Electronic Journal of Qualitative Theory of Differential Equations
Spec. Ed. I, 2009 No. **22**, 1–13; **http://www.math.u-szeged.hu/ejqtde/**

NEUMANN BOUNDARY VALUE PROBLEMS FOR IMPULSIVE DIFFERENTIAL INCLUSIONS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

In this paper, we investigate the existence of solutions for a class of second order impulsive differential inclusions with Neumann boundary conditions. By using suitable fixed point theorems, we study the case when the right hand side has convex as well as nonconvex values.

Key words and phrases: Neumann boundary value problem, differential inclusions, fixed point.
AMS (MOS) Subject Classifications: 34A60, 34B15

1 Introduction

This paper is concerned with the existence of solutions of boundary value problems (BVP for short) for second order differential inclusions with Neumann boundary conditions and impulsive effects. More precisely, in Section 3, we consider the second order impulsive Neumann BVP,

$$x''(t) + k^2 x(t) \in F(t, x(t)), \text{ a.e. } \ t \in J' := [0, 1] \setminus \{t_1, \dots, t_m\}, \tag{1}$$

$$\Delta x'|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

$$x'(0) = x'(1) = 0, \tag{3}$$

where $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact valued multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $k \in (0, \frac{\pi}{2})$, $0 < t_1 < t_2 < \ldots < t_m < 1$, $I_k \in C(\mathbb{R}, \mathbb{R})$ $(k = 1, 2, \dots, m)$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively, $k = 1, 2, \dots, m$.

In the literature there are few papers dealing with the existence of solutions for Neumann boundary value problems; see [15], [16] and the references therein. Recently in [14], the authors studied Neumann boundary value problems with impulse actions.

Motivated by the work above, this paper attempts to study existence results for impulsive Neumann boundary value problems for differential inclusions.

The aim of our paper is to present existence results for the problem (1)-(3), when the right hand side is convex as well as nonconvex valued. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. The methods used are standard, however their exposition in the framework of problem (1)-(3) is new. It is remarkable also that the results of this paper are new, even for the special case $I_k = 0$.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C([0,1],\mathbb{R})$ is the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in [0,1]\}.$$

$L^1([0,1],\mathbb{R})$ denotes the Banach space of measurable functions $x : [0,1] \longrightarrow \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|x\|_{L^1} = \int_0^1 |x(t)|dt \quad \text{for all } x \in L^1([0,1],\mathbb{R}).$$

$AC^1((0,1),\mathbb{R})$ is the space of differentiable functions $x : (0,1) \to \mathbb{R}$, whose first derivative, x', is absolutely continuous.

For a normed space $(X, |\cdot|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$

imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0,1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. For more details on multivalued maps see the books of Aubin and Cellina [1], Aubin and Frankowska [2], Deimling [6] and Hu and Papageorgiou [9].

Definition 2.1 *A multivalued map $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if*

(i) $t \longmapsto F(t,u)$ *is measurable for each* $u \in \mathbb{R}$;

(ii) $u \longmapsto F(t,u)$ *is upper semicontinuous for almost all* $t \in [0,1]$;

(iii) for each $q > 0$, there exists $\varphi_q \in L^1([0,1],\mathbb{R}_+)$ such that

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \le \varphi_q(t) \quad \text{for all } \|u\|_\infty \le q \text{ and for a.e. } t \in [0,1].$$

For each $x \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,x} = \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,x(t)) \ a.e.\ t \in [0,1]\}.$$

Let E be a Banach space, X a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times D$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in A$ and $\mathcal{J} \subset [0,1]$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in A$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.2 *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has property (BC) if*

1) N is lower semi-continuous (l.s.c.);

2) N has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C([0,1],\mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$$

by letting

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for } a.e.\ t \in [0,1]\}.$$

The operator $\mathcal{F}$ is called the Nymetzki operator associated with F.

Definition 2.3 *Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nymetzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b),\ d(a, B) = \inf_{b\in B} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [12]).

Definition 2.4 *A multivalued operator $N : X \to P_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \text{for each } x,\ y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following lemmas will be used in the sequel.

Lemma 2.1 *[13]. Let X be a Banach space. Let $F : [0,1] \times \mathbb{R} \longrightarrow P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Γ be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$, then the operator*

$$\begin{aligned} \Gamma \circ S_F : C([0,1], X) &\longrightarrow P_{cp,c}(C([0,1], X)), \\ x &\longmapsto (\Gamma \circ S_F)(x) := \Gamma(S_{F,x}) \end{aligned}$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

Lemma 2.2 *[3]. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1], \mathbb{R}))$ be a multivalued operator which has property (BC). Then N has a continuous selection; i.e., there exists a continuous function (single-valued) $g : Y \to L^1([0,1], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 2.3 *[5] Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $Fix N \neq \emptyset$.*

3 Main Results

In this section, we are concerned with the existence of solutions for the problem (1)-(3) when the right hand side has convex as well as nonconvex values. Initially, we assume that F is a compact and convex valued multivalued map.

In the following, we introduce first the space

$PC^1([0,1],\mathbb{R}) = \{x : J \longrightarrow \mathbb{R} : x(t)$ is continuously differentiable everywhere except for some t_k at which $x'(t_k^-)$ and $x'(t_k^+), k = 1,\dots,m$ exist and $x'(t_k^-) = x'(t_k)\}$.

It is clear that $PC^1([0,1],\mathbb{R})$ is a Banach space with norm

$$\|x\|_{PC^1} = \max\{\|x\|_\infty, \|x'\|_\infty\},$$

where

$$\|x\|_\infty = \sup\{|x(t)| : t \in [0,1]\}, \quad \|x'\|_\infty = \sup\{|x'(t)| : t \in [0,1]\}.$$

Definition 3.1 *A function $x \in PC^1([0,1],\mathbb{R}) \cap AC^2(J',\mathbb{R})$ is said to be a solution of (1)–(3), if there exists a function $v \in L^1([0,1],\mathbb{R})$ with $v(t) \in F(t,x(t))$, for a.e. $t \in [0,1]$, such that $x''(t) + k^2x(t) = v(t)$ a.e. on J', and for $k = 1,\dots,m$ the function x satisfies the condition $x'(t_k^+) - x'(t_k^-) = I_k(x(t_k^-))$, and the boundary conditions $x'(0) = 0 = x'(1)$.*

We need the following modified version of Lemma 2.3 from [14].

Lemma 3.1 *Suppose $\sigma : [0,1] \to \mathbb{R}$ is continuous. Then the following problem*

$$x''(t) + k^2x(t) = \sigma(t), \ a.e.\ t \in [0,1], \ k \in (0,\pi/2)$$

$$\Delta x'|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1,\dots,m,$$

$$x'(0) = 0, \ x'(1) = 0,$$

has a unique solution $x \in AC^1((0,1),\mathbb{R})$ with the representation

$$x(t) = \int_0^1 G(t,s)\sigma(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k^-)),$$

where $G(t,s)$ is the Green function associated to the correspondinh homogeneous problem

$$x''(t) + k^2x(t) = 0, \ a.e.\ t \in [0,1], \ k \in (0,\pi/2)$$

$$x'(0) = 0, \ x'(1) = 0,$$

given by

$$G(t,s) = \begin{cases} \dfrac{1}{k\sin k}\cos k(1-t)\cos ks, & 0 \le s \le t \le 1, \\ \dfrac{1}{k\sin k}\cos k(1-s)\cos kt, & 0 \le t \le s \le 1. \end{cases}$$

It is easy to prove the following properties of the Green's function:

(I) $G(t,s) \geq 0$ for any $(t,s) \in [0,1] \times [0,1]$,

(II) $G(t,s) \leq G_0 := \dfrac{1}{k \sin k}$ for any $(t,s) \in [0,1] \times [0,1]$,

Theorem 3.1 *Suppose that:*

(H1) the function $F : [0,1] \times \mathbb{R} \to P_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;

(H2) there exist a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and a function $p \in L^1([0,1],\mathbb{R}_+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,u)\} \leq p(t)\psi(\|u\|_\infty)$$

for each $(t,u) \in [0,1] \times \mathbb{R}$;

(H3) there exists a continuous non-decreasing function $\Omega : [0,\infty) \longrightarrow [0,\infty)$ such that

$$|I_k(u)| \leq \Omega(\|u\|_\infty)$$

for each $(t,u) \in [0,1] \times \mathbb{R}$, $k = 1,2,\ldots,m$;

(H4) there exists a number $M > 0$ such that

$$\frac{M}{\frac{1}{k\ \sin k}\left[\psi(M)\|p\|_{L^1} + m\Omega(M)\right]} > 1.$$

Then the BVP (1)–(3) has at least one solution.

Proof. Consider the operator

$$N(x) := \left\{ h \in C([0,1],\mathbb{R}) : h(t) = \int_0^1 G(t,s)v(s)ds + \sum_{k=1}^m G(t,t_k)I_k(x(t_k^-)),\ \ v \in S_{F,x} \right\}.$$

We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

Step 1: *$N(x)$ is convex for each $x \in C([0,1],\mathbb{R})$.*

Indeed, if $h_1,\ h_2$ belong to $N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in [0,1]$ we have

$$h_i(t) = \int_0^1 G(t,s)v_i(s)ds + \sum_{k=1}^m G(t,t_k)I_k(x(t_k^-)),\ (i = 1,2).$$

Let $0 \le d \le 1$. Then, for each $t \in [0,1]$, we have

$$(dh_1 + (1-d)h_2)(t) = \int_0^1 G(t,s)[dv_1(s) + (1-d)v_2(s)]ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k^-)).$$

Since $S_{F,x}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in N(x).$$

Step 2: *N maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.*

Let $B_q = \{x \in C([0,1],\mathbb{R}) : \|x\|_\infty \le q\}$ be a bounded set in $C([0,1],\mathbb{R})$ and $x \in B_q$. Then for each $h \in N(x)$, there exists $v \in S_{F,x}$ such that

$$h(t) = \int_0^1 G(t,s)v(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k^-)).$$

Then we have

$$\begin{aligned} |h(t)| &\le \int_0^1 |G(t,s)||v(s)|ds + \sum_{k=1}^{m} |G(t,t_k)||I_k(x(t_k^-))| \\ &\le \frac{1}{k\,\sin k}\left[\int_0^1 |v(s)|ds + m\Omega(q)\right] \\ &\le \frac{1}{k\,\sin k}\left[\int_0^1 \varphi_q(s)ds + m\Omega(q)\right]. \end{aligned}$$

Thus

$$\|h\|_\infty \le \frac{1}{k\,\sin k}\left[\int_0^1 \varphi_q(s)ds + m\Omega(q)\right].$$

Step 3: *N maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$.*

Let $r_1, r_2 \in [0,1]$, $r_1 < r_2$ and B_q be a bounded set of $C([0,1],\mathbb{R})$ as in Step 2 and $x \in B_q$. For each $h \in N(x)$

$$\begin{aligned} |h(r_2) - h(r_1)| &\le \int_0^1 |G(r_2,s) - G(r_1,s)||v(s)|ds \\ &\quad + \sum_{k=1}^{m} |G(r_2,t_k) - G(r_1,t_k)||I_k(x(t_k^-))| \\ &\le \int_0^1 |G(r_2,s) - G(r_1,s)|\varphi_q(s)ds \\ &\quad + \sum_{k=1}^{m} |G(r_2,t_k) - G(r_1,t_k)||I_k(x(t_k^-))|. \end{aligned}$$

The right hand side tends to zero as $r_2 - r_1 \to 0$. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli Theorem, we can conclude that $N : C([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

Step 4: *N has a closed graph.*

Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(x_*)$. $h_n \in N(x_n)$ means that there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$h_n(t) = \int_0^1 G(t,s)v_n(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k^-)).$$

We must show that there exists $h_* \in S_{F,x_*}$ such that, for each $t \in [0,1]$,

$$h_*(t) = \int_0^1 G(t,s)v_*(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k^-)).$$

Clearly we have

$$\|h_n - h_*\|_\infty \longrightarrow 0 \text{ as } n \to \infty.$$

Consider the continuous linear operator

$$\Gamma : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$$

defined by

$$v \longmapsto (\Gamma v)(t) = \int_0^1 G(t,s)v(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k^-)).$$

From Lemma 2.1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$h_n(t) \in \Gamma(S_{F,x_n}).$$

Since $x_n \to x_*$, it follows from Lemma 2.1 that

$$h_*(t) = \int_0^1 G(t,s)v_*(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k^-))$$

for some $v_* \in S_{F,x_*}$.

Step 5: *A priori bounds on solutions.*

Let x be a possible solution of the problem (1)–(3). Then, there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$x(t) = \int_0^1 G(t,s)v(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k^-)).$$

This implies by (H2) that, for each $t \in [0,1]$, we have

$$\begin{aligned} |x(t)| &\leq \frac{1}{k\,\sin k}\left[\int_0^1 p(s)\psi(\|x\|_\infty)ds + \sum_{k=1}^{m}\Omega(x(t_k))\right] \\ &\leq \frac{1}{k\,\sin k}\left[\psi(\|x\|_\infty)\int_0^1 p(s)ds + m\Omega(\|x\|_\infty)\right]. \end{aligned}$$

Consequently

$$\frac{\|x\|_\infty}{\dfrac{1}{k\,\sin k}\left[\psi(\|x\|_\infty)\|p\|_{L^1} + m\Omega(\|x\|_\infty)\right]} \leq 1.$$

Then by (H3), there exists M such that $\|x\|_\infty \neq M$.

Let

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\|_\infty < M+1\}.$$

The operator $N : \overline{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [7], we deduce that N has a fixed point x in $\overline{U}$ which is a solution of the problem (1)–(3). This completes the proof. □

Next, we study the case where F is not necessarily convex valued. Our approach here is based on the nonlinear alternative of Leray Schauder type combined with the selection theorem of Bresssan and Colombo for lower semi-continuous maps with decomposable values.

Theorem 3.2 *Suppose that:*

(H5) $F : [0,1]\times\mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that:*
 a) $(t,u) \mapsto F(t,u)$ *is* $\mathcal{L}\otimes\mathcal{B}$ *measurable;*
 b) $u \mapsto F(t,u)$ *is lower semi-continuous for each* $t \in [0,1]$.

(H6) for each $\rho > 0$*, there exists* $\varphi_\rho \in L^1([0,1],\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \leq \varphi_\rho(t) \text{ for all } \|u\|_\infty \leq \rho \text{ and for a.e. } t \in [0,1].$$

In addition assume that (H2), (H3) and (H4) hold. Then the BVP (1)–(3) has at least one solution.

Proof. Note that (H5) and (H6) imply that F is of l.s.c. type. Then from Lemma 2.2, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$x''(t) + k^2 x(t) = f(x(t)), \text{ a.e. } t \in J' := [0,1] \setminus \{t_1, \dots, t_m\}, \tag{4}$$

$$\Delta x'|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \dots, m, \tag{5}$$

$$x'(0) = x'(1) = 0. \tag{6}$$

It is clear that if $x \in PC^1([0,1], \mathbb{R}) \cap AC^2(J', \mathbb{R})$ is a solution of (4)–(6), then x is a solution to the problem (1)–(3). Transform the problem (4)–(6) into a fixed point theorem. Consider the operator $\bar{N}$ defined by

$$(\bar{N}x)(t) := \int_0^1 G(t,s) f(x(s)) ds + \sum_{k=1}^{m} G(t, t_k) I_k(x(t_k^-)), \ \ t \in J.$$

We can easily show that $\bar{N}$ is continuous and completely continuous. The remainder of the proof is similar to that of Theorem 3.1. □

We present now a result for the problem (1)-(3) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem for multivalued map given by Covitz and Nadler [5].

Theorem 3.3 *Suppose that:*

(H7) $F : [0,1] \times \mathbb{R} \longrightarrow P_{cp}(\mathbb{R})$ *has the property that* $F(\cdot, u) : [0,1] \to P_{cp}(\mathbb{R})$ *is measurable for each* $u \in \mathbb{R}$*;*

(H8) $H_d(F(t,u), F(t,\overline{u})) \le l(t)|u - \overline{u}|$ *for almost all* $t \in [0,1]$ *and* $u, \ \overline{u} \in \mathbb{R}$ *where* $l \in L^1([0,1], \mathbb{R})$ *and* $d(0, F(t,0)) \le l(t)$ *for almost all* $t \in [0,1]$*;*

(H9) there exist constants c_k *such that*

$$|I_k(x) - I_k(\bar{x})| \le c_k |x - \bar{x}|, \quad k = 1, 2, \dots, m, \ \forall x, \bar{x} \in \mathbb{R}.$$

If

$$\frac{1}{k \sin k} \left[\|l\|_{L^1} + m c_k \right] < 1,$$

then the BVP (1)-(3) has at least one solution.

Remark 3.1 *For each* $x \in C([0,1], \mathbb{R})$*, the set* $S_{F,x}$ *is nonempty since by (H7), F has a measurable selection (see [4], Theorem III.6).*

Proof. We shall show that N satisfies the assumptions of Lemma 2.3. The proof will be given in two steps.

Step 1: $N(x) \in P_{cl}(C([0,1], \mathbb{R}))$ *for each* $x \in C([0,1], \mathbb{R})$.

Indeed, let $(x_n)_{n\geq 0} \in N(x)$ such that $x_n \longrightarrow \tilde{x}$ in $C([0,1],\mathbb{R})$. Then, $\tilde{x} \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$x_n(t) = \int_0^1 G(t,s)v_n(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x_n(t_k^-)).$$

Using the fact that F has compact values and from (H5), we may pass to a subsequence if necessary to get that v_n converges to v in $L^1([0,1],\mathbb{R})$ and hence $v \in S_{F,x}$. Then, for each $t \in [0,1]$,

$$x_n(t) \longrightarrow \tilde{x}(t) = \int_0^1 G(t,s)v(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(\tilde{x}(t_k^-)).$$

So, $\tilde{x} \in N(x)$.

Step 2: *There exists* $\gamma < 1$ *such that*

$$H_d(N(x), N(\overline{x})) \leq \gamma \|x - \overline{x}\|_\infty \text{ for each } x, \overline{x} \in C([0,1],\mathbb{R}).$$

Let $x, \overline{x} \in C([0,1],\mathbb{R})$ and $h_1 \in N(x)$. Then, there exists $v_1(t) \in F(t,x(t))$ such that for each $t \in [0,1]$

$$h_1(t) = \int_0^1 G(t,s)v_1(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k^-)).$$

From (H8) it follows that

$$H_d(F(t,x(t)), F(t,\overline{x}(t))) \leq l(t)|x(t) - \overline{x}(t)|.$$

Hence, there exists $w \in F(t,\overline{x}(t))$ such that

$$|v_1(t) - w| \leq l(t)|x(t) - \overline{x}(t)|,\ t \in [0,1].$$

Consider $U : [0,1] \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)|x(t) - \overline{x}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\overline{x}(t))$ is measurable (see Proposition III.4 in [4]), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2(t) \in F(t,\overline{x}(t))$, and for each $t \in [0,1]$,

$$|v_1(t) - v_2(t)| \leq l(t)|x(t) - \overline{x}(t)|.$$

Let us define for each $t \in [0,1]$

$$h_2(t) = \int_0^1 G(t,s)v_2(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k^-)).$$

We have

$$\begin{aligned}|h_1(t)-h_2(t)| &\le \int_0^1 |G(t,s)||v_1(s)-v_2(s)|ds + \sum_{k=1}^{m} |G(t,t_k)|\, c_k|x(s)-\bar{x}(s)| \\ &\le \frac{1}{k\,\sin k}\int_0^1 l(s)\|x-\overline{x}\|ds + mc_k\frac{1}{k\,\sin k}\|x-\bar{x}\|.\end{aligned}$$

Thus

$$\|h_1-h_2\|_\infty \le \frac{1}{k\,\sin k}\left[\|l\|_{L^1}+mc_k\right]\|x-\overline{x}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of x and $\overline{x}$, it follows that

$$H_d(N(x),N(\overline{x})) \le \frac{1}{k\,\sin k}\left[\|l\|_{L^1}+mc_k\right]\|x-\overline{x}\|_\infty.$$

So, N is a contraction and thus, by Lemma 2.3, N has a fixed point x which is solution to (1)–(3). The proof is complete. □

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