambda x'(t) &= g(t, x(t), w_0(t)),\, t \in (0,1) \\ x(0) &= \delta x(\eta),\, x(1) = 0. \end{aligned} \tag{23}$$

Using (21) and the definition of lower and upper solutions, we get

$$\begin{aligned} g(t, w_0(t), w_0(t)) &= f(t, w_0(t)) \ge -w_0''(t) + \lambda w_0'(t),\, t \in (0,1), \\ w_0(0) &\le \delta(w_0(\eta)),\, w_0(1) \le 0, \end{aligned}$$

$$\begin{aligned} g(t, \beta(t), w_0(t)) &\le f(t, \beta(t)) \le -\beta''(t) + \lambda\beta'(t),\, t \in (0,1), \\ \beta(0) &\ge \delta\beta(\eta),\, \beta(1) \ge 0, \end{aligned}$$

which imply that w_0 and β are lower and upper solutions of (23) respectively. Hence by Theorem 2.1, there exists a C^1 positive solution $w_1 \in C[0,1] \cap C^2(0,1)$ of (23) such that

$$w_0 \leq w_1 \leq \beta \text{ on } [0,1].$$

Using (21) and the fact that w_1 is a solution of (23), we obtain

$$\begin{aligned} -w_1''(t) + \lambda w_1'(t) &= g(t, w_1(t), w_0(t)) \leq f(t, w_1(t)),\ t \in (0,1) \\ w_1(0) &= \delta w_1(\eta),\ w_1(1) = 0, \end{aligned} \tag{24}$$

which implies that w_1 is a lower solution of (2). Similarly, in view of (A_1), (21) and (24), we can show that w_1 and β are lower and upper solutions of

$$\begin{aligned} -x''(t) + \lambda x'(t) &= g(t, x(t), w_1(t)),\ t \in (0,1) \\ x(0) &= \delta x(\eta),\ x(1) = 0. \end{aligned} \tag{25}$$

Hence by Theorem 2.1, there exists a C^1 positive solution $w_1 \in C[0,1] \cap C^2(0,1)$ of (25) such that

$$w_1 \leq w_2 \leq \beta \text{ on } [0,1].$$

Continuing in the above fashion, we obtain a bounded monotone sequence $\{w_n\}$ of $C^1[0,1]$ positive solutions of the linear problems satisfying

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq ... \leq w_n \leq \beta \text{ on } [0,1], \tag{26}$$

where the element w_n of the sequence is a solution of the linear problem

$$\begin{aligned} -x''(t) + \lambda x'(t) &= g(t, x(t), w_{n-1}(t)),\ t \in (0,1) \\ x(0) &= \delta x(\eta),\ x(1) = 0 \end{aligned}$$

and for each $t \in (0,1)$, is given by

$$\begin{aligned} w_n(t) = &\int_0^1 G(t,s) g(s, w_n(s), w_{n-1}(s)) ds + \\ &\frac{(e^\lambda - e^{\lambda t})\delta}{(e^\lambda - 1) - \delta(e^\lambda - e^{\lambda\eta})} \int_0^1 G(\eta, s) g(s, w_n(s), w_{n-1}(s)) ds. \end{aligned} \tag{27}$$

The monotonicity and uniform boundedness of the sequence $\{w_n\}$ implies the existence of a pointwise limit w on $[0,1]$. From the boundary conditions, we have

$$0 = w_n(0) - \delta w_n(\eta) \to w(0) - \delta w(\eta) \text{ and } 0 = w_n(1) \to w(1).$$

Hence w satisfy the boundary conditions. Moreover, from (22), the sequence $\{g(t, w_n, w_{n-1})\}$ is uniformly bounded by $h_3(t) \in L^1[0,1]$ on $(0,1)$. Hence, the continuity of the function g on $(0,1) \times (0, \bar{\beta}] \times (0, \bar{\beta}]$ and the uniform boundedness of the sequence

$\{g(t, w_n, w_{n-1})\}$ implies that the sequence $\{g(t, w_n, w_{n-1})\}$ converges pointwise to the function $g(t, w, w) = f(t, w)$. By Lebesgue dominated convergence theorem, for any $t \in (0, 1)$,

$$\int_0^1 G(t,s)g(s, w_n(s), w_{n-1}(s))ds \to \int_0^1 G(t,s)f(s, w(s))ds.$$

Passing to the limit as $n \to \infty$, we obtain

$$\begin{aligned} w(t) &= \int_0^1 G(t,s)g(s, w(s), w(s))ds + \frac{(e^{\lambda} - e^{\lambda t})\delta}{(e^{\lambda} - 1) - \delta(e^{\lambda} - e^{\lambda\eta})} \int_0^1 G(\eta, s)g(s, w(s), w(s))ds \\ &= \int_0^1 G(t,s)f(s, w(s))ds + \frac{(e^{\lambda} - e^{\lambda t})\delta}{(e^{\lambda} - 1) - \delta(e^{\lambda} - e^{\lambda\eta})} \int_0^1 G(\eta, s)f(s, w(s))ds, \ t \in (0,1); \end{aligned}$$

that is, w is a solution of (2).

Now, we show that the convergence is quadratic. Set $v_n(t) = w(t) - w_n(t)$, $t \in [0, 1]$, where w is a solution of (2). Then, $v_n(t) \geq 0$ on $[0, 1]$ and the boundary conditions imply that $v_n(0) = \delta v_n(\eta)$ and $v_n(1) = 0$. Now, in view of the definitions of F and g, we obtain

$$\begin{aligned} &-v_n''(t) + \lambda v_n'(t) = f(t, w(t)) - g(t, w_n(t), w_{n-1}(t)) \\ &\quad = [F(t, w(t)) - \Phi(t, w(t))] \\ &\quad - [f(t, w_{n-1}(t)) + (F_x(t, w_{n-1}(t)) - \Phi_x(t, \bar{\beta}))(w_n(t) - w_{n-1}(t))] \\ &\quad = [F(t, w(t)) - F(t, w_{n-1}(t)) - F_x(t, w_{n-1}(t))(w_n(t) - w_{n-1}(t))] \\ &\quad - [\Phi(t, w(t)) - \Phi(t, w_{n-1}(t)) - \Phi_x(t, \bar{\beta}))(w_n(t) - w_{n-1}(t))], \ t \in (0, 1). \end{aligned} \tag{28}$$

Using the mean value theorem repeatedly and the fact that $\Phi_{xx} \geq 0$ on $[0, 1] \times [0, \bar{\beta}]$, we obtain, $\Phi(t, w(t)) - \Phi(t, w_{n-1}(t)) \geq \Phi_x(t, w_{n-1}(t))(w(t) - w_{n-1}(t))$ and

$$\begin{aligned} &F(t, w(t)) - F(t, w_{n-1}(t)) - F_x(t, w_{n-1}(t))(w_n(t) - w_{n-1}(t)) \\ &\quad = F_x(t, w_{n-1}(t))(w(t) - w_{n-1}(t)) + \frac{F_{xx}(t, \xi_1)}{2}(w(t) - w_{n-1}(t))^2 \\ &\quad - F_x(t, w_{n-1}(t))(w_n(t) - w_{n-1}(t)) \\ &\quad = F_x(t, w_{n-1}(t))(w(t) - w_n(t)) + \frac{F_{xx}(t, \xi_1)}{2}(w(t) - w_{n-1}(t))^2 \\ &\quad \leq F_x(t, w_{n-1}(t))(w(t) - w_n(t)) + \frac{F_{xx}(t, \xi_1)}{2}\|v_{n-1}\|^2, \ t \in (0, 1), \end{aligned}$$

where $w_{n-1}(t) \leq \xi_1 \leq w(t)$ and $\|v\| = \max\{v(t) : t \in [0, 1]\}$. Hence the equation (28) can be rewritten as

$$
\begin{aligned}
-v_n''(t) + \lambda v_n'(t) &\le F_x(t, w_{n-1}(t))(w(t) - w_n(t)) + \frac{F_{xx}(t,\xi_1)}{2}\|v_{n-1}\|^2 \\
&- \Phi_x(t, w_{n-1}(t))(w(t) - w_{n-1}(t)) + \Phi_x(t,\bar{\beta}))(w_n(t) - w_{n-1}(t)) \\
&= f_x(t, w_{n-1}(t))(w(t) - w_n(t)) + \frac{F_{xx}(t,\xi_1)}{2}\|v_{n-1}\|^2 \\
&+ [\Phi_x(t,\bar{\beta}) - \Phi_x(t, w_{n-1}(t))](w_n(t) - w_{n-1}(t)) \\
&\le \frac{F_{xx}(t,\xi_1)}{2}\|v_{n-1}\|^2 + \Phi_{xx}(t,\xi_2)(\bar{\beta} - w_{n-1}(t))(w_n(t) - w_{n-1}(t)) \\
&\le \frac{f_{xx}(t,\xi_1) + \Phi_{xx}(t,\xi_1)}{2}\|v_{n-1}\|^2 + \Phi_{xx}(t,\xi_2)(\bar{\beta} - w_{n-1}(t))(w(t) - w_{n-1}(t)) \\
&\le \frac{f_{xx}(t,\xi_1)}{2}\|v_{n-1}\|^2 + d_1\big(\frac{\|v_{n-1}\|^2}{2} + |\bar{\beta} - w_{n-1}(t)||w(t) - w_{n-1}(t)|\big),\ t \in (0,1)
\end{aligned}
\tag{29}
$$

where $w_{n-1}(t) \le \xi_2 \le w_n(t)$, $d_1 = \max\{|\Phi_{xx}| : (t,x) \in [0,1] \times [0,\bar{\beta}]\}$ and we used the fact that $f_x \le 0$ on $(0,1) \times (0,\bar{\beta}]$. Choose $r > 1$ such that

$$|\beta(t) - w_{n-1}(t)| \le r|w(t) - w_{n-1}(t)| \text{ on } [0,1].$$

We obtain

$$-v_n''(t) + \lambda v_n'(t) \le \big(\frac{f_{xx}(t,\xi_1)}{2} + d_1(r + 1/2)\big)\|v_{n-1}\|^2 \le \big(\frac{h_3(t)}{2} + d_2\big)\|v_{n-1}\|^2,\ t \in (0,1), \tag{30}$$

where $e^{-\lambda t}h_3(t) \in L^1[0,1]$ and $d_2 = d_1(r + 1/2)$.

By the comparison result, $v_n(t) \le z(t)$, $t \in [0,1]$, where $z(t)$ is the unique solution of the linear BVP

$$
\begin{aligned}
-z''(t) + \lambda z'(t) &= \big(\frac{h_3(t)}{2} + d_2\big)\|v_{n-1}\|^2, \\
z(0) &= \delta z(\eta),\ z(1) = 0.
\end{aligned}
\tag{31}
$$

Thus,

$$
\begin{aligned}
v_n(t) \le z(t) = \Big[\int_0^1 & G(t,s)\big(\frac{h_3(s)}{2} + d_2\big)ds + \\
&\frac{(e^\lambda - e^{\lambda t})\delta}{(e^\lambda - 1) - \delta(e^\lambda - e^{\lambda\eta})}\int_0^1 G(\eta,s)\big(\frac{h_3(s)}{2} + d_2\big)ds\Big]\|v_{n-1}\|^2 \\
&\le A\|v_{n-1}\|^2,
\end{aligned}
\tag{32}
$$

where A denotes

$$\max_{t\in[0,1]}\{\int_0^1 G(t,s)\big(\frac{h_3(s)}{2} + d_2\big)ds + \frac{(e^\lambda - e^{\lambda t})\delta}{(e^\lambda - 1) - \delta(e^\lambda - e^{\lambda\eta})}\int_0^1 G(\eta,s)\big(\frac{h_3(s)}{2} + d_2\big)ds\}.$$

(32) gives quadratic convergence. □

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