Electronic Journal of Qualitative Theory of Differential Equations
Spec. Ed. I, 2009 No. **31**, 1–13; **http://www.math.u-szeged.hu/ejqtde/**

POSITIVE SOLUTIONS FOR THE (n, p) BOUNDARY VALUE PROBLEM

Bo Yang

Department of Mathematics and Statistics
Kennesaw State University, Kennesaw, GA 30144 USA
e-mail: byang@kennesaw.edu

Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We consider the (n, p) boundary value problem in this paper. Some new upper estimates to positive solutions for the problem are obtained. Existence and nonexistence results for positive solutions of the problem are obtained by using the Krasnosel'skii fixed point theorem. An example is included to illustrate the results.

Key words and phrases: Positive solution, higher order boundary value problem, fixed point.
AMS (MOS) Subject Classifications: 34B18

1 Introduction

In this paper, we consider the (n, p) boundary value problem

$$u^{(n)}(t) + g(t)f(u(t)) = 0, \quad 0 \le t \le 1, \tag{1}$$

$$u^{(i)}(0) = u^{(p)}(1) = 0, \quad i = 0, 1, \cdots, n-2. \tag{2}$$

Throughout this paper, we assume that

(H1) $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous functions, with $g(t) \not\equiv 0$ on $[0, 1]$;

(H2) $n \ge 2$ and p are fixed integers such that $1 \le p \le n - 1$.

The (n, p) problem has been considered by many authors. For example, in 1995, Eloe and Henderson [5] studied a special case of the (n, p) problem in which $p = n - 2$. In 2000, Agarwal, O'Regan, and Lakshmikantham [1] considered the existence of positive solutions for the singular (n, p) problem. In 2003, Baxley and Houmand [3] considered the existence of multiple positive solutions for the (n, p) problem.

Motivated by these works, we in this paper consider the existence and nonexistence of positive solutions to the problem (1)-(2). By a positive solution, we mean a solution $u(t)$ such that $u(t) > 0$ on $(0,1)$. The purpose of this paper is twofold. First we shall prove some new upper estimates for positive solutions of the problem (1)-(2). Then, using these new upper estimates, we obtain some new existence and nonexistence results for positive solutions of the problem (1)-(2).

The Green's function $G : [0,1] \times [0,1] \to [0,\infty)$ for the problem (1)-(2) is given by (see [1])

$$G(t,s) = \frac{1}{(n-1)!} \begin{cases} t^{n-1}(1-s)^{n-p-1} - (t-s)^{n-1}, & t \geq s, \\ t^{n-1}(1-s)^{n-p-1}, & s \geq t. \end{cases}$$

And the problem (1)-(2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds, \quad 0 \leq t \leq 1. \tag{3}$$

To prove some of our results, we will need the following fixed point theorem, which is due to Krasnosel'skii [9].

Theorem 1.1 *Let X be a Banach space over the reals, and let $P \subset X$ be a cone in X. Let H_1 and H_2 be real numbers such that $H_2 > H_1 > 0$, and let*

$$\Omega_i = \{v \in X \mid \|v\| < H_i\}, \quad i = 1,2.$$

Suppose $L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$ is a completely continuous operator such that, either

(K1) $\|Lu\| \leq \|u\|$ if $u \in P \cap \partial\Omega_1$, and $\|Lu\| \geq \|u\|$ if $u \in P \cap \partial\Omega_2$; or

(K2) $\|Lu\| \geq \|u\|$ if $u \in P \cap \partial\Omega_1$, and $\|Lu\| \leq \|u\|$ if $u \in P \cap \partial\Omega_2$.

Then L has a fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.

Throughout the paper, we let $X = C[0,1]$ be equipped with norm

$$\|v\| = \max_{t \in [0,1]} |v(t)|, \quad v \in X.$$

Obviously X is a Banach space. Also we define

$$F_0 = \limsup_{x \to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x \to 0^+} \frac{f(x)}{x},$$

$$F_\infty = \limsup_{x \to +\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x \to +\infty} \frac{f(x)}{x}.$$

These constants will be used later in the statements of the existence and nonexistence theorems.

This paper is organized as follows. In Section 2, we obtain some new upper estimates to positive solutions to the (n,p) problem. In Sections 3 and 4, we establish some new existence and nonexistence results for positive solutions of the problem. An example is given at the end of the paper to illustrate the main results of the paper.

2 Estimates for Positive Solutions

We begin with some definitions. Throughout the paper, we define the functions $a : [0, 1] \to [0, 1]$, $b : [0, 1] \to [0, 1]$, and $c : [0, 1] \to [0, 1]$ by

$$a(t) = t^{n-1},$$

$$b(t) = \frac{1}{p}((n-1)t^{n-2} - (n-1-p)t^{n-1}),$$

$$c(t) = \frac{1}{p}(nt^{n-1} - (n-p)t^n).$$

The functions $a(t)$, $b(t)$, and $c(t)$ will be used to estimate positive solutions of the problem (1)-(2). It is easy to verify the following facts

(1) $a(0) = b(0) = c(0) = 0$;

(2) $a(1) = b(1) = c(1) = 1$;

(3) $a(t)$, $b(t)$, and $c(t)$ are increasing nonnegative functions;

(4) $a(t) \le c(t) \le b(t)$ for $0 \le t \le 1$.

For example, we have

$$c(t) = \frac{t^{n-1}}{p}(n(1-t) + pt) \ge 0 \quad \text{on} \quad [0, 1],$$

and

$$c(t) - a(t) = \frac{n-p}{p} t^{n-1}(1-t) \ge 0, \quad 0 \le t \le 1.$$

We leave the other details to the reader.

The next lemma was proved by Agarwal, O'Regan, and Lakshmikantham in [1]. For details of the proof, see Theorem 1.3 of [1].

Lemma 2.1 *If $u \in C^n[0, 1]$ satisfies (2), and*

$$u^{(n)}(t) \le 0, \quad 0 \le t \le 1, \tag{4}$$

then $u'(t) \ge 0$ for $0 \le t \le 1$, and

$$a(t)u(1) \le u(t) \quad \textit{for} \quad 0 \le t \le 1.$$

As a direct consequence of Lemma 2.1, we have

Lemma 2.2 *If $u \in C^n[0, 1]$ satisfies (2) and (4), then*

(1) $u(1) = \|u\|$;

(2) If $u(1) = 0$*, then* $u(t) \equiv 0$ *on* $[0, 1]$*;*

(3) If $u(1) > 0$*, then* $u(t) > 0$ *for* $0 < t \leq 1$*.*

One implication of the above lemmas is that if $u(t)$ is a positive solution to the problem (1)-(2), then $u(t) \geq a(t)\|u\|$. This provides a nice lower estimate to positive solutions for the (n, p) problem. To our knowledge, no satisfactory upper estimates for positive solutions for the (n, p) problem have been given in the literature.

Lemma 2.3 *If* $u \in C^n[0, 1]$ *satisfies (2) and (4), then* $u(t) \leq b(t)u(1)$ *for* $0 \leq t \leq 1$*.*

Proof. Suppose that $u \in C^n[0, 1]$ satisfies (2) and (4). If we define

$$h(t) = u(1)b(t) - u(t), \quad 0 \leq t \leq 1, \tag{5}$$

then

$$h^{(n)}(t) = -u^{(n)}(t) \geq 0 \text{ on } (0, 1). \tag{6}$$

To prove the lemma, it suffices to show that $h(t) \geq 0$ for $0 \leq t \leq 1$. Assume the contrary that $h(t_0) < 0$ for some $t_0 \in (0, 1)$. If we can show that this leads to a contradiction, then we are done.

It is easy to see from (5) that

$$h^{(i)}(0) = 0, \quad i = 0, 1, 2, \cdots, n - 3.$$

By the mean value theorem, because $h(0) = 0 > h(t_0)$, there exists $t_1 \in (0, t_0)$ such that $h'(t_1) < 0$. Because $0 = h'(0) > h'(t_1)$, there exists $t_2 \in (0, t_1) \subset (0, t_0)$ such that $h''(t_2) < 0$. Because $0 = h''(0) > h''(t_2)$, there exists $t_3 \in (0, t_2) \subset (0, t_1)$ such that $h'''(t_3) < 0$. Continuing this procedure, we find a sequence of numbers

$$t_0 > t_1 > t_2 > \cdots > t_{n-2} > 0$$

such that

$$h^{(i)}(t_i) < 0, \quad 0 \leq i \leq n - 2.$$

It is easy to see that $u(1) = 0$. By the mean value theorem, because $h(t_0) < 0 = h(1)$, there exists $s_1 \in (t_0, 1) \subset (t_1, 1)$ such that $h'(s_1) > 0$. Because $h'(t_1) < 0 < h'(s_1)$, there exists $s_2 \in (t_1, s_1) \subset (t_2, s_1)$ such that $h''(s_2) > 0$. Continuing this procedure, we can find a sequence of numbers

$$s_1 > s_2 > s_3 > \cdots > s_{n-1}$$

such that

$$h^{(i)}(s_i) > 0, \quad i = 1, 2, 3, \cdots, n - 1.$$

It's easy to verify that $h^{(p)}(1) = 0$. By the mean value theorem, because $h^{(p)}(s_p) > 0 = h^{(p)}(1)$, there exists $r_{p+1} \in (s_p, 1) \subset (s_{p+1}, 1)$ such that $h^{(p+1)}(r_{p+1}) < 0$. Because

$h^{(p+1)}(s_{p+1}) > 0 > h^{(p+1)}(r_{p+1})$, there exists $r_{p+2} \in (s_{p+1}, r_{p+1}) \subset (s_{p+2}, r_{p+1})$ such that $h^{(p+2)}(r_{p+2}) < 0$. Continuing this procedure, we can find a sequence of numbers

$$r_{p+1} > r_{p+2} > \cdots > r_n$$

such that

$$h^{(i)}(r_i) < 0, \quad i = p+1, p+2, \cdots, n.$$

In particular, we have that $h^{(n)}(r_n) < 0$, which contradicts (6). The proof of the lemma is now complete.

Lemma 2.4 *If $u \in C^n[0,1]$ satisfies (2) and (4), and $u^{(n)}(t)$ is non-increasing on $[0,1]$, then $u(t) \leq c(t)u(1)$ for $0 \leq t \leq 1$.*

Proof. Though the proof of this lemma is somewhat similar to that of Lemma 2.3, we write it out for the purpose of completeness.

If we define

$$h(t) = u(1)c(t) - u(t), \quad 0 \leq t \leq 1,$$

then

$$h^{(n)}(t) = -u(1)p^{-1}(n-p)n! - u^{(n)}(t) \quad \text{on} \quad (0,1). \tag{7}$$

Therefore, $h^{(n)}(t)$ is nondecreasing.

To prove the lemma, it suffices to show that $h(t) \geq 0$ for $t \in [0,1]$. Assume the contrary that $h(t_0) < 0$ for some $t_0 \in (0,1)$.

It is easy to see that $h(0) = h'(0) = h''(0) = \cdots = h^{(n-2)}(0) = 0$. By the mean value theorem, because $h(0) = 0 > h(t_0)$, there exists $t_1 \in (0, t_0)$ such that $h'(t_1) < 0$. Because $0 = h'(0) > h'(t_1)$, there exists $t_2 \in (0, t_1) \subset (0, t_0)$ such that $h''(t_2) < 0$. Continuing this procedure, we find a sequence of numbers

$$t_0 > t_1 > t_2 > t_3 > \cdots > t_{n-1}$$

such that

$$h^{(i)}(t_i) < 0, \quad i = 1, 2, 3, \cdots, n-1.$$

It is easy to see that $h(1) = 0$. By the mean value theorem, because $h(t_0) < 0 = h(1)$, there exists $s_1 \in (t_0, 1) \subset (t_1, 1)$ such that $h'(s_1) > 0$. Because $h'(t_1) < 0 < h'(s_1)$, there exists $s_2 \in (t_1, s_1) \subset (t_2, s_1)$ such that $h''(s_2) > 0$. Continuing this procedure, we can find a sequence of numbers

$$s_1 > s_2 > s_3 > \cdots > s_n$$

such that

$$h^{(i)}(s_i) > 0, \quad i = 1, 2, 3, \cdots, n.$$

In particular, we have $h^{(n)}(s_n) > 0$.

It's easy to verify that $h^{(p)}(1) = 0$. By the mean value theorem, because $h^{(p)}(s_p) > 0 = h^{(p)}(1)$, there exists $r_{p+1} \in (s_p, 1) \subset (s_{p+1}, 1)$ such that $h^{(p+1)}(r_{p+1}) < 0$. Because $h^{(p+1)}(s_{p+1}) > 0 > h^{(p+1)}(r_{p+1})$, there exists $r_{p+2} \in (s_{p+1}, r_{p+1}) \subset (s_{p+2}, r_{p+1})$ such that $h^{(p+2)}(r_{p+2}) < 0$. Continuing this procedure, we can find a sequence of numbers

$$r_{p+1} > r_{p+2} > \cdots > r_n$$

such that

$$r_i > s_{i-1} > s_i, \quad i = p+1, p+2, \cdots, n$$

and

$$h^{(i)}(r_i) < 0, \quad i = p+1, p+2, \cdots, n.$$

In particular, we have $h^{(n)}(r_n) < 0$ and $r_n > s_n$.

Now we have $h^{(n)}(s_n) > 0 > h^{(n)}(r_n)$ and $s_n < r_n$, which contradicts the fact that $h^{(n)}$ is nondecreasing. The proof is now complete.

Theorem 2.1 *Suppose that, in addition to (H1) and (H2), the following condition holds.*

(H3) Both f and g are non-decreasing functions.

If $u \in C^n[0, 1]$ is a non-negative solution of the problem (1)-(2), then $u(t) \le c(t)u(1)$ for $0 \le t \le 1$.

Proof. Suppose that $u \in C^n[0, 1]$ is a non-negative solution of the problem (1)-(2). Obviously $u(t)$ satisfies (2) and (4). From Lemma 2.1 we see that $u(t)$ is nondecreasing. If (H3) holds, then

$$u^{(n)}(t) = -g(t)f(u(t))$$

is nonincreasing on $[0, 1]$. Now it follows immediately from Lemma 2.4 that $u(t) \le c(t)u(1)$ for $0 \le t \le 1$. The proof is now complete.

Theorem 2.2 *Suppose that (H1) and (H2) hold. If $u \in C^n[0, 1]$ is a non-negative solution of the problem (1)-(2), then $u(t) \le b(t)u(1)$ for $0 \le t \le 1$.*

Theorem 2.2 follows directly from Lemma 2.3. Note that Theorems 2.1 and 2.2 provide some upper estimates for positive solutions for the (n, p) boundary value problem. These upper estimates are new and have not been obtained before.

Now we define

$$P = \{v \in X \mid v(1) \geq 0,\ a(t)v(1) \leq v(t) \leq b(t)v(1) \text{ on } [0,1]\},$$

and

$$Q = \left\{ v \in X \;\middle|\; \begin{array}{c} v(1) \geq 0, v(t) \text{ is non-decreasing, and} \\ a(t)v(1) \leq v(t) \leq c(t)v(1) \text{ on } [0,1] \end{array} \right\}.$$

Then it is easily seen that both P and Q are positive cones of the Banach space X. And we have

Lemma 2.5 *If $u \in P$ or $u \in Q$, then $u(1) = \|u\|$.*

Proof. If $u \in Q$, then $u(0) = 0$, and u is non-decreasing. Therefore, $u(1) = \|u\|$.
If $u \in P$, then for each $t \in (0,1)$, we have

$$u(t) \leq b(t)u(1) \leq b(1)u(1) = u(1),$$

where the second inequality follows from the fact that $b(t)$ is nondecreasing. Therefore, $u(1) = \|u\|$. The proof is complete.

With the definition of P and Q, we can restate Theorems 2.1 and 2.2 as follows.

Theorem 2.3 *Suppose that (H1) and (H2) hold. If $u(t)$ is a non-negative solution to the problem (1)-(2), then $u \in P$.*

Theorem 2.4 *Suppose that (H1), (H2), and (H3) hold. If $u(t)$ is a non-negative solution to the problem (1)-(2), then $u \in Q$.*

Define an operator $T : P \to X$ by

$$(Tu)(t) = \int_0^1 G(t,s)g(s)f(u(s))ds, \quad 0 \leq t \leq 1.$$

Now the integral equation (3) is equivalent to the equality

$$Tu = u, \quad u \in P.$$

It is well known that $T : P \to X$ is a completely continuous operator. In order to solve the problem (1)-(2) we need only to find a fixed point of T.

By similar arguments to those of Theorems 2.1 and 2.2, we can prove the next two theorems without any difficulty.

Theorem 2.5 *If (H1) and (H2) hold, then $T(P) \subset P$.*

Theorem 2.6 *If (H1), (H2), and (H3) hold, then $T(Q) \subset Q$.*

3 Existence Results

Throughout we define

$$A = \int_0^1 G(1,s)g(s)a(s)\,ds, \quad B = \int_0^1 G(1,s)g(s)b(s)\,ds,$$

and

$$C = \int_0^1 G(1,s)g(s)c(s)\,ds.$$

The next theorem is our first existence result.

Theorem 3.1 *Suppose that (H1) and (H2) hold. If $BF_0 < 1 < Af_\infty$, then the problem (1)-(2) has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B < 1$. There exists $H_1 > 0$ such that

$$f(x) \le (F_0 + \varepsilon)x \quad \text{for} \quad 0 < x \le H_1.$$

For each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned} (Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &\le \int_0^1 G(1,s)g(s)(F_0+\varepsilon)u(s)\,ds \\ &\le (F_0+\varepsilon)\|u\| \int_0^1 G(1,s)g(s)b(s)ds \\ &= (F_0+\varepsilon)\|u\|B \\ &\le \|u\|, \end{aligned}$$

which means $\|Tu\| \le \|u\|$. So, if we let $\Omega_1 = \{u \in X |\ \|u\| < H_1\}$, then

$$\|Tu\| \le \|u\|, \quad \text{for} \quad u \in P \cap \partial\Omega_1.$$

To construct Ω_2, we choose $\beta \in (0, 1/4)$ and $\delta > 0$ such that

$$(f_\infty - \delta) \int_\beta^1 G(1,s)g(s)a(s)\,ds > 1.$$

There exists $H_3 > 0$ such that

$$f(x) \ge (f_\infty - \delta)x \quad \text{for} \quad x \ge H_3.$$

Let $H_2 = \max\{H_3\beta^{1-n},\ 2H_1\}$. If $u \in P$ with $\|u\| = H_2$, then

$$u(t) \ge a(t)H_2 = t^{n-1}H_2 \ge \beta^{n-1}H_2 \ge H_3 \quad \text{for} \quad \beta \le t \le 1.$$

Therefore, if $u \in P$ with $\|u\| = H_2$, then

$$\begin{aligned}(Tu)(1) &\geq \int_\beta^1 G(1,s)g(s)f(u(s))ds \\ &\geq \int_\beta^1 G(1,s)g(s)(f_\infty - \delta)u(s)\,ds \\ &\geq \int_\beta^1 G(1,s)g(s)a(s)\,ds \cdot (f_\infty - \delta)\|u\| \\ &\geq \|u\|,\end{aligned}$$

which means $\|Tu\| \geq \|u\|$. So, if we let $\Omega_2 = \{u \in X|\ \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$ and then

$$\|Tu\| \geq \|u\|, \quad \text{for} \ \ u \in P \cap \partial\Omega_2.$$

Then the condition (K1) of Theorem 1.1 is satisfied. So there exists a fixed point of T in P. The proof is complete.

Theorem 3.2 *Suppose that (H1) and (H2) hold. If $BF_\infty < 1 < Af_0$, then the problem (1)-(2) has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ such that $(f_0 - \varepsilon)A \geq 1$. There exists $H_1 > 0$ such that

$$f(x) \geq (f_0 - \varepsilon)x \quad \text{for} \quad 0 < x \leq H_1.$$

So, for $u \in P$ with $\|u\| = H_1$ we have

$$\begin{aligned}(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &\geq \int_0^1 G(1,s)g(s)u(s)\,ds \cdot (f_0 - \varepsilon) \\ &\geq \int_0^1 G(1,s)g(s)a(s)\,ds \cdot (f_0 - \varepsilon)\|u\| \\ &= A(f_0 - \varepsilon)\|u\| \\ &\geq \|u\|,\end{aligned}$$

which means $\|Tu\| \geq \|u\|$. So, if we let $\Omega_1 = \{u \in X \mid \|u\| < H_1\}$, then

$$\|Tu\| \geq \|u\|, \quad \text{for} \ \ u \in P \cap \partial\Omega_2.$$

To construct Ω_2, we choose $\delta \in (0,1)$ such that $((F_\infty + \delta)B + \delta) \leq 1$. There exists an $H_3 > 0$ such that

$$f(x) \leq (F_\infty + \delta)x \ \ \text{for} \ \ x \geq H_3.$$

Let $M = \max_{0 \le x \le H_3} f(x)$. Then

$$f(x) \le M + (F_\infty + \delta)x \quad \text{for} \quad x \ge 0.$$

Let

$$K = M \int_0^1 G(1,s)g(s)ds,$$

and let

$$H_2 = \max\{2H_1, K(1-(F_\infty+\delta)B)^{-1}\}.$$

If $u \in P$ with $\|u\| = H_2$, then we have

$$\begin{aligned}
(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\
&\le \int_0^1 G(1,s)g(s)(M + (F_\infty+\delta)u(s))\,ds \\
&\le K + (F_\infty+\delta)\int_0^1 G(1,s)g(s)u(s)\,ds \\
&\le K + (F_\infty+\delta)H_2 \int_0^1 G(1,s)g(s)b(s)\,ds \\
&= K + (F_\infty+\delta)BH_2 \\
&\le H_2,
\end{aligned}$$

which means $\|Tu\| \le \|u\|$. So, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then

$$\|Tu\| \le \|u\|, \quad \text{for} \quad u \in P \cap \partial\Omega_2.$$

Now from Theorem 1.1 we see that problem (1)-(2) has at least one positive solution. The proof is complete.

Theorem 3.3 *Suppose that (H1), (H2), and (H3) hold. If $CF_0 < 1 < Af_\infty$, then the problem (1)-(2) has at least one positive solution.*

Theorem 3.4 *Suppose that (H1), (H2), and (H3) hold. If $CF_\infty < 1 < Af_0$, then the problem (1)-(2) has at least one positive solution.*

The proofs of Theorems 3.3 and 3.4 are very similar to those of Theorems 3.1 and 3.2. The only difference is that we use the positive cone Q, instead of P, in the proofs of Theorems 3.3 and 3.4.

4 Nonexistence Results

In this section, we give some nonexistence results for positive solutions to the (n,p) problem.

Theorem 4.1 *Suppose that (H1) and (H2) hold. If $Bf(x) < x$ for all $x > 0$, then the problem (1)-(2) has no positive solutions.*

Proof. Assume to the contrary that $u(t)$ is a positive solution of the problem (1)-(2). Then $u \in P$, $u(t) > 0$ on $(0,1]$, and

$$\begin{aligned} u(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(1,s)g(s)u(s)\,ds \\ &\leq B^{-1}\int_0^1 G(1,s)g(s)b(s)\,ds \cdot u(1) \\ &= u(1), \end{aligned}$$

which is a contradiction. The proof is now complete.

Theorem 4.2 *Suppose that (H1) and (H2) hold. If $Af(x) > x$ for all $x > 0$, then the problem (1)-(2) has no positive solutions.*

Theorem 4.3 *Suppose that (H1), (H2), and (H3) hold. If $Cf(x) < x$ for all $x > 0$, then the problem (1)-(2) has no positive solutions.*

The proofs of Theorems 4.2 and 4.3 are quite similar to that of Theorem 4.1 and are therefore omitted.

Example 4.1 Consider the boundary value problem

$$u''''(t) + g(t)f(u(t)) = 0, \quad 0 < t < 1, \tag{8}$$

$$u(0) = u'(0) = u''(0) = u''(1) = 0, \tag{9}$$

where

$$g(t) = 1 + 2t, \quad 0 \leq t \leq 1,$$

$$f(x) = \frac{\lambda x(1+3x)}{1+x}, \quad x \geq 0.$$

Here $\lambda > 0$ is a parameter. The problem (8)-(9) is a special case of the problem (1)-(2), in which $n = 4$ and $p = 2$. It is easy to see that, for the problem (8)-(9), we have $f_0 = F_0 = \lambda$ and $f_\infty = F_\infty = 3\lambda$. Also we have

$$a(t) = t^3, \quad b(t) = \frac{1}{2}(3t^2 - t^3), \quad c(t) = 2t^3 - t^4,$$

and

$$G(1, s) = \frac{1}{6}s(s-1)(s-2).$$

It is easy to see that $\lambda x < f(x) < 3\lambda x$ for $x > 0$. Using *Maple* or *Mathematica*, we can easily compute the constants:

$$A = \frac{43}{2520}, \quad B = \frac{147}{5040}, \quad C = \frac{331}{15120}.$$

From Theorem 3.1 we see that if

$$19.53 \approx \frac{2520}{129} < \lambda < \frac{5040}{147} \approx 34.286,$$

then the problem (8)-(9) has at least one positive solution. From Theorems 4.1 and 4.2 we see that if

$$\text{either} \quad \lambda \leq \frac{5040}{441} \approx 11.43 \quad \text{or} \quad \lambda \geq \frac{2520}{43} \approx 58.60,$$

then the problem (8)-(9) has no positive solutions.

Note that the function $g(t)$ is increasing in t, and $f(x)$ is increasing in x for each fixed $\lambda > 0$, therefore Theorems 3.3 and 4.3 apply. From Theorem 3.3 we see that if

$$19.53 \approx \frac{2520}{129} < \lambda < \frac{15120}{331} \approx 45.68,$$

then the problem (8)-(9) has at least one positive solution. From Theorem 4.3 we see that if

$$\lambda \leq \frac{5040}{331} \approx 15.277,$$

then the problem (8)-(9) has no positive solutions.

Acknowledgment

The author is grateful to the anonymous referee for his/her careful reading of the manuscript and valuable suggestions for its improvement.

References

[1] R. P. Agarwal, D. O'Regan, and V. Lakshmikantham, Singular $(p, n-p)$ focal and (n, p) higher order boundary value problems, *Nonlinear Analysis* **42** (2000), 215–228.

[2] D. R. Anderson and J. M. Davis, Multiple solutions and eigenvalues for third-order right focal boundary value problems, *J. Math. Anal. Appl.* **267** (1) (2002), 135–157.

[3] J. V. Baxley and C. R. Houmand, Nonlinear higher order boundary value problems with multiple positive solutions, *J. Math. Anal. Appl.* **286** (2003), 682–691.

[4] C. J. Chyan and J. Henderson, Multiple solutions for (n, p) boundary value problems, *Dynam. Systems Appl.* **10** (1) (2001), 53–61.

[5] P. W. Eloe and J. Henderson, Positive solutions for higher order ordinary differential equations, *Electron. J. Differential Equations* **1995** (3) (1995), 1–10.

[6] J. R. Graef and B. Yang, Boundary value problems for second order nonlinear differential equations, *Communications in Applied Analysis* **6** (2) (2002), 273–288.

[7] J. Henderson and H. Wang, Positive solutions for nonlinear eigenvalue problems, *J. Math. Anal. Appl.* **208** (1997), 252–259.

[8] E. R. Kaufmann, Multiple positive solutions for higher order boundary value problems, *Rocky Mountain J. Math.* **28** (3) (1998), 1017–1028.

[9] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[10] H. Wang, On the existence of positive solutions for semilinear elliptic equation in the annulus, *J. Differential Equations* **109** (1994), 1–7.

[11] B. Yang, Estimates of positive solutions for higher order right focal boundary value problem; *Communications in Mathematical Analysis*, **4** (1) (2008), 1–9.