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SOLUTIONS FOR SINGULAR VOLTERRA INTEGRAL EQUATIONS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We consider the system of Volterra integral equations

$$u_i(t) = \int_0^t g_i(t,s)[P_i(s,u_1(s),u_2(s),\cdots,u_n(s)) \\ + \ Q_i(s,u_1(s),u_2(s),\cdots,u_n(s))]ds, \quad t\in[0,T],\ 1\le i\le n$$

where $T>0$ is fixed and the nonlinearities $P_i(t,u_1,u_2,\cdots,u_n)$ can be *singular* at $t=0$ and $u_j=0$ where $j\in\{1,2,\cdots,n\}$. Criteria are offered for the existence of *fixed-sign solutions* $(u_1^*,u_2^*,\cdots,u_n^*)$ to the system of Volterra integral equations, i.e., $\theta_i u_i^*(t)\ge 0$ for $t\in[0,1]$ and $1\le i\le n$, where $\theta_i\in\{1,-1\}$ is fixed. We also include an example to illustrate the usefulness of the results obtained.

Key words and phrases: Fixed-sign solutions, singularities, Volterra integral equations.
AMS (MOS) Subject Classifications: 45B05

1 Introduction

In this paper we shall consider the system of Volterra integral equations

$$u_i(t) = \int_0^t g_i(t,s)[P_i(s,u_1(s),u_2(s),\cdots,u_n(s)) + Q_i(s,u_1(s),u_2(s),\cdots,u_n(s))]ds, \\ t\in[0,T],\ 1\le i\le n \tag{1.1}$$

where $T>0$ is fixed. The nonlinearities $P_i(t,u_1,u_2,\ \cdots,u_n)$ can be *singular* at $t=0$ and $u_j=0$ where $j\in\{1,2,\cdots,n\}$.

Throughout, let $u=(u_1,u_2,\cdots,u_n)$. We are interested in establishing the existence of solutions u of the system (1.1) in $(C[0,T])^n = C[0,T]\times C[0,T]\times\cdots\times C[0,T]$ (n

times). Moreover, we are concerned with *fixed-sign* solutions u, by which we mean $\theta_i u_i(t) \geq 0$ for all $t \in [0,T]$ and $1 \leq i \leq n$, where $\theta_i \in \{1,-1\}$ is fixed. Note that *positive* solution is a special case of fixed-sign solution when $\theta_i = 1$ for $1 \leq i \leq n$.

The system (1.1) when $P_i = 0,\ 1 \leq i \leq n$ reduces to

$$u_i(t) = \int_0^t g_i(t,s) Q_i(s, u_1(s), u_2(s), \cdots, u_n(s)) ds, \quad t \in [0,T],\ 1 \leq i \leq n. \tag{1.2}$$

This equation when $n = 1$ has received a lot of attention in the literature [12, 13, 14, 16, 17, 18, 19], since it arises in real-world problems. For example, astrophysical problems (e.g., the study of the density of stars) give rise to the Emden differential equation

$$\begin{cases} y'' - t^p y^q = 0, \quad t \in [0,T] \\ y(0) = y'(0) = 0, \quad p \geq 0, \quad 0 < q < 1 \end{cases}$$

which reduces to $(1.2)|_{n=1}$ when $g_1(t,s) = (t-s)s^p$ and $Q_1(t,y) = y^q$. Other examples occur in nonlinear diffusion and percolation problems (see [13, 14] and the references cited therein), and here we get (1.2) where g_i is a convolution kernel, i.e.,

$$u_i(t) = \int_0^t g_i(t-s) Q_i(s, u_1(s), u_2(s), \cdots, u_n(s)) ds, \quad t \in [0,T],\ 1 \leq i \leq n.$$

In particular, Bushell and Okrasiński [13] investigated a special case of the above system given by

$$y(t) = \int_0^t (t-s)^{\gamma-1} Q(y(s)) ds, \quad t \in [0,T]$$

where $\gamma > 1$.

In the literature, the conditions placed on the kernels g_i are not natural. A new approach is thus employed in this paper to present new results for (1.1). In particular, new "lower type inequalities" on the solutions are presented. Our results extend, improve and complement the existing theory in the literature [1, 2, 3, 4, 11, 15, 20, 21, 22]. We have generalized the problems to (i) *systems*, (ii) *general* form of nonlinearities $P_i,\ 1 \leq i \leq n$ that can be singular in *both* independent and dependent variables, (iii) existence of *fixed-sign* solutions, which include *positive* solutions as special case. Other related work on *systems* of integral equations can be found in [5, 6, 7, 8, 9, 10, 23]. Note that the technique employed in singular integral equations [10] is entirely different from the present work.

2 Main Results

Let the real Banach space $B = (C[0,T])^n$ be equipped with the norm

$$\|u\| = \max_{1 \leq i \leq n} \sup_{t \in [0,T]} |u_i(t)|.$$

Our main tool is the following theorem.

Theorem 2.1 *Consider the system*

$$u_i(t) = c_i(t) + \int_0^t g_i(t,s) f_i(s, u(s))ds, \quad t \in [0,T], \ 1 \le i \le n \tag{2.1}$$

where $T > 0$ is fixed. Let $1 \le p \le \infty$ be an integer and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume the following conditions hold for each $1 \le i \le n$:

(C_1) $c_i \in C[0,T]$;

(C_2) $f_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is a L^q-Carathéodory function, i.e., (i) the map $u \mapsto f_i(t,u)$ is continuous for almost all $t \in [0,T]$, (ii) the map $t \mapsto f_i(t,u)$ is measurable for all $u \in \mathbb{R}^n$, (iii) for any $r > 0$, there exists $\mu_{r,i} \in L^q[0,T]$ such that $\|u\| \le r$ ($\|u\|$ denotes the norm in $\mathbb{R}^n$) implies $|f_i(t,u)| \le \mu_{r,i}(t)$ for almost all $t \in [0,T]$;*

(C_3) $g_i(t,s) : \Delta \to \mathbb{R}$, *where* $\Delta = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le T\}$, $g_i^t(s) = g_i(t,s) \in L^p[0,t]$ *for each* $t \in [0,T]$, *and*

$$\sup_{t \in [0,T]} \int_0^t |g_i^t(s)|^p \, ds < \infty, \quad 1 \le p < \infty$$
$$\sup_{t \in [0,T]} \ ess \sup_{s \in [0,t]} |g_i^t(s)| < \infty, \quad p = \infty;$$

and

(C_4) for any $t, t' \in [0,T]$ with $t^* = \min\{t, t'\}$, we have

$$\int_0^{t^*} |g_i^t(s) - g_i^{t'}(s)|^p \, ds \to 0 \ as \ t \to t', \quad 1 \le p < \infty$$
$$ess \sup_{s \in [0,t^*]} |g_i^t(s) - g_i^{t'}(s)|^p \to 0 \ as \ t \to t', \quad p = \infty.$$

In addition, suppose there is a constant $M > 0$, independent of λ, with $\|u\| \ne M$ for any solution $u \in (C[0,T])^n$ to

$$u_i(t) = c_i(t) + \lambda \int_0^t g_i(t,s) f_i(s, u(s))ds, \quad t \in [0,T], \ 1 \le i \le n \tag{2.2$_\lambda$}$$

for each $\lambda \in (0,1)$. Then, (2.1) has at least one solution in $(C[0,T])^n$.

Proof. For each $1 \le i \le n$, define

$$g_i^*(t,s) = \begin{cases} g_i(t,s), & 0 \le s \le t \le T \\ 0, & 0 \le t \le s \le T. \end{cases}$$

Then, (2.1) is equivalent to

$$u_i(t) = c_i(t) + \int_0^T g_i^*(t,s) f_i(s, u(s)) ds, \quad t \in [0,T], \ 1 \le i \le n. \tag{2.3}$$

Now, the system (2.3) (or equivalently (2.1)) has at least one solution in $(C[0,T])^n$ by Theorem 2.1 in [23], which is stated as follows: Consider the system below

$$u_i(t) = c_i(t) + \int_0^T g_i(t,s) f_i(s, u(s)) ds, \quad t \in [0,T], \ 1 \le i \le n \tag{$*$}$$

where the following conditions hold for each $1 \le i \le n$ and for some integers p, q such that $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$: ($C_1$), ($C_2$), $g_i(t,s) \in [0,T]^2 \to \mathbb{R}$, and $g_i^t(s) = g_i(t,s) \in L^p[0,T]$ for each $t \in [0,T]$. Further, suppose there is a constant $M > 0$, independent of λ, with $\|u\| \ne M$ for any solution $u \in (C[0,T])^n$ to

$$u_i(t) = c_i(t) + \lambda \int_0^T g_i(t,s) f_i(s, u(s)) ds, \quad t \in [0,T], \ 1 \le i \le n$$

for each $\lambda \in (0,1)$. Then, ($*$) has at least one solution in $(C[0,T])^n$. □

Remark 2.1 If (C_4) is changed to

(C_4)′ for any $t, t' \in [0,T]$ with $t^* = \min\{t, t'\}$ and $t^{**} = \max\{t, t'\}$, we have for $1 \le p < \infty$,

$$\int_0^{t^*} |g_i(t,s) - g_i(t',s)|^p \ ds + \int_{t^*}^{t^{**}} |g_i(t^{**},s)|^p \ ds \to 0$$

as $t \to t'$, and for $p = \infty$,

$$\text{ess} \sup_{s \in [0,t^*]} |g_i(t,s) - g_i(t',s)| + \text{ess} \sup_{s \in [t^*,t^{**}]} |g_i(t^{**},s)| \to 0$$

as $t \to t'$,

then automatically we have the inequalities in (C_3).

We shall now apply Theorem 2.1 to obtain an existence result for (1.1). Let $\theta_i \in \{-1, 1\}$, $1 \le i \le n$ be fixed. For each $1 \le j \le n$, we define

$$[0,\infty)_j = \begin{cases} [0,\infty), & \theta_j = 1 \\ (-\infty, 0], & \theta_j = -1 \end{cases}$$

and $(0,\infty)_j$ is similarly defined.

Theorem 2.2 *Let $\theta_i \in \{-1, 1\}$, $1 \le i \le n$ be fixed and let the following conditions be satisfied for each $1 \le i \le n$:*

(I_1) $P_i : (0,T] \times (\mathbb{R}\backslash\{0\})^n \to \mathbb{R}$, $\theta_i P_i(t,u) > 0$ *and is continuous for* $(t,u) \in (0,T] \times \prod_{j=1}^n (0,\infty)_j$, $Q_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $\theta_i Q_i(t,u) \geq 0$ *and is continuous for* $(t,u) \in [0,T] \times \prod_{j=1}^n [0,\infty)_j$;

(I_2) $\theta_i P_i$ *is 'nonincreasing' in* u, *i.e., if* $\theta_j u_j \geq \theta_j v_j$ *for some* $j \in \{1,2,\cdots,n\}$, *then*

$$\theta_i P_i(t,u_1,\cdots,u_j,\cdots,u_n) \leq \theta_i P_i(t,u_1,\cdots,v_j,\cdots,u_n), \quad t \in (0,T];$$

(I_3) *there exist nonnegative* r_i *and* γ_i *such that* $r_i \in C(0,T]$, $\gamma_i \in C(0,\infty)$, $\gamma_i > 0$ *is nonincreasing, and*

$$\theta_i P_i(t,u) \geq r_i(t)\gamma_i(|u_i|), \quad (t,u) \in (0,T] \times \prod_{j=1}^n (0,\infty)_j;$$

(I_4) *there exist nonnegative* d_i *and* h_{ij}, $1 \leq j \leq n$ *such that* $d_i \in C[0,T]$, $h_{ij} \in C(0,\infty)$, h_{ij} *is nondecreasing, and*

$$\frac{Q_i(t,u)}{P_i(t,u)} \leq d_i(t)h_{i1}(|u_1|)h_{i2}(|u_2|)\cdots h_{in}(|u_n|), \quad (t,u) \in (0,T] \times \prod_{j=1}^n (0,\infty)_j;$$

(I_5) $g_i(t,s) : \Delta \to \mathbb{R}$, $g_i^t(s) = g_i(t,s) \in L^1[0,t]$ *for each* $t \in [0,T]$, *and*

$$\sup_{t\in[0,T]} \int_0^t |g_i^t(s)|ds < \infty;$$

(I_6) *for any* $t, t' \in [0,T]$ *with* $t^* = \min\{t,t'\}$, *we have* $\int_0^{t^*} |g_i^t(s) - g_i^{t'}(s)|ds \to 0$ *as* $t \to t'$;

(I_7) *for each* $t \in [0,T]$, $g_i(t,s) \geq 0$ *for a.e.* $s \in [0,t]$;

(I_8) *for any* $t_1, t_2 \in (0,T]$ *with* $t_1 < t_2$, *we have*

$$g_i(t_1,s) \leq g_i(t_2,s), \quad a.e.\ s \in [0,t_1];$$

(I_9) *for any* $k_j \in \{1,2,...\}$, $1 \leq j \leq n$, *we have*

$$\sup_{t\in[0,T]} \int_0^t g_i(t,s)\theta_i P_i\left(s, \frac{\theta_1}{k_1}, \cdots, \frac{\theta_n}{k_n}\right) ds < \infty,$$

$$\sup_{s\in[0,T]} \int_0^s g_i(s,x)r_i(x)dx < \infty,$$

$$\int_0^s g_i(s,x)r_i(x)dx > 0, \quad a.e.\ s \in [0,T],$$

$$\sup_{t\in[0,T]}\int_0^t g_i(t,s)\theta_i P_i(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s))ds<\infty$$

where

$$\beta_i(s)=G_i^{-1}\left(\int_0^s g_i(s,x)r_i(x)dx\right)$$

for $s\in[0,T]$ *and*

$$G_i(z)=\frac{z}{\gamma_i(z)}$$

for $z>0$*, with* $G_i(0)=0=G_i^{-1}(0)$*;*

(I_{10}*) there exists* $\rho_i\in C[0,T]$ *such that for* $t,x\in[0,T]$ *with* $t<x$*, we have*

$$\int_0^t [g_i(x,s)-g_i(t,s)]\theta_i P_i(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s))ds\le|\rho_i(x)-\rho_i(t)|;$$

and

(I_{11}*) if* $z>0$ *satisfies*

$$z\le K+L\left\{1+\max_{1\le i\le n}\left[\sup_{t\in[0,T]}d_i(t)\right]\left[\prod_{j=1}^n h_{ij}(z)\right]\right\}$$

for some constants $K,L\ge 0$*, then there exists a constant* M *(which may depend on* K *and* L*) such that* $z\le M$*.*

Then, (1.1) has a fixed-sign solution $u\in(C[0,T])^n$ *with*

$$\theta_i u_i(t)\ge\beta_i(t)$$

for $t\in[0,T]$ *and* $1\le i\le n$ *(*β_i *is defined in (*I_9*)).*

Proof. Let $N=\{1,2,\cdots\}$ and $k=(k_1,k_2,\cdots,k_n)\in N^n$. First, we shall show that the nonsingular system

$$u_i(t)=\frac{\theta_i}{k_i}+\int_0^t g_i(t,s)\left[P_i^*(s,u(s))+Q_i^*(s,u(s))\right]ds,\quad t\in[0,T],\ 1\le i\le n \qquad (2.4)^k$$

has a solution for each $k\in N^n$, where

$$P_i^*(t,u_1,\cdots,u_n)=\begin{cases}P_i(t,v_1,\cdots,v_n), & t\in(0,T]\\ 0, & t=0\end{cases}$$

with

$$v_j=\begin{cases}u_j, & \theta_j u_j\ge\dfrac{1}{k_j}\\[2mm] \dfrac{\theta_j}{k_j}, & \theta_j u_j\le\dfrac{1}{k_j}\end{cases}$$

and

$$Q_i^*(t, u_1, \cdots, u_n) = Q_i(t, w_1, \cdots, w_n), \; t \in [0, T]$$

with

$$w_j = \begin{cases} u_j, & \theta_j u_j \ge 0 \\ 0, & \theta_j u_j \le 0. \end{cases}$$

Let $k \in N^n$ be fixed. We shall use Theorem 2.1 to show that $(2.4)^k$ has a solution. Note that conditions (C_1)–(C_4) are satisfied with $p = 1$ and $q = \infty$. We need to consider the family of problems

$$u_i(t) = \frac{\theta_i}{k_i} + \lambda \int_0^t g_i(t,s) \left[P_i^*(s, u(s)) + Q_i^*(s, u(s))\right] ds, \quad t \in [0, T], \; 1 \le i \le n \quad (2.5)_\lambda^k$$

where $\lambda \in (0, 1)$. Let $u \in (C[0, T])^n$ be any solution of $(2.5)_\lambda^k$. Clearly, for each $1 \le i \le n$,

$$\theta_i u_i(t) \ge \frac{1}{k_i} > 0, \; t \in [0, T]$$

and so $P_i^*(t, u(t)) = P_i(t, u(t))$ for $t \in (0, T]$ and $Q_i^*(t, u(t)) = Q_i(t, u(t))$ for $t \in [0, T]$. Applying (I_2) and (I_4), we find for $t \in [0, T]$ and $1 \le i \le n$,

$$\begin{aligned}
&|u_i(t)| = \theta_i u_i(t) \\
&= \frac{1}{k_i} + \lambda \int_0^t g_i(t,s) \left[\theta_i P_i^*(s, u(s)) + \theta_i Q_i^*(s, u(s))\right] ds \\
&= \frac{1}{k_i} + \lambda \int_0^t g_i(t,s) \theta_i P_i(s, u(s)) \left[1 + \frac{Q_i(s, u(s))}{P_i(s, u(s))}\right] ds \\
&\le 1 + \int_0^t g_i(t,s) \theta_i P_i\left(s, \frac{\theta_1}{k_1}, \cdots, \frac{\theta_n}{k_n}\right) \left[1 + d_i(s) \prod_{j=1}^n h_{ij}(|u_j(s)|)\right] ds \\
&\le 1 + C_i(1 + D_i)
\end{aligned}$$

where

$$C_i = \sup_{t \in [0,T]} \int_0^t g_i(t,s) \theta_i P_i\left(s, \frac{\theta_1}{k_1}, \cdots, \frac{\theta_n}{k_n}\right) ds$$

and

$$D_i = \left[\sup_{s \in [0,T]} d_i(s)\right] \prod_{j=1}^n h_{ij}(\|u\|).$$

Thus,

$$\|u\| \le 1 + \left(\max_{1 \le i \le n} C_i\right) \left(1 + \max_{1 \le i \le n} D_i\right)$$

and so by (I_{11}) there exists a constant M_k with $\|u\| \le M_k$. Theorem 2.1 now guarantees that $(2.4)^k$ has a solution $u^k \in (C[0, T])^n$ with $\theta_i u_i^k(t) \ge \frac{1}{k_i}$ for $t \in [0, T]$ and $1 \le i \le n$.

Consequently, $P_i^*(t,u^k(t)) = P_i(t,u^k(t))$, $Q_i^*(t,u^k(t)) = Q_i(t,u^k(t))$ and u^k is a solution of the system

$$u_i(t) = \frac{\theta_i}{k_i} + \int_0^t g_i(t,s)[P_i(s,u(s)) + Q_i(s,u(s))]ds, \quad t \in [0,T],\ 1 \le i \le n. \tag{2.6}$$

Moreover, $\theta_i u_i^k$ is nondecreasing on $(0,T)$, since for $t, x \in (0,T)$ with $t < x$,

$$\begin{aligned}
&\theta_i u_i^k(x) - \theta_i u_i^k(t) \\
&= \int_0^t [g_i(x,s) - g_i(t,s)] \left[\theta_i P_i(s,u^k(s)) + \theta_i Q_i(s,u^k(s))\right] ds \\
&\quad + \int_t^x g_i(x,s) \left[\theta_i P_i(s,u^k(s)) + \theta_i Q_i(s,u^k(s))\right] ds \\
&\ge 0
\end{aligned}$$

where we have made use of (I_1), (I_7) and (I_8).

Next, we shall obtain a solution to (1.1) by means of the Arzéla-Ascoli theorem, as a limit of solutions of $(2.4)^k$ (as $k_i \to \infty,\ 1 \le i \le n$). For this we shall show that

$$\{u^k\}_{k \in N^n} \text{ is a bounded and equicontinuous family on } [0,T]. \tag{2.7}$$

To proceed we need to obtain a lower bound for $\theta_i u_i^k(t),\ t \in [0,T],\ 1 \le i \le n$. Using (I_3) and the fact that $\theta_i u_i^k = |u_i^k|$ is nondecreasing on $(0,T)$, we get

$$\begin{aligned}
|u_i^k(t)| &= \theta_i u_i^k(t) \\
&= \frac{1}{k_i} + \int_0^t g_i(t,s)[\theta_i P_i(s,u^k(s)) + \theta_i Q_i(s,u^k(s))]ds \\
&\ge \int_0^t g_i(t,s)\theta_i P_i(s,u^k(s))ds \\
&\ge \int_0^t g_i(t,s) r_i(s) \gamma_i(|u_i^k(s)|)ds \\
&\ge \gamma_i(|u_i^k(t)|) \int_0^t g_i(t,s) r_i(s) ds
\end{aligned}$$

or equivalently

$$G_i(|u_i^k(t)|) = \frac{|u_i^k(t)|}{\gamma_i(|u_i^k(t)|)} \ge \int_0^t g_i(t,s) r_i(s) ds.$$

Noting that G_i is an increasing function (since γ_i is nonincreasing), we have

$$\theta_i u_i^k(t) = |u_i^k(t)| \ge G_i^{-1}\left(\int_0^t g_i(t,s) r_i(s) ds\right) = \beta_i(t), \quad t \in [0,T],\ 1 \le i \le n \tag{2.8}$$

for each $k \in N^n$.

We shall now show that $\{u^k\}_{k\in N^n}$ is a bounded family on $[0,T]$. Fix $k \in N^n$. Using (I_2), (2.8) and (I_4), we obtain for $t \in [0,T]$ and $1 \le i \le n$,

$$\begin{aligned}
|u_i^k(t)| &= \theta_i u_i^k(t) \\
&= \frac{1}{k_i} + \int_0^t g_i(t,s)\theta_i P_i(s,u^k(s)) \left[1 + \frac{Q_i(s,u^k(s))}{P_i(s,u^k(s))}\right] ds \\
&\le 1 + \int_0^t g_i(t,s)\theta_i P_i\left(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s)\right) \left[1 + d_i(s)\prod_{j=1}^n h_{ij}(|u_j^k(s)|)\right] ds \\
&\le 1 + E_i(1+D_i)
\end{aligned}$$

where

$$E_i = \sup_{t\in[0,T]} \int_0^t g_i(t,s)\theta_i P_i\left(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s)\right) ds.$$

It follows that

$$\|u^k\| \le 1 + \left(\max_{1\le i\le n} E_i\right)\left(1 + \max_{1\le i\le n} D_i\right)$$

and by (I_{11}) there exists a constant M (independent of k) with $\|u^k\| \le M$. Thus, $\{u^k\}_{k\in N^n}$ is bounded.

Next, we shall show that $\{u^k\}_{k\in N^n}$ is equicontinuous. Let $k \in N^n$ be fixed. For $t, x \in [0,T]$ with $t < x$, using the fact that $\theta_i u_i^k$ is nondecreasing and an earlier technique, we obtain for each $1 \le i \le n$,

$$\begin{aligned}
|u_i^k(x) - u_i^k(t)| &= \theta_i u_i^k(x) - \theta_i u_i^k(t) \\
&= \int_0^t [g_i(x,s) - g_i(t,s)]\theta_i P_i(s,u^k(s)) \left[1 + \frac{Q_i(s,u^k(s))}{P_i(s,u^k(s))}\right] ds \\
&\quad + \int_t^x g_i(x,s)\theta_i P_i(s,u^k(s)) \left[1 + \frac{Q_i(s,u^k(s))}{P_i(s,u^k(s))}\right] ds \\
&\le \left\{\int_0^t [g_i(x,s) - g_i(t,s)]\theta_i P_i\left(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s)\right) ds \right. \\
&\quad \left. + \int_t^x g_i(x,s)\theta_i P_i\left(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s)\right) ds\right\} \\
&\quad \times \left\{1 + \left[\sup_{s\in[0,T]} d_i(s)\right]\prod_{j=1}^n h_{ij}(M)\right\} \\
&\le \left[|\rho_i(x) - \rho_i(t)| + \int_t^x g_i(T,s)\theta_i P_i\left(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s)\right) ds\right] \\
&\quad \times \left\{1 + \left[\sup_{s\in[0,T]} d_i(s)\right]\prod_{j=1}^n h_{ij}(M)\right\}
\end{aligned}$$

where we have used (I_{10}) and (I_8) in the last inequality. This shows that $\{u^k\}_{k\in N^n}$ is an equicontinuous family on $[0,T]$.

Now, the Arzéla-Ascoli theorem guarantees the existence of a subsequence N^* of N, and a function $u^* \in (C[0,T])^n$ with u^k converging uniformly on $[0,T]$ to u^* as $k_i \to \infty,\ 1 \le i \le n$ through N^*. Further,

$$\beta_i(t) \le \theta_i u_i^*(t) \le M, \quad t \in [0,T],\ 1 \le i \le n. \tag{2.9}$$

It remains to show that u^* is indeed a solution of (1.1). Fix $t \in [0,T]$. Then, from (2.6) we have for each $1 \le i \le n$,

$$u_i^k(t) = \frac{\theta_i}{k_i} + \int_0^t g_i(t,s)[P_i(s,u^k(s)) + Q_i(s,u^k(s))]ds.$$

Let $k_i \to \infty$ through N^*, and use the Lebesgue dominated convergence theorem with (I_9), to obtain for each $1 \le i \le n$,

$$u_i^*(t) = \int_0^t g_i(t,s)[P_i(s,u^*(s)) + Q_i(s,u^*(s))]ds.$$

This argument holds for each $t \in [0,T]$, hence u^* is indeed a solution of (1.1). $\square$

Remark 2.2 If (I_6) is changed to

(I_6)$'$ for any $t, t' \in [0,T]$ with $t^* = \min\{t,t'\}$ and $t^{**} = \max\{t,t'\}$, we have

$$\int_0^{t^*} |g_i(t,s) - g_i(t',s)|ds + \int_{t^*}^{t^{**}} |g_i(t^{**},s)|ds \to 0$$

as $t \to t'$,

then automatically we have $\sup_{t\in[0,T]} \int_0^t |g_i^t(s)|ds < \infty$ which appears in (I_5).

Remark 2.3 If $Q_i \equiv 0$, then we can pick $d_i = 0$ in (I_4), and trivially (I_{11}) is satisfied with $M = K + L$.

Remark 2.4 Let p and q be as in Theorem 2.1. Suppose (C_4) and

(C_5) $\displaystyle\int_0^T [\theta_i P_i(s, \theta_1\beta_1(s), \cdots, \theta_n\beta_n(s))]^q\, ds < \infty$

are satisfied. Then, (I_{10}) is not required in Theorem 2.2. In fact, (I_{10}) is only needed to show that $\{u^k\}_{k\in N^n}$ is equicontinuous. Let $k \in N^n$ be fixed. For $t, x \in [0,T]$ with

$t < x$, from the proof of Theorem 2.2 we have for each $1 \le i \le n$,

$$\begin{aligned}
|u_i^k(x) - u_i^k(t)| \;&=\; \theta_i u_i^k(x) - \theta_i u_i^k(t) \\
&\le \left\{ \int_0^t [g_i(x,s) - g_i(t,s)] \theta_i P_i\left(s, \theta_1\beta_1(s), \cdots, \theta_n\beta_n(s)\right) ds \right. \\
&\quad \left. + \int_t^x g_i(x,s) \theta_i P_i\left(s, \theta_1\beta_1(s), \cdots, \theta_n\beta_n(s)\right) ds \right\} \\
&\quad \times \left\{ 1 + \left[\sup_{s\in[0,T]} d_i(s) \right] \prod_{j=1}^n h_{ij}(M) \right\} \\
&\le \left\{ \left(\int_0^t [g_i(x,s) - g_i(t,s)]^p \; ds \right)^{\frac{1}{p}} \left(\int_0^T [\theta_i P_i\left(s, \theta_1\beta_1(s), \cdots, \theta_n\beta_n(s)\right)]^q \; ds \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \int_t^x g_i(T,s) \theta_i P_i\left(s, \theta_1\beta_1(s), \cdots, \theta_n\beta_n(s)\right) ds \right\} \\
&\quad \times \left\{ 1 + \left[\sup_{s\in[0,T]} d_i(s) \right] \prod_{j=1}^n h_{ij}(M) \right\}.
\end{aligned}$$

Hence, in view of (C_4) and (C_5), we see that $\{u^k\}_{k\in N^n}$ is an equicontinuous family on $[0,T]$.

3 Example

Consider the system of singular Volterra integral equations

$$\begin{cases} u_1(t) = \displaystyle\int_0^t (t-s) \left\{ [u_1(s)]^{-a_1} + [u_2(s)]^{-a_2} + [u_1(s)]^{a_3} [u_2(s)]^{a_4} \right\} ds, \;\; t \in [0,T] \\ u_2(t) = \displaystyle\int_0^t (t-s) \left\{ [u_1(s)]^{-b_1} + [u_2(s)]^{-b_2} + [u_1(s)]^{b_3} [u_2(s)]^{b_4} \right\} ds, \;\; t \in [0,T] \end{cases} \tag{3.1}$$

where $a_i, b_i > 0, \; i = 1,2,3,4$ and $T > 0$ are fixed with

$$\begin{gathered} a_1 < 1, \quad b_2 < 1, \quad 2a_2 < b_2 + 1, \\ 2b_1 < a_1 + 1, \quad a_1 + a_3 + a_4 = b_1 + b_3 + b_4 = \frac{1}{3}. \end{gathered} \tag{3.2}$$

(Many a_i and $b_i, \; i = 1,2,3,4$ fulfill (3.2), for instance $a_1 = \frac{1}{6}, \; a_2 < \frac{7}{12}, \; a_3 = \frac{1}{8}, \; a_4 = \frac{1}{24}$, $b_1 = \frac{1}{24}, \; b_2 = b_3 = \frac{1}{6}, \; b_4 = \frac{1}{8}$.)

Here, (3.1) is of the from (1.1) with

$$\begin{aligned} g_1(t,s) &= g_2(t,s) = t-s,\\ P_1(t,u_1,u_2) &= u_1^{-a_1}+u_2^{-a_2},\quad Q_1(t,u_1,u_2)=u_1^{a_3}\ u_2^{a_4},\\ P_2(t,u_1,u_2) &= u_1^{-b_1}+u_2^{-b_2},\quad Q_2(t,u_1,u_2)=u_1^{b_3}\ u_2^{b_4}. \end{aligned}$$

It is clear that $g_i,\ i=1,2$ fulfill (I_5)–(I_8). Suppose we are interested in *positive solutions* of (3.1), i.e., when $\theta_1=\theta_2=1$. Clearly, (I_1) and (I_2) are satisfied. Further, (I_3) and (I_4) are fulfilled if we choose

$$\begin{aligned} r_1 = r_2 &= d_1 = d_2 = 1,\\ \gamma_1(z) = z^{-a_1}, &\quad \gamma_2(z)=z^{-b_2},\\ h_{11}(z) = z^{a_1+a_3}, &\quad h_{12}(z)=z^{a_4},\\ h_{21}(z) = z^{b_1+b_3}, &\quad h_{22}(z)=z^{b_4}. \end{aligned}$$

Hence, we have

$$\begin{aligned} G_1(z) = \frac{z}{\gamma_1(z)} = z^{a_1+1},&\quad G_2(z)=\frac{z}{\gamma_2(z)}=z^{b_2+1},\\ G_1^{-1}(z)=z^{\frac{1}{a_1+1}},&\quad G_2^{-1}(z)=z^{\frac{1}{b_2+1}} \end{aligned}$$

and subsequently

$$\begin{aligned} \beta_1(t) &= G_1^{-1}\left(\int_0^t g_1(t,x)r_1(x)dx\right)\\ &= \left(\int_0^t (t-x)dx\right)^{\frac{1}{a_1+1}} = \left(\frac{t^2}{2}\right)^{\frac{1}{a_1+1}},\\ \beta_2(t) &= G_2^{-1}\left(\int_0^t g_2(t,x)r_2(x)dx\right)\\ &= \left(\int_0^t (t-x)dx\right)^{\frac{1}{b_2+1}} = \left(\frac{t^2}{2}\right)^{\frac{1}{b_2+1}}. \end{aligned} \tag{3.3}$$

Now, noting (3.2) we see that

$$\begin{aligned} &\int_0^T P_1(s,\beta_1(s),\beta_2(s))ds\\ &= \int_0^T \left[\left(\frac{s^2}{2}\right)^{\frac{-a_1}{a_1+1}}+\left(\frac{s^2}{2}\right)^{\frac{-a_2}{b_2+1}}\right]ds<\infty \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\int_0^T P_2(s,\beta_1(s),\beta_2(s))ds\\&=\int_0^T\left[\left(\frac{s^2}{2}\right)^{\frac{-b_1}{a_1+1}}+\left(\frac{s^2}{2}\right)^{\frac{-b_2}{b_2+1}}\right]ds<\infty.\end{aligned}\tag{3.5}$$

Applying (3.4) and (3.5), we find for $i=1,2$,

$$\begin{aligned}&\sup_{t\in[0,T]}\int_0^t g_i(t,s)P_i(s,\beta_1(s),\beta_2(s))ds\\&\le T\int_0^T P_i(s,\beta_1(s),\beta_2(s))ds<\infty.\end{aligned}$$

Thus, the condition (I_9) is satisfied.

Next, to check condition (I_{10}), we note that for $t,x\in[0,T]$ with $t<x$, on using (3.4) and (3.5) we have

$$\begin{aligned}&\int_0^t [g_1(x,s)-g_1(t,s)]P_1(s,\beta_1(s),\beta_2(s))ds\\&\le (x-t)\int_0^T P_1(s,\beta_1(s),\beta_2(s))ds \ \le\ (x-t)K_1\end{aligned}$$

and

$$\begin{aligned}&\int_0^t [g_2(x,s)-g_2(t,s)]P_2(s,\beta_1(s),\beta_2(s))ds\\&\le (x-t)\int_0^T P_2(s,\beta_1(s),\beta_2(s))ds \ \le\ (x-t)K_2\end{aligned}$$

where K_1 and K_2 are some finite constants. Hence, condition (I_{10}) is satisfied.

Finally, the condition (I_{11}) is equivalent to

$$\begin{cases}\text{if } z>0 \text{ satisfies } z\le K+L\left(1+z^{\frac{1}{3}}\right)\\ \text{for some constants } K,L\ge 0,\ \text{then there exists}\\ \text{a constant } M (\text{which may depend on } K \text{ and } L)\\ \text{such that } z\le M,\end{cases}\tag{3.6}$$

which is true since if z is unbounded, then obviously $z>K+L\left(1+z^{\frac{1}{3}}\right)$ for any $K,L\ge 0$. As an illustration, pick $K=L=1$, then the inequality in (3.6) becomes

$$z\le 1+\left(1+z^{\frac{1}{3}}\right)$$

which can be solved to obtain

$$0 < z \leq 3.5213 = M.$$

It now follows from Theorem 2.2 that the system (3.1), (3.2) has a *positive solution* $u \in (C[0,T])^2$ with $u_i(t) \geq \beta_i(t)$ for $t \in [0,T]$ and $i = 1, 2$, where $\beta_i(t)$ is given by (3.3).

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