Electronic Journal of Qualitative Theory of Differential Equations
Proc. 10th Coll. Qualitative Theory of Diff. Equ. (July 1–4, 2015, Szeged, Hungary)
2016, No. **18**, 1–18; doi: 10.14232/ejqtde.2016.8.18 http://www.math.u-szeged.hu/ejqtde/

# A system of delay difference equations with continuous time with lag function between two known functions

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Appeared 11 August 2016
Communicated by Tibor Krisztin

**Abstract.** The asymptotic behavior of solutions of the system of difference equations with continuous time and lag function between two known real functions is studied. The cases when the lag function is between two linear delay functions, between two power delay functions and between two constant delay functions are observed and illustrated by examples. The asymptotic estimates of solutions of the considered system are obtained.

**Keywords:** functional equations, difference equations with continuous time, asymptotic behavior.

**2010 Mathematics Subject Classification:** 39A21, 39B72.

## 1 Introduction

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+$ the set of positive real numbers. Assume that $t_0 > 0$ is a given real number, $n$ is a positive integer and $A, B : [t_0, \infty) \to \mathbb{R}^{n\times n}$ are given $n \times n$ real matrix valued functions. Let $\sigma : [t_0, \infty) \to \mathbb{R}$ be given such that $\sigma(t) < t$ holds for all $t \geq t_0$, and $\lim_{t\to\infty} \sigma(t) = \infty$.

This paper discusses the asymptotic behavior of solutions of the system of difference equations

$$x(t) = A(t)x(t-1) + B(t)x(\sigma(t)), \quad t \geq t_0 \tag{1.1}$$

with the *initial condition*

$$x^{\phi}(t) = \phi(t) \quad \text{for} \quad t_{-1} \leq t < t_0, \quad t_{-1} = \min\left\{\inf\{\sigma(s) : s \geq t_0\}, t_0 - 1\right\}, \tag{1.2}$$

where $\phi : [t_{-1}, t_0) \to \mathbb{R}^n$, $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))$ is a given function.

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**Definition 1.1.** By a *solution* of the system (1.1) we mean a vector function $x^{\phi} : [t_{-1}, \infty) \to \mathbb{R}^n$ which satisfies initial condition (1.2) for $t_{-1} \le t < t_0$ and satisfies the system (1.1) for $t \ge t_0$.

The asymptotic behavior of equation (1.1) in the scalar case has been investigated by Medina and Pituk [8], Péics [14], Philos and Purnaras [15], Zhou and Yu [17]. For the system case with discrete arguments see Čermak and Jánský [2], Gilyazev and Kipnis [3], Kaslik [4], Matsunaga [7], and the references therein. Papers by Blizorukov [1], Pelyukh [10,11], Korenevskii and Kaizer [5,6], Shaiket [16] generalise some fundamental results for solutions of difference equations with continuous arguments. Results given here generalize results in [12] and [13] in the sense of their application for some new type of lag functions.

For given positive integer $m$, $t \in \mathbb{R}_+$ and a function $f : \mathbb{R} \to \mathbb{R}$ we use the standard notation

$$\prod_{\ell=t}^{t-1} f(\ell) = 1, \qquad \prod_{\ell=t-m}^{t} f(\ell) = f(t-m)f(t-m+1)\cdots f(t)$$

and

$$\sum_{\tau=t}^{t-1} f(\tau) = 0, \qquad \sum_{\tau=t-m}^{t} f(\tau) = f(t-m) + f(t-m+1) + \cdots + f(t).$$

We shall say that the infinite product $\prod_{k=1}^{\infty} a_k$ *converges* if only a finite number of the factors $a_k$ are zero and if $n$ is an integer with the property that $a_m \neq 0$ for all $m \ge n$, then the sequence $a_n$, $a_n a_{n+1}$, $a_n a_{n+1} a_{n+2}, \ldots$ converges to a limit distinct from zero. If an infinite product does not converge we shall say it *diverges*.

If $\prod_{n=1}^{\infty} a_n$ represents a convergent infinite product, then it is convenient to write it in the form $\prod_{n=1}^{\infty} (1 + b_n)$, where $a_n = 1 + b_n$ and $\lim_{n\to\infty} b_n = 0$. If the product $\prod_{n=1}^{\infty} (1 + |b_n|)$ converges, we shall say that the product of $\prod_{n=1}^{\infty} (1 + b_n)$ *converges absolutely*.

We can find the following theorem in [9], as Theorem 3 on page 45.

**Theorem A.** *A necessary and sufficient condition that the infinite product $\prod_{n=1}^{\infty} (1 + b_n)$ converges absolutely is that the infinite series $\sum_{n=1}^{\infty} b_n$ converges absolutely.*

The difference operator $\Delta$ is defined by

$$\Delta f(t) = f(t+1) - f(t).$$

For a function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, the difference operator $\Delta_t$ is given by

$$\Delta_t g(t, a) = g(t+1, a) - g(t, a).$$

For a given function $\sigma : [t_0, \infty) \to \mathbb{R}$ with $\sigma(t) < t$ and $\lim_{t\to\infty} \sigma(t) = \infty$, set

$$t_m = \inf\{s : \sigma(s) > t_{m-1}\} \quad \text{for all } m = 1, 2, \ldots$$

In Figure 1.1 we can see the special case of creating the points $\{t_m\}$ when the delay function is $\sigma(t) = \frac{t}{2}$.

For a given sequence of points $\{t_m\}$, fix a point $t \ge t_0$, and define natural numbers $k_m(t)$ such that $k_m(t) := [t - t_m]$, $m = 0, 1, 2 \ldots$ For some $t \in \mathbb{R}$, $[t]$ denotes the integer part of $t$. Set

$$T_m(t) := \{t - k_m(t), t - k_m(t) + 1, \ldots, t - 1, t\}, \quad m = 0, 1, 2, \ldots$$

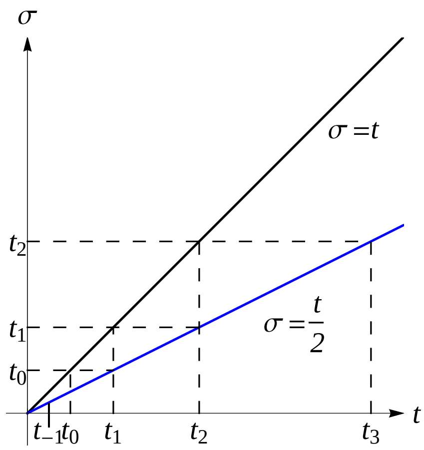


Figure 1.1: The $\{t_m\}$ points for the $\sigma(t) = \dfrac{t}{2}$ delay function.

For the given functions $g_i : [t_{-1}, \infty) \to (0, \infty)$ and $a_i : [t_0, \infty) \to (0, 1)$, $i = 1, 2, \ldots, n$, and for the given non-negative integer $m$ we define the numbers

$$R_{im} := \sup_{t_m \le t < t_{m+1}} \left\{ g_i(t) \sum_{\tau = t - k_m(t)}^{t} \frac{\Delta_\tau g_i(\tau - 1)}{g_i(\tau) g_i(\tau - 1)} \prod_{\ell = \tau+1}^{t} a_i(\ell) \right\}. \tag{1.3}$$

For the given functions $g_i : [t_{-1}, \infty) \to (0, \infty)$ and the given initial functions $\phi_i$, $i = 1, 2, \ldots, n$, we set

$$M_{i0} = \sup_{t_{-1} \le t < t_0} g_i(t) |\phi_i(t)| \quad \text{for} \quad i = 1, 2, \ldots, n \quad \text{and} \quad M_0 = \max\{M_{10}, M_{20}, \ldots, M_{n0}\}. \tag{1.4}$$

We discuss the case when matrix $A$ is diagonal and its components are between 0 and 1. Consider the following hypotheses.

$(H_1)$ For every $t \ge t_0$, $A(t) = \operatorname{diag}(a_1(t), \ldots, a_n(t))$ is an $n \times n$ diagonal matrix with real entries satisfying $0 < a_i(t) < 1$, for all $t \ge t_0$, $i = 1, 2, \ldots, n$.

$(H_2)$ $B(t) = (b_{ij}(t))$ is an $n \times n$ matrix with real entries for all $t \ge t_0$.

$(H_3)$ There exists a diagonal $n \times n$ matrix $G(t) = \operatorname{diag}(g_1(t), \ldots, g_n(t))$ for all $t \ge t_{-1}$ so that the diagonal entries $g_i : [t_{-1}, \infty) \to (0, \infty)$ are bounded on the initial interval $[t_{-1}, t_0)$, $i = 1, 2, \ldots, n$, and such that

$$\sum_{j=1}^{n} \frac{|b_{ij}(t)|}{g_j(\sigma(t))} \le \frac{(1 - a_i(t))}{g_i(t)} \quad \text{for} \quad t \ge t_0, \quad i = 1, 2, \ldots, n.$$

$(H_4)$ There are real numbers $R_i$, $i = 1, 2, \ldots, n$, such that

$$\prod_{m=1}^{j} (1 + R_{im}) \le R_i, \quad \text{for all positive integers } j \text{ and } i = 1, 2, \ldots, n,$$

where the numbers $R_{im}$ are defined by (1.3).

$(H_5)$ $\sigma : [t_0, \infty) \to \mathbb{R}$ is a given function with the property that $\sigma(t) < t$ for every $t \ge t_0$ and $\lim_{t \to \infty} \sigma(t) = \infty$.

The next theorem, which was proven in [13], gives asymptotic estimates for the rate of convergence of the components of solutions of equation (1.1).

**Theorem B.** *Suppose that conditions* ($H_1$)*,* ($H_2$)*,* ($H_3$) ($H_4$) *and* ($H_5$) *hold. Let* $x^\phi$ *be the solution of the initial value problem* (1.1) *and* (1.2) *with bounded components* $\phi_i$*,* $i = 1, 2, \dots, n$*, in* (1.2)*. Then*

$$\left| x_i^\phi(t) \right| \leq \frac{M_0 R_i}{g_i(t)} \quad \textit{for all} \quad t \geq t_0 \quad \textit{and} \quad i = 1, 2, \dots, n,$$

*where* $M_0$ *is defined by* (1.4)*.*

**Remark 1.2.** In Theorem B, let the functions $g_i$, $i = 1, 2, \dots, n$, defined by ($H_3$), be monotone increasing. Then the sequences $\{R_{im}\}_m$, $i = 1, 2, \dots, n$, defined by (1.3), have only positive members and the assumption

$$\prod_{m=1}^{\infty} (1 + R_{im}) < \infty \quad \text{for all } i = 1, 2, \dots, n,$$

implies the existence of real numbers $R_1, R_2, \dots R_n$, which satisfies condition ($H_4$).

If the functions $g_i$, $i = 1, 2, \dots, n$, are monotone decreasing, then condition ($H_4$) is satisfied with $R_1 = R_2 = \cdots = R_n = 1$.

## 2 Main results

In [13] it is illustrated how the rate of convergence of the components of the solutions can be estimated by a power function in the particular case when the lag function is $\sigma(t) = ct$, $0 < c < 1$, $t > 0$. In this paper we generalize these results to the case when the lag function is squeezed between two linear functions, i.e. we show how the rate of convergence of the components of the solutions can be estimated by a power function when the lag function $\sigma$ has the property

$$ct \leq \sigma(t) \leq Ct, \quad c, C \in \mathbb{R},\ 0 < c \leq C < 1,\ t > 0.$$

Moreover, we present how the rate of convergence of the components of the solutions can be estimated by a power of logarithmic function and by an exponential function, for the lag function with the property

$$t^c \leq \sigma(t) \leq t^C, \quad c, C \in \mathbb{R},\ 0 < c \leq C < 1,\ t \geq 1$$

and

$$\sigma(t) = t - \delta(t), \quad 1 \leq c \leq \delta(t) \leq C,\ c, C \in \mathbb{R},\ c \leq C,\ t > 0, \quad \text{where} \quad \delta(t) \not\equiv 1,$$

respectively.

In Figure 2.1 we can see the special case of the delay function, when

$$ct \leq \sigma(t) \leq Ct, \quad t > 0,$$

for real numbers $c$ and $C$ such that $0 < c \leq C < 1$.

We shall need the following hypothesis.

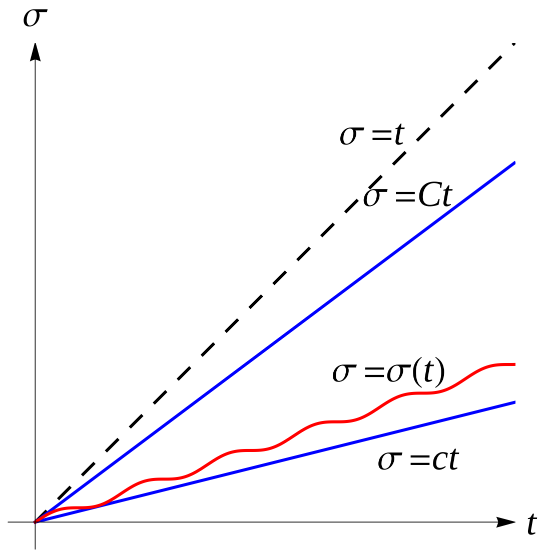


Figure 2.1: The delay function such that $ct \le \sigma(t) \le Ct,\ t > 0$, for $0 < c \le C < 1$.

$(H_6)$ There exist real numbers $Q$ and $\alpha_i$, $i = 1, 2, \dots, n$, such that $0 < Q \le 1$, $0 < \alpha_i < 1$ and

$$\sum_{j=1}^{n} |b_{ij}(t)| \le Q(1 - a_i(t)), \quad \alpha_i \le 1 - a_i(t)$$

for $t \ge t_0$, where the functions $a_i$ and $b_{ij}$, $i, j = 1, 2, \dots, n$, are given in $(H_1)$ and $(H_2)$.

**Theorem 2.1.** *Suppose that conditions $(H_1)$, $(H_2)$ and $(H_6)$ hold. Let $\sigma : [t_0, \infty) \to \mathbb{R}$ be a real function such that $ct \le \sigma(t) \le Ct$ for all $t \ge t_0 > 1$, for real numbers $c$ and $C$ such that $0 < c \le C < 1$ and $C^{1+K} < Q$, where $K = \log_c Q$. Let $x = x^{\phi}$ be a solution of the initial value problem (1.1) and (1.2) with bounded components $\phi_i$, $i = 1, 2, \dots, n$. Then*

$$|x_i(t)| \le \frac{M_0 R_i}{t^K} \quad \textit{for all} \quad t \ge t_0, \quad i = 1, 2, \dots, n,$$

*where, for $i = 1, 2, \dots, n$,*

$$M_0 = \max_{1 \le i \le n} \left\{ \sup_{t_{-1} \le t < t_0} \left\{ t^K |\phi_i(t)| \right\} \right\}, \quad R_i = \prod_{m=1}^{\infty} \left( 1 + \frac{K t_0^K}{\alpha_i c^K (t_0 - C^m)^{K+1}} \left( \frac{C^{K+1}}{c^K} \right)^m \right).$$

*Proof.* Let $t_0 > 1$ be a real number. The relations $\sigma(t_{m+1}) = t_m$ and $ct \le \sigma(t) \le Ct$ imply that

$$\frac{t_0}{C^m} \le t_m \le \frac{t_0}{c^m} \quad \text{for} \quad m = 1, 2, \dots$$

Set

$$t_{-1} = \min\{t_0 - 1, ct_0\} \quad \text{and} \quad g_i(t) = t^K,\ i = 1, 2, \dots, n.$$

Since $Q = c^K$, it follows that

$$\sum_{j=1}^{n} \frac{|b_{ij}(t)|}{g_j(\sigma(t))} = \frac{\sum_{j=1}^{n} |b_{ij}(t)|}{\sigma(t)^K} \le \frac{Q(1 - a_i(t))}{c^K t^K} = \frac{1 - a_i(t)}{g_i(t)} \quad \text{for} \quad i = 1, 2, \dots$$

Therefore, condition $(H_3)$ of Theorem B is valid. Moreover,

$$\begin{aligned} R_{im} &\leq \sup_{t_m \leq t < t_{m+1}} \left\{ t^K \sum_{\tau = t - k_m(t)}^{t} \frac{\tau^K - (\tau-1)^K}{\tau^K (\tau-1)^K} (1-\alpha_i)^{t-\tau} \right\} \\ &\leq \sup_{t_m \leq t < t_{m+1}} \left\{ t^K (1-\alpha_i)^t \sum_{\tau = t - k_m(t)}^{t} \frac{K}{(\tau-1)^{K+1}} \left( \frac{1}{1-\alpha_i} \right)^{\tau} \right\} \\ &\leq \sup_{t_m \leq t < t_{m+1}} \left\{ \frac{t^K (1-\alpha_i)^t K}{(t - k_m(t) - 1)^{K+1}} \frac{1-\alpha_i}{\alpha_i} \left( \frac{1}{1-\alpha_i} \right)^{t+1} \right\} \\ &\leq \sup_{t_m \leq t < t_{m+1}} \left\{ \frac{K t^K}{\alpha_i (t - k_m(t) - 1)^{K+1}} \right\} \\ &\leq \frac{K t_0^K}{\alpha_i c^K (t_0 - C^m)^{K+1}} \left( \frac{C^{K+1}}{c^K} \right)^m \end{aligned}$$

for all $m = 1, 2, \ldots$, $i = 1, 2, \ldots, n$. Applying d'Alembert's ratio test for the series $\sum_{m=1}^{\infty} a_m$, where

$$a_m = \frac{K t_0^K}{\alpha_i c^K (t_0 - C^m)^{K+1}} \left( \frac{C^{K+1}}{c^K} \right)^m,$$

we obtain that

$$L = \lim_{m \to \infty} \frac{a_{m+1}}{a_m} = \frac{C^{K+1}}{c^K} < \frac{Q}{c^K} = 1.$$

The relation $L < 1$ means that the series $\sum_{m=1}^{\infty} a_m$ is convergent. Since $0 < R_{im} \leq a_m$ for all $m = 1, 2, \ldots$, $i = 1, 2, \ldots, n$, hence the series $\sum_{m=1}^{\infty} R_{im}$ is also convergent for $i = 1, 2, \ldots, n$. Applying Theorem A, it follows that the infinite product $\prod_{m=1}^{\infty} (1 + R_{im})$ is convergent for $i = 1, 2, \ldots, n$, and the numbers $R_1, R_2, \ldots, R_n$ exist. Then, Theorem B implies the assertion. □

In Figure 2.2 we can see the special case of the delay function, when

$$t^c \leq \sigma(t) \leq t^C$$

for real numbers $c$ and $C$ such that $0 < c \leq C < 1$.

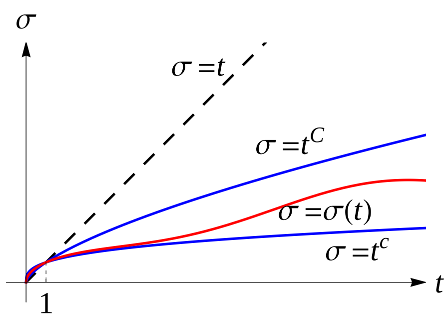


Figure 2.2: The delay function such that $t^c \leq \sigma(t) \leq t^C$, $0 < c \leq C < 1$.

**Theorem 2.2.** *Suppose that conditions $(H_1)$, $(H_2)$ and $(H_6)$ hold. Let $\sigma : [t_0, \infty) \to \mathbb{R}$ be a real function such that $t^c \leq \sigma(t) \leq t^C$ for all $t \geq t_0 > 1$ for real numbers $c$ and $C$ such that $0 < c \leq C < 1$.*

*Let $x = x^{\phi}$ be a solution of the initial value problem (1.1) and (1.2) with bounded components $\phi_i$, $i = 1, 2, \ldots, n$. Then*

$$|x_i(t)| \le \frac{M_0 R_i}{\ln^K t} \quad \textit{for} \quad \textit{all} \quad t \ge t_0, \quad i = 1, 2, \ldots, n,$$

*where $K = \log_c Q$, and for $i = 1, 2, \ldots, n$, we set*

$$M_0 = \max_{1 \le i \le n} \left\{ \sup_{t_{-1} \le t < t_0} \left\{ \ln^K t |\phi_i(t)| \right\} \right\},$$

$$R_i = \prod_{m=1}^{\infty} \left( 1 + \frac{K \left(c^{-K}\right)^{m+1} \ln^K t_0}{\alpha_i \left(t_0^{(1/C)^m} - 1\right) \ln^{K+1} \left(t_0^{(1/C)^m} - 1\right)} \right).$$

*Proof.* Let $t_0 > 1$ be a real number. The relations

$$\sigma(t_{m+1}) = t_m \quad \text{and} \quad t^c \le \sigma(t) \le t^C$$

imply that

$$t_0^{\left(\frac{1}{C}\right)^m} \le t_m \le t_0^{\left(\frac{1}{c}\right)^m} \quad \text{for} \quad m = 0, 1, 2, \ldots$$

Let $g_i(t) = \ln^K t$, $i = 1, 2, \ldots, n$. Since $Q = c^K$, it follows that

$$\sum_{j=1}^{n} \frac{|b_{ij}(t)|}{g_j(\sigma(t))} = \frac{\sum_{j=1}^{n} |b_{ij}(t)|}{\ln^K \sigma(t)} \le \frac{Q(1 - a_i(t))}{\ln^K t^c} = \frac{Q(1 - a_i(t))}{c^K \ln^K t} = \frac{1 - a_i(t)}{g_i(t)}.$$

Therefore, condition $(H_3)$ of Theorem B is valid. Moreover,

$$\begin{aligned}
R_{im} &\le \sup_{t_m \le t < t_{m+1}} \left\{ \ln^K t \sum_{\tau = t - k_m(t)}^{t} \frac{\ln^K \tau - \ln^K (\tau - 1)}{\ln^K \tau \ln^K (\tau - 1)} (1 - \alpha_i)^{t - \tau} \right\} \\
&\le \sup_{t_m \le t < t_{m+1}} \left\{ \ln^K t (1 - \alpha_i)^t \sum_{\tau = t - k_m(t)}^{t} \frac{K}{(\tau - 1) \ln^{K+1} (\tau - 1)} \left( \frac{1}{1 - \alpha_i} \right)^{\tau} \right\} \\
&\le \sup_{t_m \le t < t_{m+1}} \left\{ \frac{\ln^K t (1 - \alpha_i)^t K}{(t - k_m(t) - 1) \ln^{K+1} (t - k_m(t) - 1)} \frac{1 - \alpha_i}{\alpha_i} \left( \frac{1}{1 - \alpha_i} \right)^{t+1} \right\} \\
&\le \sup_{t_m \le t < t_{m+1}} \left\{ \frac{K \ln^K t}{\alpha_i (t - k_m(t) - 1) \ln^{K+1} (t - k_m(t) - 1)} \right\} \\
&\le \frac{K \left( \left( \frac{1}{c} \right)^K \right)^{m+1} \ln^K t_0}{\alpha_i \left( t_0^{(1/C)^m} - 1 \right) \ln^{K+1} \left( t_0^{(1/C)^m} - 1 \right)} \sim \frac{C^{m(K+1)}}{c^{K(m+1)} t_0^{(1/C)^m}}
\end{aligned}$$

for $m = 1, 2, \ldots$, $i = 1, 2, \ldots, n$. Now, apply d'Alembert's ratio test for the series $\sum_{m=1}^{\infty} a_m$, where

$$a_m = \frac{C^{m(K+1)}}{c^{K(m+1)} t_0^{(1/C)^m}}.$$

After some transformation we get that

$$L = \lim_{m \to \infty} \frac{a_{m+1}}{a_m} = \lim_{m \to \infty} \frac{C^{K+1}}{c^K t_0^{(1/C)^m \left(\frac{1}{C} - 1\right)}} = 0 < 1.$$

The relation $L < 1$ means, as in the proof of Theorem 2.1, that the infinite product $\prod_{m=1}^{\infty}(1+R_{im})$ is convergent for $i = 1, 2, \dots, n$, and the numbers $R_1, R_2, \dots, R_n$ exist.

Then, Theorem B implies the assertion. □

In Figure 2.3 we can see the special case of the delay function $\sigma(t) = t - \delta(t)$, where $c \le \delta(t) \le C$ for real numbers $c$ and $C$ such that $1 \le c \le C$ and $\delta(t) \not\equiv 1$ for $t > 0$.

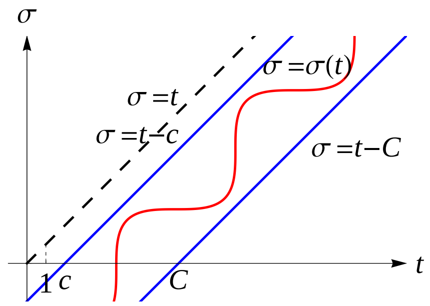


Figure 2.3: The delay function $\sigma(t) = t - \delta(t)$ such that $c \le \delta(t) \le C$, $1 \le c \le C$.

**Theorem 2.3.** *Let $\sigma(t) = t - \delta(t)$ be a real function such that $c \le \delta(t) \le C$ for real numbers $c$ and $C$, where $1 \le c \le C$ for all $t \ge t_0 > C$, and $\delta(t) \not\equiv 1$ for $t \ge t_0$. Suppose that conditions $(H_1)$ and $(H_2)$ hold and there exists a real number $\lambda > 1$ such that*

$$\sum_{j=1}^{n} |b_{ij}(t)| \le \frac{1 - \lambda a_i(t)}{\lambda^C} \quad \textit{for all } t \ge t_0,\ i = 1, 2, \dots, n.$$

*Let $x = x^{\phi}$ be a solution of the initial value problem (1.1) and (1.2) with bounded components $\phi_i$, $i = 1, 2, \dots, n$. Then*

$$|x_i(t)| \le \frac{M_0}{\lambda^t} \quad \textit{for all} \quad t \ge t_0, \quad i = 1, 2, \dots, n,$$

*where we set*

$$M_0 = \max_{1 \le i \le n} \left\{ \sup_{t_{-1} \le t < t_0} \{\lambda^t |\phi_i(t)|\} \right\}, \quad i = 1, 2, \dots, n.$$

*Proof.* Let $t_0 \ge C$ be a real number. The relations

$$t_{m+1} - \delta(t_{m+1}) = t_m \quad \text{and} \quad t - C \le t - \delta(t) \le t - c$$

imply that

$$t_0 + mc \le t_m \le t_0 + mC \quad \text{for} \quad m = 1, 2, \dots$$

Introduce the transformation $y_i(t) = x_i(t)\lambda^t$, $i = 1, 2, \dots, n$. Let $t \in [t_m, t_{m+1})$ and $\tau \in T_m(t)$. Then, system (1.1) is equivalent to

$$\Delta_\tau \left( y_i(\tau - 1) \prod_{\ell=t-k_m(t)}^{\tau-1} \frac{1}{\lambda a_i(\ell)} \right) = \sum_{j=1}^{n} b_{ij}(\tau)\lambda^{\delta(\tau)} y_j(\tau - \delta(\tau)) \prod_{\ell=t-k_m(t)}^{\tau} \frac{1}{\lambda a_i(\ell)},$$

for $i = 1, 2, \dots, n$. Summing up both sides of these equation from $t - k_m(t)$ to $t$ gives that, for $i = 1, 2, \dots, n$,

$$y_i(t) = y_i(t - k_m(t) - 1) \prod_{\ell=t-k_m(t)}^{t} \lambda a_i(\ell) + \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} b_{ij}(\tau)\lambda^{\delta(\tau)} y_j(\tau - \delta(\tau)) \prod_{\ell=\tau+1}^{t} \lambda a_i(\ell).$$

Define

$$\mu_{im} := \sup_{t_{m-1}\le t<t_m} |y_i(t)|, \quad M_{im} := \max\{\mu_{i0}, \mu_{i1}, \dots, \mu_{im}\} \quad \text{for} \quad m = 0, 1, 2, \dots, \ i = 1, 2, \dots, n$$

and let

$$M_m = \max_{1\le i\le n} \{M_{im}\} \quad \text{for} \quad m = 0, 1, 2, \dots$$

Since $|y_i(p(\tau))| \le M_m$, for $i = 1, 2, \dots, n$, $\tau \in T_m(t)$ and $t_m \le t < t_{m+1}$, from the hypotheses of the theorem it follows that

$$\begin{aligned} |y_i(t)| &\le M_m \left( \prod_{\ell=t-k_m(t)}^{t} \lambda a_i(\ell) + \sum_{\tau=t-k_m(t)}^{t} (1-\lambda a_i(\tau)) \prod_{\ell=\tau+1}^{t} \lambda a_i(\ell) \right) \\ &= M_m \left( \prod_{\ell=t-k_m(t)}^{t} \lambda a_i(\ell) + \sum_{\tau=t-k_m(t)}^{t} \Delta_\tau \left( \prod_{\ell=\tau}^{t} \lambda a_i(\ell) \right) \right) = M_m \end{aligned}$$

for $i = 1, 2, \dots, n$. The above inequality implies that

$$M_{m+1} \le M_m \quad \text{for} \quad m = 0, 1, 2, \dots, \quad \text{and} \quad |y_i(t)| \le M_0 \quad \text{for} \quad i = 1, 2, \dots, n.$$

Therefore,

$$|x_i(t)| \le \frac{M_0}{\lambda^t} \quad \text{for} \quad t \ge t_0 \quad \text{and} \quad i = 1, 2, \dots, n$$

and the proof is complete. □

## 3 Examples and remarks

In this section we give some examples with the characteristic cases of the delay functions to illustrate the main results. The following three examples illustrate Theorems 2.1, 2.2, 2.3 in the case when the lag function is between two linear delay functions, or between two power delay functions, or between two constant delay functions. Let be

$$A(t) = \begin{bmatrix} \frac{1}{(1+t)^2} & 0 \\ 0 & \frac{1}{(1+t)^2} \end{bmatrix}, \qquad B(t) = \begin{bmatrix} \frac{1}{3} - \frac{1}{3(1+t)^2} & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} - \frac{1}{2(1+t)^2} \end{bmatrix}, \tag{3.1}$$

$$t_0 = 4, \quad \phi_1(t) = \phi_2(t) = 1.5 \sin 6t. \tag{3.2}$$

It is obvious that the hypotheses $(H_1)$ and $(H_2)$ are fulfilled. The hypothesis $(H_6)$ is satisfied with

$$Q = \frac{41}{48}, \quad \alpha_1 = \alpha_2 = \frac{24}{25}, \tag{3.3}$$

since

$$\begin{aligned} |b_{11}(t)| + |b_{12}(t)| &= \frac{1}{3}\left(1 - \frac{1}{(1+t)^2}\right) + \frac{1}{2} \\ &= \left(1 - \frac{1}{(1+t)^2}\right) \left( \frac{1}{3} + \frac{1}{2} \left(1 - \frac{1}{(1+t)^2}\right)^{-1} \right) \\ &\le \left(1 - \frac{1}{(1+t)^2}\right) \left( \frac{1}{3} + \frac{1}{2} \left(1 - \frac{1}{(1+t_0)^2}\right)^{-1} \right) \\ &= \frac{41}{48}\left(1 - \frac{1}{(1+t)^2}\right) = Q(1 - a_1(t)) \quad \text{and similarly,} \end{aligned}$$

$$
\begin{aligned}
|b_{21}(t)| + |b_{22}(t)| &= \frac{1}{3} + \frac{1}{2}\left(1 - \frac{1}{(1+t)^2}\right) \\
&= \left(1 - \frac{1}{(1+t)^2}\right)\left(\frac{1}{3}\left(1 - \frac{1}{(1+t)^2}\right)^{-1} + \frac{1}{2}\right) \\
&\leq \left(1 - \frac{1}{(1+t)^2}\right)\left(\frac{1}{3}\left(1 - \frac{1}{(1+t_0)^2}\right)^{-1} + \frac{1}{2}\right) \\
&= \frac{61}{72}\left(1 - \frac{1}{(1+t)^2}\right) \leq Q(1 - a_2(t)),
\end{aligned}
$$

$$
1 - a_i(t) = 1 - \frac{1}{(1+t)^2} \geq 1 - \frac{1}{(1+t_0)^2} = \frac{24}{25} = \alpha_i > 0, \quad i = 1,2. \tag{3.4}
$$

**Example 3.1.** Let

$$
\sigma(t) = \frac{1}{3}(\sin t + t)
$$

be the lag function. Let the matrix functions $A$ and $B$ be defined by (3.1), the initial point $t_0$ and the initial functions be defined by (3.2). Now, it is

$$
ct \leq \sigma(t) \leq Ct \quad \text{for} \quad c = \frac{1}{4} \quad \text{and} \quad C = \frac{3}{4},
$$

so the lag function is between two linear functions. Set

$$
t_{-1} = \frac{1}{3}(4 + \sin 4) \approx 1.08107,
$$

so the initial interval is

$$
\left[\frac{1}{3}(4 + \sin 4), 4\right).
$$

Since

$$
\left(\frac{3}{4}\right)^{1+\log_4 \frac{48}{41}} \approx 0.725864 < \frac{41}{48} \approx 0.85417, \quad \text{for} \quad K = \log_4 \frac{48}{41} \approx 0.11371,
$$

the condition $C^{1+K} < Q$ is satisfied. So, the conditions of Theorem 2.1 are satisfied for values $Q$, $\alpha_1$, $\alpha_2$ defined by (3.3). Therefore, with values

$$
M_0 = M_{10} = M_{20} = \sup_{\frac{1}{3}(4+\sin 4) \leq t < 4} \left\{1.5t^K |\sin 6t|\right\} \approx 1.75244, \quad R_1 = R_2 \approx 1.23841,
$$

for the solution of the system (1.1) it follows that

$$
|x_1(t)| \leq M_0 \frac{R_1}{t^K} \quad \text{and} \quad |x_2(t)| \leq M_0 \frac{R_2}{t^K} \quad \text{for all} \quad t \geq 4.
$$

That means the function

$$
\gamma_i(t) = M_0 \frac{R_i}{t^K}
$$

is a cover function of the component $x_i$ of the solution, $i = 1,2$. The graphs of the functions $x_1$ (blue curve), $\gamma_1$ and $-\gamma_1$ (black curves) are shown in left picture of Figure 3.1. Red curve in right picture of Figure 3.1 is the graph of the component $x_2$ and the black curves are the graphs of the cover functions $\gamma_2$ and $-\gamma_2$.

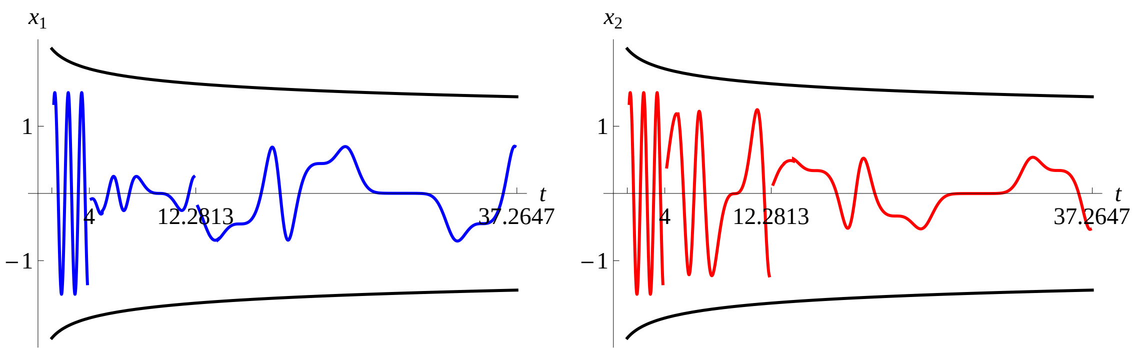


Figure 3.1: Graphs of the components of the solution and their cover functions for system with the lag function between two linear functions.

**Remark 3.2.** In this example the lag function is between two linear functions, and we got that the rate of convergence of the components of the solutions can be estimated by a power function. The components of solution are decaying functions because their cover functions are decaying functions. Since the initial functions are continuous functions on the initial interval $[t_{-1}, t_0]$, therefore the components of solution are piecewise continuous, i.e. the components of solution are continuous on the appropriate intervals $(t_m, t_{m+1})$, $m = 0, 1, 2, \ldots$

**Example 3.3.** Let

$$\sigma(t) = \left(\frac{1}{6}\cos t + 1\right)\sqrt{t}$$

be the lag function. Let the matrix functions $A$ and $B$ be defined by (3.1), and the initial point $t_0$ and the initial functions be defined by (3.2). Now, it is

$$t^c \le \sigma(t) \le t^C \quad \text{for} \quad c = \frac{1}{3} \quad \text{and} \quad C = \frac{2}{3},$$

so the lag function is between two power functions. Set

$$t_{-1} = \left(\frac{1}{6}\cos t_0 + 1\right)\sqrt{t_0} \approx 1.78212,$$

so the initial interval is

$$\left[2\left(\frac{1}{6}\cos 4 + 1\right), 4\right).$$

The conditions of Theorem 2.2 are satisfied for the values $Q$, $\alpha_1$, $\alpha_2$ defined by (3.3), so with the values

$$K = \log_3 \frac{48}{41} \approx 0.14348, \quad M_0 = M_{10} = M_{20} \approx 1.56897, \quad R_1 = R_2 \approx 1.01811,$$

for the solution of the system (1.1) it follows that

$$|x_1(t)| \le M_0 \frac{R_1}{\ln^K t} \quad \text{and} \quad |x_2(t)| \le M_0 \frac{R_2}{\ln^K t} \quad \text{for all} \quad t \ge 4.$$

Therefore the function

$$\gamma_i(t) = M_0 \frac{R_i}{\ln^K t}$$

is cover function of the $x_i$, $i = 1, 2$. The graphs of the functions $x_1$ (blue curve), $\gamma_1$ and $-\gamma_1$ (black curves) are shown in left picture of Figure 3.2, and the functions $x_2$ (red curve), $\gamma_2$ and $-\gamma_2$ (black curves) are presented in right picture of Figure 3.2.

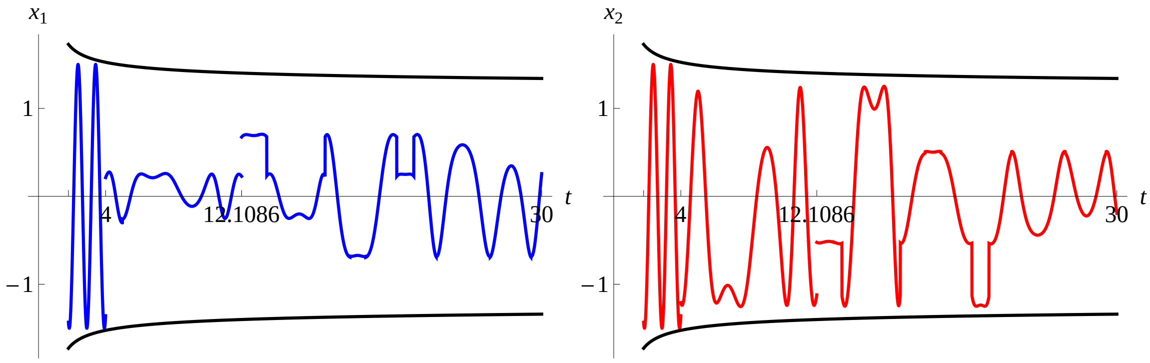


Figure 3.2: Graphs of the components of the solution and their cover functions for system with the lag function between two power functions.

**Remark 3.4.** In the above example the lag function is between two power functions, and we got that the rate of convergence of the components of the solutions can be estimated by a power of logarithmic function. Now, the components of solution are decaying functions because their cover functions are decaying functions. Since the initial functions are continuous on the initial interval, hence the components of solution are piecewise continuous functions for $t \geq t_0$. We can observe that, in this case, the convergence to zero is much slower than the convergence in the case when the cover function is only a power function.

**Example 3.5.** Let

$$\sigma(t) = t - \sin 2t - 2$$

be the lag function. Let the matrix functions $A$ and $B$ be defined by (3.1), the initial point $t_0$ and the initial functions be defined by (3.2). Now, for $c = 1$ and $C = 3$, $t - C \leq \sigma(t) \leq t - c$ is satisfied, so the lag function is between two constant delay functions. Notice that function $\delta(t) = \sin 2t + 2$ has value 1 for infinitely many points, but $\delta(t) \not\equiv 1$. Set $t_{-1} = 2 - \sin 4 \approx 1.01064$, hence the initial interval is $[2 - \sin 4, 4)$. The conditions of Theorem 2.3 are satisfied with the values

$$\lambda \approx 1.03914, \quad M_0 = M_{10} = M_{20} \approx 1.74412.$$

Hence, for the solution of the system (1.1) it follows that

$$|x_1(t)| \leq \frac{M_0}{\lambda^t} \quad \text{and} \quad |x_2(t)| \leq \frac{M_0}{\lambda^t} \quad \text{for all} \quad t \geq 4,$$

so the components of the solution have the same cover function

$$\gamma(t) = \frac{M_0}{\lambda^t}.$$

The graphs of the components $x_1$ and $x_2$ (blue and red curves) with functions $\gamma$ and $-\gamma$ (black curves) are presented in Figure 3.3.

**Remark 3.6.** In this example the lag function is between two constant delay functions, and we got that the rate of convergence of the components of the solutions can be estimated by an exponential function. In view of the fact that the cover functions are decaying functions, components of solution are also decaying functions. According to the continuity of initial functions on the initial interval, the components of solution are piecewise continuous functions for $t \geq t_0$. For this example, we can observe that the convergence to zero is much faster than the convergence in the case when the cover function is only a power function.

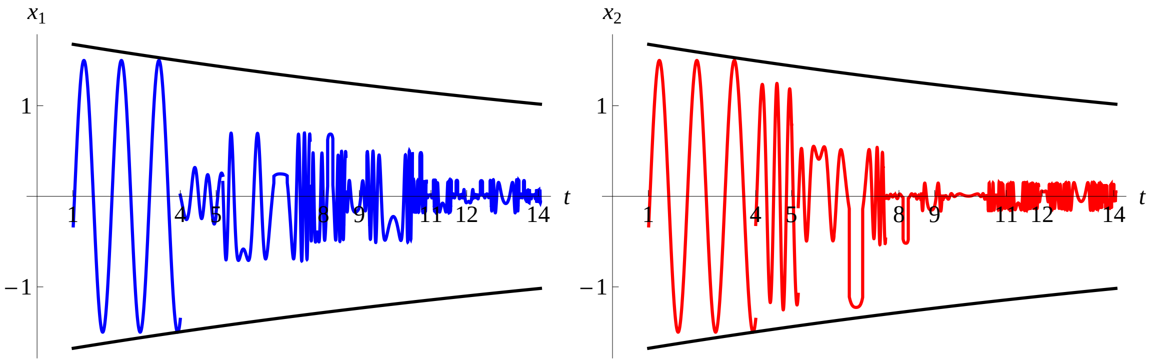


Figure 3.3: Graphs of the components of the solution and their cover functions for system with the lag function between two constant delay functions.

The following example presents the case when Theorem 2.1 gives only boundary condition.

**Example 3.7.** Let be

$$A(t) = \begin{bmatrix} \frac{1}{(1+t)^2} & 0 \\ 0 & \frac{1}{(1+t)^2} \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} \frac{1}{2} - \frac{1}{2(1+t)^2} & \frac{1}{2} - \frac{1}{2(1+t)^2} \\ \frac{1}{3} - \frac{1}{3(1+t)^2} & \frac{1}{3} - \frac{1}{3(1+t)^2} \end{bmatrix}$$

and let $\sigma(t) = \frac{t}{2}$ be the linear delay function. Now, $c = C = \frac{1}{2}$. For the initial point $t_0 = 4$ we get $t_{-1} = 2$ and the initial interval $[2, 4)$. For the initial functions we choose

$$\phi_1(t) = \phi_2(t) = 1.5 \sin 6t.$$

Due to (3.4),

$$|b_{11}(t)| + |b_{12}(t)| = 1 - \frac{1}{(1+t)^2} = 1 \cdot (1 - a_1(t)) \quad \text{and}$$

$$|b_{21}(t)| + |b_{22}(t)| = \frac{2}{3}\left(1 - \frac{1}{(1+t)^2}\right) \leq 1 \cdot \left(1 - \frac{1}{(1+t)^2}\right) = 1 \cdot (1 - a_2(t)),$$

the hypothesis $(H_6)$ is satisfied for the values $Q = 1$, $\alpha_1 = \alpha_2 = \frac{24}{25}$, and it follows that $K = 0$. Hence, for the values

$$M_0 = M_{10} = M_{20} = 1.5, \quad R_1 = R_2 = \prod_{m=1}^{\infty} \left(1 + \frac{K t_0^K}{\alpha_1 (t_0 - c^m)^{K+1}} c^{m-K}\right) = 1$$

the conditions of Theorem 2.1 are fulfilled. Now, for the solution of the system (1.1) it follows that

$$|x_1(t)| \leq M_{10} R_1 \quad \text{and} \quad |x_2(t)| \leq M_{20} R_2 \quad \text{for all} \quad t \geq 4.$$

In this case the cover function $\gamma_i(t) = M_{10} R_i$ of component $x_i$, $i = 1, 2$, is a constant function. The graphs of first component (blue curve) and functions $\gamma_1$ and $-\gamma_1$ (black curves) are plotted in left picture of Figure 3.4. The graphs of second component and its cover functions are shown in right picture of Figure 3.4.

**Remark 3.8.** In the previous example the value $K = 0$ means that Theorem 2.1 gives us only the boundedness of the solution of the considered system of difference equation. That is, the components of solution does not necessary decay. We can get a similar example for Theorem 2.2, too.

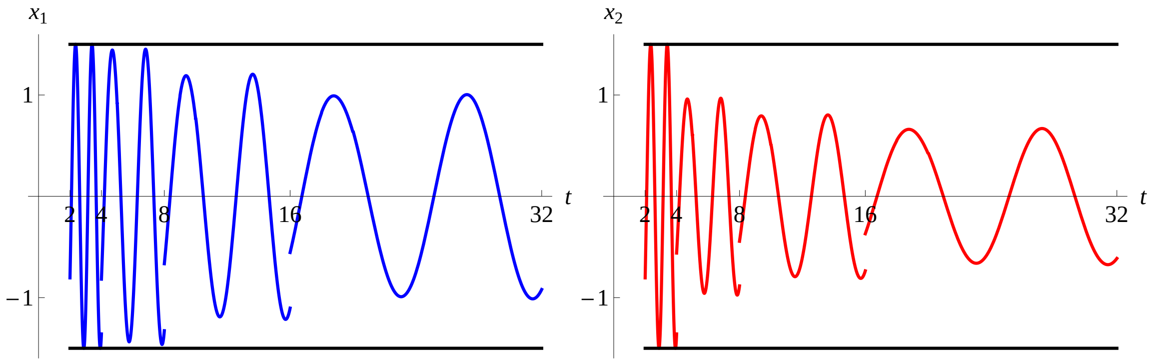


Figure 3.4: Graphs of the components of the solution and their non-decaying cover functions for system with linear lag function.

We created examples that show how the rate of convergence of the components of solutions can be estimated by properly selected auxiliary functions. In the following example we consider the components of solutions of the system of difference equations for the case $n = 2$, $c = C$, with the same coefficients, the same initial points, the same initial intervals, the same initial functions and different types of lag functions.

**Example 3.9.** Let

$$A(t) = \begin{bmatrix} \frac{1}{(1+t)^2} & 0 \\ 0 & \frac{1}{2(1+t)^2} \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} \frac{1}{2}\left(1 - \frac{1}{(1+t)^2}\right) & \frac{1}{3}\left(1 - \frac{1}{(1+t)^2}\right) \\ \frac{1}{2}\left(1 - \frac{1}{2(1+t)^2}\right) & \frac{1}{3}\left(1 - \frac{1}{2(1+t)^2}\right) \end{bmatrix}.$$

We compare the components of solutions of the system of difference equations given for the linear delay function $\sigma(t) = \frac{t}{2}$, for the power delay function $\sigma(t) = \sqrt{t}$ and for the constant delay function $\sigma(t) = t - 2$, with the initial point $t_0 = 4$, $t_{-1} = 2$ and the initial interval $[2, 4)$, and with the initial functions

$$\phi_1(t) = 1.5\left(t^2 - 2\right), \quad \phi_2(t) = 1.5\sqrt{t}.$$

In the cases of the linear delay function and the power delay function it is $c = C = \frac{1}{2}$. Since

$$\sum_{j=1}^{2} |b_{1j}(t)| = \frac{1}{2}\left(1 - \frac{1}{(1+t)^2}\right) + \frac{1}{3}\left(1 - \frac{1}{(1+t)^2}\right) = \frac{5}{6}\left(1 - \frac{1}{(1+t)^2}\right) = \frac{5}{6}(1 - a_1(t)),$$

$$\sum_{j=1}^{2} |b_{2j}(t)| = \frac{1}{2}\left(1 - \frac{1}{2(1+t)^2}\right) + \frac{1}{3}\left(1 - \frac{1}{2(1+t)^2}\right) = \frac{5}{6}\left(1 - \frac{1}{2(1+t)^2}\right) = \frac{5}{6}(1 - a_2(t)),$$

$$1 - a_1(t) = 1 - \frac{1}{(1+t)^2} \geq 1 - \frac{1}{(1+t_0)^2} = \frac{24}{25}$$

and

$$1 - a_2(t) = 1 - \frac{1}{2(1+t)^2} \geq 1 - \frac{1}{2(1+t_0)^2} = \frac{49}{50},$$

hence the hypothesis $(H_6)$ is satisfied with $Q = \frac{5}{6}$, $\alpha_1 = \frac{24}{25}$ and $\alpha_2 = \frac{49}{50}$.

For the values

$$K = \log_2 \frac{6}{5} \approx 0.263034, \quad R_1 \approx 1.09481, \quad R_2 \approx 1.09282,$$

$$M_{10} \approx 30.24, \quad M_{20} \approx 4.32, \quad M_0 = \max\{M_{10}, M_{20}\}$$

and for the lag function $\sigma(t) = \frac{t}{2}$, the conditions of Theorem 2.1 are satisfied and it follows that

$$|x_1(t)| \leq M_0 \frac{R_1}{t^K} \quad \text{and} \quad |x_2(t)| \leq M_0 \frac{R_2}{t^K} \quad \text{for all} \quad t \geq 4.$$

The graphs of the component $x_i$ of solution, cover functions $\gamma_i$ and $-\gamma_i$, where $\gamma_i(t) = M_0 \frac{R_i}{t^K}$, are plotted by blue color in Figure 3.5 (left picture for $i = 1$ and right picture for $i = 2$).

For the values

$$K = \log_2 \frac{6}{5} \approx 0.263034, \quad R_1 \approx 1.00838, \quad R_2 \approx 1.00821,$$

$$M_{10} \approx 22.88401, \quad M_{20} \approx 3.26914, \quad M_0 = \max\{M_{10}, M_{20}\}$$

and for the lag function $\sigma(t) = \sqrt{t}$, the conditions of Theorem 2.2 are also satisfied and it follows that

$$|x_1(t)| \leq M_0 \frac{R_1}{\ln^K t}, \quad \text{and} \quad |x_2(t)| \leq M_0 \frac{R_2}{\ln^K t} \quad \text{for all} \quad t \geq 4.$$

Red curves in Figure 3.5 are the graphs of the component $x_i$ of solution, cover functions $\gamma_i$ and $-\gamma_i$,

$$\gamma_i(t) = M_0 \frac{R_i}{\ln^K t}, \quad \text{for} \quad i = 1 \quad \text{and} \quad i = 2$$

respectively.

For the constant delay function $\sigma(t) = t - 2$ the conditions of Theorem 2.3 are satisfied for the values

$$\lambda = \frac{13\sqrt{111} - 3}{125} \approx 1.07171, \quad M_{10} \approx 27.70290, \quad M_{20} \approx 3.95756, \quad M_0 = \max\{M_{10}, M_{20}\}$$

and it follows that

$$|x_1(t)| \leq \frac{M_0}{\lambda^t} \quad \text{and} \quad |x_2(t)| \leq \frac{M_0}{\lambda^t} \quad \text{for all} \quad t \geq 4.$$

The graphs of the components of solution, cover functions $\gamma$ and $-\gamma$, $\gamma(t) = \frac{M_0}{\lambda^t}$, are shown by green curves in Figure 3.5.

**Remark 3.10.** The example presented above shows that the components of solutions of the system of difference equations with the linear delay lag function are power-low decaying, those with the power delay lag function are logarithmic decaying and those with the constant delay lag function are exponentially decaying. The components of the solution tend to zero for all observed lag functions. The convergence is the fastest in the case of constant delay lag function and it is the slowest in the case of power delay lag function.

**Remark 3.11.** In the above examples all the values $M_{10}$, $M_{20}$, $R_1$ and $R_2$ were determined using the software *Mathematica*.

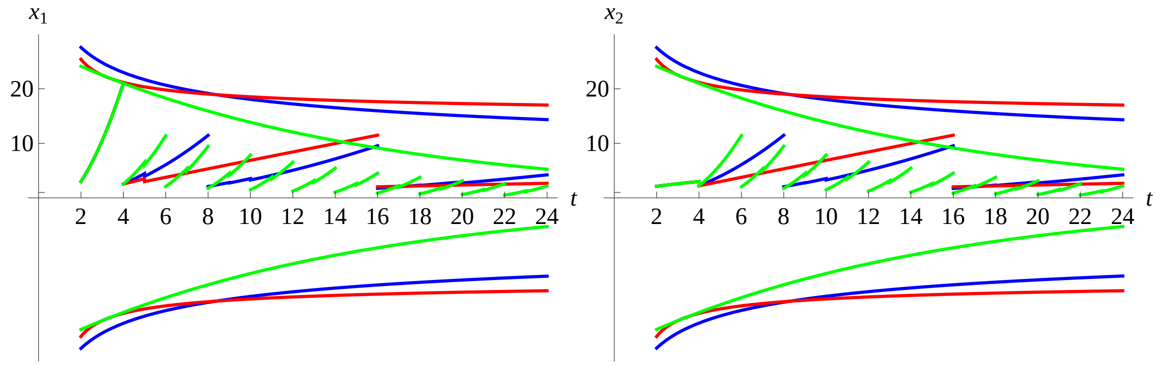


Figure 3.5: Comparison of the solutions of the system of difference equations with the linear delay, the power delay and the constant delay lag functions.

## 4 Conclusions

The system of delay difference equations with continuous time (1.1) with the initial condition (1.2) is an initial value problem. Using the step by step method, the unique solution of the initial value problem (1.1), (1.2) exists for $t \geq t_0$. Furthermore, the solution of the initial value problem (1.1), (1.2) is continuous if and only if the initial function defined by (1.2) is a continuous function and satisfies the condition

$$\phi(t_0) = A(t)\phi(t_0 - 1) + B(t_0)\phi(\sigma(t_0)). \tag{4.1}$$

If condition (4.1) is violated, then we can only speak about the existence of a piecewise continuous solution, as in the case of the examples in the previous section.

We have shown that the solutions of the initial value problem (1.1), (1.2) with lag functions squeezed between two linear functions or two power functions or two constant delay functions can be estimated by functions which tend to zero. Therefore, those solutions converge to zero.

The definition of asymptotic stability of solutions of system (1.1) can be introduced by analogy with definitions given for difference equations with continuous time and can be found, for example, in [6], pp. 193–194, [16], pp. 985–986.

**Definition 4.1.** The trivial solution $x_i(t) \equiv 0$, $i = 1, 2, \ldots, n$, of system (1.1) is called stable if for any $\varepsilon > 0$ and $t_0 > 0$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that if

$$\sup_{i=1}^{n} |\phi_i(t)| < \delta(\varepsilon, t_0), \quad \text{for} \quad t_{-1} \leq t < t_0,$$

then the solution $x^{\phi}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ of the initial value problem (1.1), (1.2) satisfies the inequality

$$\sup_{i=1}^{n} |x_i(t)| < \varepsilon \quad \text{for} \quad t \geq t_0.$$

**Definition 4.2.** The trivial solution $x_i(t) \equiv 0$, $i = 1, 2, \ldots, n$, of system (1.1) is said to be asymptotically stable if it is stable in the sense of Definition 4.1 and

$$\lim_{t\to\infty} |x_i(t)| = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

According to the properties of the received cover functions and in a sense of the above definitions we can conclude that the conditions of Theorems 2.1 and 2.2 with $K \neq 0$ and

Theorem 2.3 lead to the existence of the asymptotically stable solutions of the considered equation. These results can be motivation to further investigations for getting new conditions for existence of asymptotically stable solutions of difference equations with continuous time.

## Acknowledgements

The research of H. Péics is supported by Serbian Ministry of Science, Technology and Development for Scientific Research Grant no. III44006. The authors thank the referee for the valuable comments.

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