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Notes on spectrum and exponential decay in nonautonomous evolutionary equations

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Abstract. We first determine the dichotomy (Sacker–Sell) spectrum for certain nonautonomous linear evolutionary equations induced by a class of parabolic PDE systems. Having this information at hand, we underline the applicability of our second result: If the widths of the gaps in the dichotomy spectrum are bounded away from 0, then one can rule out the existence of super-exponentially decaying (i.e. slow) solutions of semi-linear evolutionary equations.

Keywords: nonautonomous evolutionary equation, dichotomy spectrum, Sacker–Sell spectrum, slow solutions.

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1 Motivation

When investigating evolutionary equations near non-constant reference solutions, in the vicinity of compact invariant sets (e.g. nontrivial attractors, homo- or heteroclinic solutions) or under time-variant parameters, one is confronted with a nonautonomous problem: the variational equation becomes explicitly time-dependent and an appropriate spectral theory turns out indispensable in order to determine e.g. linear stability. Due to its ambient robustness properties, uniform asymptotic stability is a favorable concept and can be determined by means of the dichotomy (dynamical or Sacker–Sell) spectrum (cf. [3, 14, 16]). Actually, applications of the aforesaid dichotomy spectrum $\Sigma \subseteq \mathbb{R}$ reach further than basic stability issues. For instance gaps in Σ allow to construct the entire hierarchy of stable and unstable manifolds, as well as their invariant foliations. Under particular assumptions on the spectrum it is even possible to extend Lu's topological linearization result [11] to a class of nonautonomous evolutionary equations in Banach spaces. Yet, specifically this endeavor requests certain preparations concerning the dichotomy spectrum.

First, for the sake of relevant examples, Σ has to be known (at least qualitatively) in various types of evolutionary differential equations. For instance, delay differential equations were considered in [12]. Building on previous results from [4, 9, 10], in our Section 3 we determine

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the dichotomy spectrum for linear evolutionary equations whose infinitesimal generator is sectorial with compact resolvent. Canonical examples include uniformly elliptic differential operators or the poly-Laplacian under the standard boundary conditions.

Second, extending the topological linearization argument of [11] requires evolutionary equations without nontrivial small solutions. This class of functions decays to 0 faster than any exponential function and typically occurs for delay differential equations (cf. [6, pp. 74ff]). Parabolic PDEs, on the other hand, cannot have slow solutions and [1, Lemma 5] serves as standard reference. In Section 4 we generalize this technical, but helpful and interesting result to semi-linear equations and allow a time-dependent linear part; furthermore, our proof is slightly simpler. This necessitates to impose two central assumptions: (a) The invariant projectors associated to the dichotomy spectrum are complete. In the autonomous special case this means that the infinitesimal generator has a complete set of eigenvectors. (b) Moreover, the width of the gaps in Σ needs to be uniformly bounded away from 0.

Indeed, the note at hand is essentially a supplement to [13] and provides preparations being crucial there. Nonetheless, we feel the present examples and results are both handy and of independent interest when dealing with nonautonomous parabolic PDEs, their geometric theory and beyond. Our approach to nonautonomous dynamics is via evolution families and 2-parameter semigroups, rather than skew-product semiflows as used in [4, 9, 10]. We feel this is more appropriate in the present situation since one can omit e.g. uniform continuity properties of the coefficient functions (in order to guarantee a compact base space). Finally, compared to [4,9,10] more general time-dependencies and a wider flexibility on the differential operator is allowed.

Notation. The kernel of a linear operator A on a Banach space X is denoted by $N(A)$, $R(A)$ is its range and id_X the identity. We write $\sigma(A)$ for the spectrum and $\sigma_p(A)$ for the point spectrum of A. The Kronecker symbol is denoted as $\delta_{kl} \in \{0,1\}$, $k,l \in \mathbb{N}$.

Given nonempty subsets $B, C \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$ it is convenient to abbreviate

$$\lambda + B := \{\lambda + b \in \mathbb{R} : b \in B\}, \qquad B + C := \{b + c \in \mathbb{R} : b \in B, c \in C\}.$$

2 Evolution families, dichotomies and Bohl exponents

For an unbounded interval $J \subseteq \mathbb{R}$ and a Banach space $(\mathcal{X}, \|\cdot\|)$, let us introduce our central notions: We begin with a useful generalization of the semigroup concept when dealing with time-dependent problems: An *evolution family* $\mathcal{T} : \{(t,s) \in J \times J : s \le t\} \to L(\mathcal{X})$ on $\mathcal{X}$ is defined as a mapping such that $(t,s) \mapsto \mathcal{T}(t,s)u$ is continuous for all $u \in \mathcal{X}$, which furthermore fulfills

- $\mathcal{T}(t,t) = \mathrm{id}_{\mathcal{X}}$ and $\mathcal{T}(t,s)\mathcal{T}(s,\tau) = \mathcal{T}(t,\tau)$ for all $\tau \le s \le t$
- there exist reals $K_0 \ge 1$, $\alpha_0 \in \mathbb{R}$ such that $\|\mathcal{T}(t,s)\| \le K_0 e^{\alpha_0(t-s)}$ for all $s \le t$.

One says the evolution family $\mathcal{T}$ has an *exponential dichotomy* (ED for short) on J, if there exists a projector $\mathcal{P} : J \to L(\mathcal{X})$ and reals $K \ge 1$, $\alpha > 0$ such that

- $\mathcal{T}(t,s)\mathcal{P}(s) = \mathcal{P}(t)\mathcal{T}(t,s)$ for all $s \le t$ ($\mathcal{P}$ is an *invariant projector*)
- $\bar{\mathcal{T}}(t,s) := \mathcal{T}(t,s)|_{N(\mathcal{P}(s))} : N(\mathcal{P}(s)) \to N(\mathcal{P}(t))$ is an isomorphism for $s < t$
- $\|\mathcal{T}(t,s)\mathcal{P}(s)\| \le Ke^{-\alpha(t-s)}$ and $\|\bar{\mathcal{T}}(s,t)\,[\mathrm{id}_{\mathcal{X}} - \mathcal{P}(t)]\| \le Ke^{\alpha(s-t)}$ for $s \le t$.

Let $N \subseteq \mathbb{N}$ be an index set. A family of projectors $\mathcal{P}_k : J \to L(\mathcal{X})$, $k \in N$, is called *complete*, if one has

$$u = \sum_{k \in N} \mathcal{P}_k(\tau)u \quad \text{for all } (\tau, u) \in J \times \mathcal{X}.$$

With $\gamma \in \mathbb{R}$ we write $\mathcal{T}_\gamma(t,s) := e^{\gamma(s-t)}\mathcal{T}(t,s)$ for the associated *scaled evolution family*. The *dichotomy spectrum* Σ_J of $\mathcal{T}$ is

$$\Sigma_J := \{\gamma \in \mathbb{R} : \mathcal{T}_\gamma \text{ admits no ED on } J\}.$$

For evolution families as defined above, Σ_J is a closed subset of $(-\infty, \alpha_0]$. In general, the spectrum depends on the interval J and at any rate it holds $\Sigma_J \subseteq \Sigma_{\mathbb{R}}$. If the evolution family $\mathcal{T}$ is the evolution operator of an abstract nonautonomous differential equation

$$\boxed{\dot{u} = \mathcal{A}(t)u,} \tag{L}$$

then we write $\Sigma_J(\mathcal{A})$ for the dichotomy spectrum. By definition, one has the relation

$$\Sigma_J(\mathcal{A} + \lambda) = \lambda + \Sigma_J(\mathcal{A}) \quad \text{for all } \lambda \in \mathbb{R}. \tag{2.1}$$

As a prototype example let us constitute

Example 2.1 (dichotomy spectrum in finite dimensions)**.** In case $\mathcal{X} = \mathbb{R}^n$, a linear ODE

$$\boxed{\dot{u} = B(t)u} \tag{2.2}$$

with continuous coefficient operator $B : J \to L(\mathbb{R}^n)$ and bounded growth generates an evolution family $T(t,s) \in L(\mathbb{R}^d)$, $t, s \in J$. Its dichotomy spectrum has the form

$$\Sigma_J(B) = \bigcup_{j=1}^{m} \left[\lambda_j^-, \lambda_j^+\right],$$

where the reals λ_j^-, λ_j^+ are ordered according to $\lambda_m^- \le \lambda_m^+ < \cdots < \lambda_1^- \le \lambda_1^+$. Each of the $m \le n$ *spectral intervals* $[\lambda_j^-, \lambda_j^+]$ corresponds to an invariant vector bundle

$$\mathcal{X}_j = \left\{(t,x) \in J \times \mathbb{R}^n : x \in R(p_j(t))\right\} \quad \text{for } 1 \le j \le m,$$

where $p_j : J \to L(\mathbb{R}^n)$ is an invariant projector and one has the Whitney sum (cf. [14,16])

$$J \times \mathbb{R}^d = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_m.$$

For the further special case of scalar ODEs $\dot{u} = a(t)u$ the dichotomy spectrum allows an explicit representation. Thereto, given a continuous $a : J \to \mathbb{R}$, its *upper Bohl* resp. *lower Bohl exponent* is defined as

$$\overline{\beta}_J(a) := \limsup_{T \to \infty} \sup_{\substack{\tau \in J \\ [\tau, \tau+T] \subseteq J}} \frac{1}{T} \int_\tau^{\tau+T} a, \qquad \underline{\beta}_J(a) := \liminf_{T \to \infty} \inf_{\substack{\tau \in J \\ [\tau, \tau+T] \subseteq J}} \frac{1}{T} \int_\tau^{\tau+T} a.$$

One obviously has $\underline{\beta}_J(a) \le \overline{\beta}_J(a)$ and the integrability conditions

$$\sup_{0 \le t-s \le 1} \int_s^t a < \infty, \qquad \sup_{0 \le s-t \le 1} \int_s^t a < \infty \tag{2.3}$$

are necessary and sufficient for finite Bohl exponents, i.e. $\overline{\beta}_J(a) < \infty$ resp. $-\infty < \underline{\beta}_J(a)$ (for this, see [7, p. 259, Proposition 3.3.14]). The boundedness $\gamma := \sup_{t\in J}|a(t)| < \infty$ even guarantees that $-\gamma \le \underline{\beta}_J(a) \le \overline{\beta}_J(a) \le \gamma$. Moreover, Bohl exponents satisfy

$$\underline{\beta}_J(\lambda + a) = \lambda + \underline{\beta}_J(a), \qquad \overline{\beta}_J(\lambda + a) = \lambda + \overline{\beta}_J(a) \quad \text{for all } \lambda \in \mathbb{R}.$$

Example 2.2. (1) If $a(t) \equiv \alpha$, then $\underline{\beta}_J(a) = \overline{\beta}_J(a) = \alpha$ for all $\alpha \in \mathbb{R}$.

(2) If $a : J \to \mathbb{R}$ is θ-periodic with some $\theta > 0$, i.e. $a(t+\theta) = a(t)$ holds for all $t \in J$ satisfying $t + \theta \in J$, then Bohl exponents are the means

$$\underline{\beta}_J(a) = \overline{\beta}_J(a) = \frac{1}{\theta}\int_t^{t+\theta} a(r)\,\mathrm{d}r \quad \text{for all } t \in J.$$

(3) If $a(t) = \frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2}\sin\ln t$ with reals $\alpha \le \beta$, then $\underline{\beta}_J(a) = \alpha$ and $\overline{\beta}_J(a) = \beta$ holds for every unbounded subinterval $J \subseteq (0,\infty)$.

(4) If $a : \mathbb{R} \to \mathbb{R}$ is continuous and fulfills $\lim_{t\to\pm\infty} a(t) = \alpha^\pm$ for reals α^-, α^+, then

$$\underline{\beta}_{(-\infty,\tau]}(a) = \overline{\beta}_{(-\infty,\tau]}(a) = \alpha^-, \qquad \underline{\beta}_{[\tau,\infty)}(a) = \overline{\beta}_{[\tau,\infty)}(a) = \alpha^+ \quad \text{for all } \tau \in \mathbb{R},$$
$$\underline{\beta}_{\mathbb{R}}(a) = \min\left\{\alpha^-, \alpha^+\right\}, \qquad \overline{\beta}_{\mathbb{R}}(a) = \max\left\{\alpha^-, \alpha^+\right\}.$$

Equations with such asymptotically constant or periodic coefficients were studied in [10].

The importance of Bohl exponents is due to their role in stability theory and as boundary points of the dichotomy spectrum. Our above Example 2.2 can be used in the following lemma.

Lemma 2.3. *If a continuous function $a : J \to \mathbb{R}$ fulfills the integrability conditions (2.3), then the ordinary differential equation*

$$\boxed{\dot{u} = a(t)u} \tag{2.4}$$

in $\mathcal{X}$ possesses the dichotomy spectrum $\Sigma_J(a) = [\underline{\beta}_J(a), \overline{\beta}_J(a)]$.

Proof. Since (2.4) has the evolution operator $\mathcal{T}(t,s) = \left(\exp\int_s^t a\right)\mathrm{id}_{\mathcal{X}}$ for all $t, s \in J$, the claim follows from [8, Proposition A.2]. □

3 Dichotomy spectrum of parabolic equations

Let X be a separable Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$.

3.1 Generators with discrete spectrum

Let us suppose $A : D(A) \subseteq X \to X$ is a linear unbounded operator generating a C_0-semigroup $S : [0,\infty) \to L(X)$ and

- $\sigma(A) = \sigma_p(A) = \{\lambda_k : k \in \mathbb{N}\} \subseteq \mathbb{R}$ with

  $$\lambda_1 \ge \lambda_2 \ge \cdots,$$

  where every eigenvalue λ_k is repeated as many times as its finite *multiplicity* given by

  $$\mu_k = \dim N(A - \lambda_k\,\mathrm{id}_X)$$

- each corresponding eigenspace

$$N(A-\lambda_k\,\mathrm{id}_X)=\mathrm{span}\left\{e_1^k,\dots,e_{\mu_k}^k\right\}$$

is spanned by orthogonal eigenvectors $e_1^k,\dots,e_{\mu_k}^k\in X$ of A. Thus,

$$\Pi_k:=\sum_{j=1}^{\mu_k}\langle\cdot,e_j^k\rangle e_j^k\in L(X) \tag{3.1}$$

defines an orthogonal projection on X for every $k\in\mathbb{N}$

- $\left\{e_j^k\in X:\,1\le j\le\mu_k \text{ and } k\in\mathbb{N}\right\}$ is a complete orthonormal set in X.

The typical examples for such operators are as follows, where $\Omega\subseteq\mathbb{R}^d$ denotes a bounded domain with piecewise smooth boundary.

Example 3.1 (uniformly elliptic differential operators)**.** Consider a uniformly elliptic differential operator in divergence form (see e.g. [5, pp. 354ff.])

$$Lu(x)=\sum_{i,j=1}^{d}D_j\big(a^{ij}(x)D_iu(x)\big)\quad\text{for all } x\in\Omega$$

with coefficients $a^{ij}\in C^\infty(\bar\Omega)$ satisfying $a^{ij}=a^{ji}$ for all $1\le i,j\le d$. If we define the operator

$$(Au)(x)=Lu(x)$$

on $X=L^2(\Omega)$, then the above properties hold:

(1) In order to capture Dirichlet boundary conditions $u(x)\equiv 0$ on $\partial\Omega$, choose

$$D(A)=H^2(\Omega)\cap H_0^1(\Omega).$$

The principle eigenvalue $\lambda_1<0$ is negative; the eigenfunctions are contained in $H_0^1(\Omega)$.

(2) Concerning Neumann boundary conditions $D_\nu u(x)\equiv 0$ on $\partial\Omega$, choose the domain

$$D(A)=\left\{u\in H^2(\Omega):\,D_\nu u(x)\equiv 0\text{ on }\partial\Omega\right\}.$$

For the Laplacian one has $\lambda_1=0$.

In particular, if L is the Laplacian Δ equipped with Dirichlet, Neumann or Robin boundary conditions (i.e. $au(x)+bD_\nu u(x)\equiv 0$ on $\partial\Omega$), then according to Weyl's Law the eigenvalues behave asymptotically as $\lambda_k\sim C_d(\Omega)k^{2/d}$ for $k\to\infty$.

Example 3.2 (poly-Laplacian)**.** Given $m\in\mathbb{N}$ let us consider the poly-Laplacian

$$Lu(x)=-(-\Delta)^m u(x)\quad\text{for all } x\in\Omega$$

with homogeneous boundary conditions $u(x)\equiv D_\nu u(x)\equiv\dots\equiv D_\nu^{m-1}u(x)\equiv 0$ on $\partial\Omega$. It yields an operator

$$(Au)(x):=Lu(x)$$

on $X=L^2(\Omega)$ with $D(A)=H^{2m}(\Omega)\cap H_0^m(\Omega)$ fulfilling our above properties. The principle eigenvalue is $\lambda_1<0$ and thanks to [2, p. 12, Theorem 1.11], the eigenvalues behave asymptotically as $\lambda_k\sim C_d(\Omega)k^{2m/d}$ for $k\to\infty$.

Example 3.3 (Laplacian in 1*d*)**.** The special cases with $\Omega = (0, \ell)$, $\ell > 0$, and

$$Lu(x) = u_{xx}(x) \quad \text{for all } 0 < x < \ell$$

allow more explicit results:

(1) For Dirichlet boundary conditions,

$$D(A) = H^2(0, \ell) \cap H_0^1(0, \ell)$$

yields simple eigenvalues $\lambda_k = -(\frac{\pi k}{\ell})^2$ with eigenfunctions $e^k(x) = \sqrt{\frac{2}{\ell}} \sin(\frac{\pi k}{\ell} x)$ for $k \in \mathbb{N}$.

(2) For Neumann boundary conditions,

$$D(A) = \left\{ u \in H^2(0, \ell) : u_x(0) = u_x(\ell) = 0 \right\},$$

all eigenvalues $\lambda_k = -\left(\frac{\pi(k-1)}{\ell}\right)^2$ are simple with the eigenfunctions

$$e^1(x) \equiv \frac{1}{\sqrt{\ell}}, \qquad e^k(x) = \sqrt{\frac{2}{\ell}} \cos\left(\frac{\pi(k-1)}{\ell} x\right) \quad \text{for all } k \geq 2.$$

Finally, in the above examples $-A : D(A) \subseteq X \to X$ is a sectorial operator and A generates an analytic semigroup $S : [0, \infty) \to L(X)$ on X (cf. [15, p. 106, Theorem 38.1]) allowing the Fourier representation

$$S(t)u = \sum_{k \in \mathbb{N}} e^{\lambda_k t} \Pi_k u \quad \text{for all } t \geq 0,\ u \in X. \tag{3.2}$$

3.2 Systems of parabolic equations

Let L denote a differential operator from the previous Examples 3.1–3.3. Consider the n-dimensional system of PDEs

$$\begin{cases} u_t^1 = d_{11} L u^1 + \cdots + d_{1n} L u^n, \\ \quad \vdots \\ u_t^n = d_{n1} L u^1 + \cdots + d_{nn} L u^n, \end{cases} \tag{3.3}$$

which briefly can be written as

$$U_t = D\mathcal{L}U$$

with $U = (u^1, \ldots, u^n)^T$, $\mathcal{L}U := (Lu^1, \ldots, Lu^n)^T$. The "diffusion matrix" $D \in L(\mathbb{R}^n)$ has the entries d_{ij} and is supposed to be symmetric positive-definite.

In order to formulate (3.3) as an abstract evolutionary equation in a separable Hilbert space, we equip the cartesian product $\mathcal{X} := X^n$ with the inner product

$$\langle\langle U, V \rangle\rangle := \sum_{j=1}^{n} \langle u^j, v^j \rangle \quad \text{for all } U, V \in \mathcal{X}.$$

Endowing the PDE system (3.3) with ambient boundary conditions allows us to define

$$(\mathcal{A}U)(x) := D\mathcal{L}U(x) \quad \text{for all } x \in \Omega \tag{3.4}$$

as an operator on $\mathcal{X} = L^2(\Omega)^n$ and the domain $D(\mathcal{A}) = D(A)^n$. The diagonalizability assumption on D shows that also $-\mathcal{A}$ is sectorial and thus $\mathcal{A}$ generates an analytical semigroup $\mathcal{S} : [0, \infty) \to L(\mathcal{X})$ on $\mathcal{X}$. Thanks to (3.2) it allows the Fourier representation

$$\mathcal{S}(t)U = \sum_{k \in \mathbb{N}} e^{\lambda_k t D} P_k U \tag{3.5}$$

with $U \in \mathcal{X}$ and the complete family $(P_k)_{k\in\mathbb{N}}$ of orthogonal projections on $\mathcal{X}$ given by

$$P_k = \operatorname{diag}(\Pi_k, \dots, \Pi_k).$$

Here, $\Pi_k \in L(X)$ are the orthogonal projections from (3.1) and one has

$$P_k P_l = \delta_{kl} P_l \quad \text{for all } k, l \in \mathbb{N}. \tag{3.6}$$

Lemma 3.4. *Under the above assumptions it is $\mathcal{S}(t)P_k = P_k\mathcal{S}(t)$ for all $t \geq 0$, $k \in \mathbb{N}$.*

Proof. Because D is a positive-definite matrix, there exists an invertible $S \in L(\mathbb{R}^n)$ such that

$$SDS^{-1} = \operatorname{diag}(d_1, \dots, d_n)$$

with eigenvalues $d_j > 0$ for all $1 \leq j \leq n$. Suppose that the entries of S^{-1} are denoted by $\tilde{s}_{ij}$. Given $U \in \mathcal{X}$ we first obtain

$$P_j S^{-1} U = P_j \sum_{l=1}^{n} \begin{pmatrix} \tilde{s}_{1l} u^l \\ \vdots \\ \tilde{s}_{nl} u^l \end{pmatrix} = \sum_{l=1}^{n} \begin{pmatrix} \tilde{s}_{1l} \Pi_j u^l \\ \vdots \\ \tilde{s}_{nl} \Pi_j u^l \end{pmatrix} = S^{-1} \begin{pmatrix} \Pi_j u^1 \\ \vdots \\ \Pi_j u^n \end{pmatrix} U = S^{-1} P_j U \tag{3.7}$$

for all $j \in \mathbb{N}$ and this implies

$$\begin{aligned}
S(P_j\mathcal{S}(t)U) &\overset{(3.5)}{=} S\left(P_j \sum_{k\in\mathbb{N}} e^{\lambda_k t S^{-1}\operatorname{diag}(d_1,\dots,d_n)S} P_k U\right) \\
&= S\left(P_j S^{-1} \sum_{k\in\mathbb{N}} e^{\lambda_k t \operatorname{diag}(d_1,\dots,d_n)} S P_k U\right) \\
&\overset{(3.7)}{=} \sum_{k\in\mathbb{N}} P_j P_k \begin{pmatrix} e^{\lambda_k t d_1} & & \\ & \ddots & \\ & & e^{\lambda_k t d_n} \end{pmatrix} \sum_{l=1}^{n} \begin{pmatrix} s_{1l}\Pi_k u^l \\ \vdots \\ s_{nl}\Pi_k u^l \end{pmatrix} \\
&\overset{(3.6)}{=} P_j \sum_{l=1}^{n} \begin{pmatrix} e^{\lambda_j t d_1} s_{1l}\Pi_j u^l \\ \vdots \\ e^{\lambda_j t d_n} s_{nl}\Pi_j u^l \end{pmatrix} = \sum_{l=1}^{n} \begin{pmatrix} e^{t\lambda_j d_1} s_{1l}\Pi_j u^l \\ \vdots \\ e^{t\lambda_j d_n} s_{nl}\Pi_j u^l \end{pmatrix} \\
&= e^{t\lambda_j \operatorname{diag}(d_1,\dots,d_n)} S P_j U = S e^{t\lambda_j D} P_j U \overset{(3.6)}{=} S\left(\sum_{k\in\mathbb{N}} e^{\lambda_k t D} P_k P_j U\right) \\
&\overset{(3.5)}{=} S(\mathcal{S}(t)P_j U) \quad \text{for all } t \geq 0.
\end{aligned}$$

Thus, the claim is established by multiplying with S^{-1} from the left. □

We first capture the effect of a scalar multiplicative and time-dependent perturbation on the dichotomy spectrum of (3.3). Thereto, assume that $a : J \to (0, \infty)$ is a continuous function fulfilling the integrability conditions (2.3). Endowed with ambient boundary conditions the system of parabolic equations

$$U_t = a(t) D \mathcal{L} U$$

can be formulated as nonautonomous abstract evolutionary equation

$$\boxed{\dot{u} = a(t)\mathcal{A}u,} \tag{3.8}$$

whose evolution operator $\mathcal{T}(t,s) \in L(\mathcal{X})$ is given by

$$\mathcal{T}(t,s) = \mathcal{S}\left(\int_s^t a\right) \overset{(3.5)}{=} \sum_{k\in\mathbb{N}} e^{\lambda_k \int_s^t aD} P_k \quad \text{for all } s \le t. \tag{3.9}$$

This representation allows us to obtain

Theorem 3.5 (multiplicative perturbation)**.** *The dichotomy spectrum of the evolutionary equation* (3.8) *is*

$$\Sigma_J(a\mathcal{A}) = \bigcup_{k\in\mathbb{N}} \bigcup_{j=1}^{n} \left[\underline{\beta}_J(\lambda_k d_j a), \overline{\beta}_J(\lambda_k d_j a)\right]$$

with $\sigma(D) = \{d_1, \dots, d_n\}$ *and complete invariant projectors* $P_k : J \to L(\mathcal{X})$, $k \in \mathbb{N}$.

Proof. Thanks to Lemma 3.4 and (3.9) we obtain for every $k \in \mathbb{N}$ that

$$P_k\mathcal{T}(t,s) = P_k\mathcal{S}\left(\int_s^t a\right) = \mathcal{S}\left(\int_s^t a\right) P_k = \mathcal{T}(t,s)P_k \quad \text{for all } s \le t.$$

Hence, the finite-dimensional vector bundles $\mathcal{X}_k := \{(t,U) \in J \times \mathcal{X} : U \in R(P_k)\}$ are invariant w.r.t. (3.8). Thanks to (3.9), inside of each $\mathcal{X}_k$ the dynamics is determined by

$$\dot{u} = \lambda_k a(t) Du, \tag{3.10}$$

having an evolution operator satisfying $T^k(t,s) := \mathcal{T}(t,s)P_k$. It consequently follows

$$\mathcal{T}(t,s) = \sum_{k\in\mathbb{N}} T^k(t,s) \quad \text{for all } s \le t$$

and thus

$$\Sigma_J(a\mathcal{A}) = \bigcup_{k\in\mathbb{N}} \Sigma_J(a\lambda_k D).$$

Because D is assumed to be symmetric, the ODEs (3.10) are kinematically similar to the diagonal systems $\dot{u} = \lambda_k a(t)\,\mathrm{diag}(d_1, \dots, d_n)u$ for all $k \in \mathbb{N}$. Since kinematic similarity leaves the dichotomy spectrum invariant, one obtains

$$\Sigma_J(a\lambda_k D) = \Sigma_J(a\lambda_k\,\mathrm{diag}(d_1, \dots, d_n)) = \bigcup_{l=1}^{n} \Sigma_J(a\lambda_k d_l) \quad \text{for all } k \in \mathbb{N}$$

due to the fact that the spectrum of diagonal systems is the union of their diagonal spectra. Then the assertion follows with Lemma 2.3. □

Example 3.6 (periodic case)**.** If $a : J \to (0,\infty)$ is θ-periodic, then Example 2.2 (2) guarantees

$$\Sigma_J(a\mathcal{A}) = \frac{1}{\theta}\left(\int_t^{t+\theta} a\right) \bigcup_{j=1}^{n} d_j \bigcup_{k\in\mathbb{N}} \{\lambda_k\},$$

i.e. one obtains a discrete spectrum preserving the asymptotics of λ_k for $k \to \infty$, provided that the mean $\int_t^{t+\theta} a \neq 0$ does not vanish.

Example 3.7 (asymptotically autonomous case)**.** Let $\sigma(D) = \{1\}$ and suppose the eigenvalues of $\mathcal{L}$ form a strictly decreasing sequence in $(-\infty, 0]$ with

$$\lim_{k\to\infty} \lambda_k = -\infty, \qquad \lim_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} = 1.$$

For $a : \mathbb{R} \to (0,\infty)$ satisfying the limit relations $\lim_{t\to\pm\infty} a(t) = \alpha^\pm$ with $0 < \alpha^-, \alpha^+$ we set $\underline{\mu} := \min\{\alpha^+, \alpha^-\}$, $\overline{\mu} := \max\{\alpha^-, \alpha^+\}$ and obtain from Theorem 3.5 that

$$\Sigma_{\mathbb{R}}(a\mathcal{A}) = \bigcup_{k\in\mathbb{N}} \left[\lambda_k\overline{\mu}, \lambda_k\underline{\mu}\right].$$

If $\alpha^+ = \alpha^-$, then $\Sigma_{\mathbb{R}}(a\mathcal{A}) = \alpha^+ \bigcup_{k\in\mathbb{N}} \{\lambda_k\}$. Otherwise, since the sequence $\left(\frac{\lambda_{k+1}}{\lambda_k}\right)_{k\in\mathbb{N}}$ approaches its limit from above, there is a minimal $k^* \in \mathbb{N}$ with $\lambda_{k+1}/\lambda_k < \overline{\mu}/\underline{\mu}$ for all $k \geq k^*$. We derive $\underline{\mu}\lambda_{k+1} > \overline{\mu}\lambda_k$ and hence successive spectral intervals $[\lambda_k\overline{\mu}, \lambda_k\underline{\mu}]$ overlap for every $k \geq k^*$. Thus, the dichotomy spectrum consists only of finitely many intervals

$$\Sigma_{\mathbb{R}}(a\mathcal{A}) = \Big(-\infty, \lambda_{k^*}\underline{\mu}\Big] \cup \bigcup_{k=1}^{k^*-1} \left[\lambda_k\overline{\mu}, \lambda_k\underline{\mu}\right].$$

Our second aim is to describe the impact of linear-homogeneous perturbations in (3.3). Given a continuous matrix-valued function $B : J \to L(\mathbb{R}^n)$ we consider the PDEs

$$U_t = a(t)\mathcal{L}U + B(t)U.$$

After fixing ambient boundary conditions, it gives rise to the abstract nonautonomous evolutionary equation

$$\boxed{\dot{u} = \left[a(t)\mathcal{A} + \mathcal{B}(t)\right]u} \tag{3.11}$$

with $(\mathcal{A}U)(x) := \mathcal{L}U(x)$ and $(\mathcal{B}(t)U)(x) := B(t)U(x)$ for all $t \in J$, $U \in \mathcal{X}$ and $x \in \Omega$.

Theorem 3.8 (homogeneous perturbation)**.** *The dichotomy spectrum of the evolutionary equation* (3.11) *is*

$$\Sigma_J(a\mathcal{A} + \mathcal{B}) = \bigcup_{k\in\mathbb{N}} \Sigma_J(\lambda_k a + B)$$

and possesses complete projectors $p_l(\cdot)P_k : J \to L(\mathcal{X})$, $1 \leq l \leq m$, $k \in \mathbb{N}$, where $p_j : J \to L(\mathbb{R}^d)$ are the invariant projectors from Example 2.1.

Proof. We subdivide the proof into two steps.

(I) For $t \in J$ we set $\mathcal{P}_k^l(t) := p_l(t)P_k \in L(\mathcal{X})$ and write p_{ij}^l for the components of p_l. Because the orthogonal projections $\Pi_k \in L(X)$ are linear mappings, we obtain

$$\mathcal{P}_k^l(t)U = p_l(t)\begin{pmatrix}\Pi_k u^1\\ \vdots \\ \Pi_k u^n\end{pmatrix} = \sum_{j=1}^n \begin{pmatrix}p_{1j}^l(t)\Pi_k u^1\\ \vdots \\ p_{nj}^l(t)\Pi_k u^n\end{pmatrix} = P_k\sum_{j=1}^n \begin{pmatrix}p_{1j}^l(t)u^1\\ \vdots \\ p_{nj}^l(t)u^n\end{pmatrix} = P_k p_l(t)U$$

for all $U = (u^1, \ldots, u^n) \in \mathcal{X}$ and thus

$$\mathcal{P}_k^l(t) = P_k p_l(t). \tag{3.12}$$

Since P_k is a projection and p_l a projector, this allows us to show

$$\mathcal{P}_k^l(t)^2 = p_l(t)P_k p_l(t)P_k \overset{(3.12)}{=} p_l(t)P_k p_l(t) \overset{(3.12)}{=} \mathcal{P}_k^l(t)$$

and therefore $p_l(\cdot)P_k : J \to L(\mathcal{X})$ is a projector for all $k \in \mathbb{N}$, $1 \le l \le m$.

(II) Let $\mathcal{T}_B(t,s) \in L(\mathcal{X})$, $s,t \in J$, denote the evolution family generated by the ODE

$$\dot{u} = \mathcal{B}(t)u$$

in $\mathcal{X}$. If the evolution family $T_B(t,s) \in L(\mathbb{R}^n)$ of (2.2) has the components $t_{ij}(t,s) \in \mathbb{R}$, $1 \le i,j \le n$, then it follows

$$\begin{aligned} \mathcal{P}_k^l(t)\mathcal{T}_B(t,s)U &= p_l(t)\sum_{j=1}^n \begin{pmatrix} \Pi_k t_{1j}(t,s)u^j \\ \vdots \\ \Pi_k t_{nj}(t,s)u^j \end{pmatrix} = p_l(t)\mathcal{T}_B(t,s)P_k U \\ &= \mathcal{T}_B(t,s)\mathcal{P}_k^l(s)U \quad \text{for all } s,t \in J, \end{aligned} \tag{3.13}$$

because $p_l : J \to L(\mathbb{R}^n)$ is an invariant projector for (2.2). Since the matrix function B does not depend on $x \in \Omega$, the operators $\mathcal{A}$ and $\mathcal{B}$ commute and consequently the product representation $\mathcal{T}(t,s) = \mathcal{T}_A(t,s)\mathcal{T}_B(t,s)$ holds for all $s \le t$. We arrive at

$$\begin{aligned} \mathcal{T}(t,s)\mathcal{P}_i^l(s)U &\overset{(3.9)}{=} \mathcal{S}\left(\int_s^t a\right)\mathcal{T}_B(t,s)\mathcal{P}_i^l(s)U \overset{(3.13)}{=} \mathcal{S}\left(\int_s^t a\right)\mathcal{P}_i^l(t)\mathcal{T}_B(t,s)U \\ &\overset{(3.12)}{=} \mathcal{S}\left(\int_s^t a\right)P_i p_l(t)\mathcal{T}_B(t,s)U = P_i\mathcal{S}\left(\int_s^t a\right)p_l(t)\mathcal{T}_B(t,s)U \end{aligned}$$

due to Lemma 3.4. This allows us to continue

$$\begin{aligned} \mathcal{T}(t,s)\mathcal{P}_i^l(s)U &\overset{(3.9)}{=} P_i \sum_{k\in\mathbb{N}} e^{\lambda_k \int_s^t a} P_k p_l(t)\mathcal{T}_B(t,s)U \\ &\overset{(3.12)}{=} P_i \sum_{k\in\mathbb{N}} e^{\lambda_k \int_s^t a} p_l(t)P_k\mathcal{T}_B(t,s)U \\ &= P_i p_l(t) \sum_{k\in\mathbb{N}} e^{\lambda_k \int_s^t a} P_k\mathcal{T}_B(t,s)U \\ &\overset{(3.12)}{=} \mathcal{P}_i^l(t) \sum_{k\in\mathbb{N}} e^{\lambda_k \int_s^t a} P_k\mathcal{T}_B(t,s)U \\ &\overset{(3.9)}{=} \mathcal{P}_i^l(t)\mathcal{T}_A(t,s)\mathcal{T}_B(t,s)U = \mathcal{P}_i^l(t)\mathcal{T}(t,s)U \quad \text{for all } s \le t. \end{aligned}$$

Consequently, $\mathcal{X}_i^l := \{(t,U) \in J \times \mathcal{X} : U \in R(\mathcal{P}_i^l(t))\}$ are finite-dimensional vector bundles being invariant w.r.t. (3.11). On every Whitney sum $\mathcal{X}_k^1 \oplus \cdots \oplus \mathcal{X}_k^m \subseteq J \times \mathcal{X}$ the dynamics is determined by the linear ODE

$$\dot{u} = [\lambda_k a(t) + B(t)]\, u \quad \text{for all } k \in \mathbb{N}$$

in $\mathbb{R}^n$ with evolution operator $T^k(t,s) := T(t,s)P_k$. It consequently follows that

$$\mathcal{T}(t,s) = \sum_{k\in\mathbb{N}} T^k(t,s) \quad \text{for all } s \le t$$

and thus $\Sigma_J(a\mathcal{A} + \mathcal{B}) = \bigcup_{k\in\mathbb{N}} \Sigma_J(a\lambda_k + B)$. □

Corollary 3.9. *If $a(t) \equiv \alpha > 0$ on J, then*

$$\Sigma_J(a\mathcal{A} + \mathcal{B}) = \alpha \bigcup_{k\in\mathbb{N}} \{\lambda_k\} + \Sigma_J(B).$$

Proof. For such constant functions a the dichotomy spectrum of (3.11) becomes

$$\Sigma_J(\alpha\mathcal{A} + \mathcal{B}) = \bigcup_{k\in\mathbb{N}} \Sigma_J(\alpha\lambda_k + B) \overset{(2.1)}{=} \bigcup_{k\in\mathbb{N}} \Sigma_J(\alpha\lambda_k) + \Sigma_J(B)$$

and this implies the claim. □

4 Exponential decay

Our spectral theory obtained above provides examples well-suited to illustrate a nonautonomous version of [1, Lemma 5]. This vital result ensures that forward solutions to time-variant parabolic evolutionary equations cannot decay to 0 faster than exponentially.

We actually consider abstract semi-linear evolutionary equations

$$\boxed{\dot{u} = \mathcal{A}(t)u + \mathcal{F}(t,u)} \tag{E}$$

in a Banach space $\mathcal{X}$. Here, $t \in J$ is from an interval $J \subseteq \mathbb{R}$ unbounded above. Let us suppose that the linear part (L) induces an evolution family $\mathcal{T}(t,s) \in L(\mathcal{X})$, $s \leq t$, with the properties:

($\mathbf{L_1}$) $\Sigma_J(\mathcal{A}) = \bigcup_{k\in\mathbb{N}}[\lambda_k^-, \lambda_k^+] \subseteq (-\infty, \alpha_0]$ for some $\alpha_0 \in \mathbb{R}$;

($\mathbf{L_2}$) there exist real sequences $(\alpha_k)_{k\in\mathbb{N}}$, $(\beta_k)_{k\in\mathbb{N}}$ with

$$\cdots < \alpha_2 < \beta_2 < \alpha_1 < \beta_1 < \alpha_0$$

such that $\Sigma_J(\mathcal{A}) \subseteq \bigcup_{k\in\mathbb{N}}(\beta_k, \alpha_{k-1})$ (cf. Fig. 4.1);

($\mathbf{L_3}$) the invariant projectors $\mathcal{P}_k : J \to L(\mathcal{X})$ associated to the spectral intervals $[\lambda_k^-, \lambda_k^+]$, $k \in \mathbb{N}$, are complete.

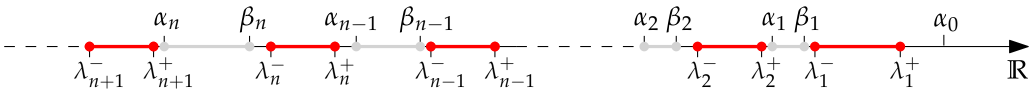

Figure 4.1: Dichotomy spectrum $\Sigma_J(\mathcal{A})$ of the linear part (L) (in red) and the gap intervals $[\alpha_n, \beta_n]$, $n \in \mathbb{N}$

Concerning the continuous nonlinearity $\mathcal{F} : J \times \mathcal{X} \to \mathcal{X}$ in (E) let us assume that

(**N**) $\mathcal{F}(t,0) \equiv 0$ on J and there exists an $L \geq 0$ such that

$$\|\mathcal{F}(t,u) - \mathcal{F}(t,\bar{u})\| \leq L\,\|u - \bar{u}\| \quad \text{for all } t \in J,\ u, \bar{u} \in \mathcal{X}.$$

The mild solution to (E) satisfying $u(\tau) = u_0$ is denoted by $u(\cdot; \tau, u_0) : [\tau, \infty) \to \mathcal{X}$ for an initial time $\tau \in J$ and an initial state $u_0 \in \mathcal{X}$.

Let the center of the gap intervals $[\alpha_k, \beta_k]$ (cf. Fig. 4.1) be denoted by $\gamma_k := \frac{\alpha_k+\beta_k}{2}$ and we introduce the *pseudo-stable fiber bundles*

$$\mathcal{W}_k := \left\{(\tau, u_0) \in J \times \mathcal{X} : \lim_{t\to\infty} e^{\gamma_k(\tau-t)} \|u(t;\tau,u_0)\| = 0\right\} \quad \text{for all } k \in \mathbb{N}.$$

These sets clearly satisfy the inclusions $\mathcal{W}_{k+1} \subseteq \mathcal{W}_k$ for all $k \in \mathbb{N}$.

Notice that a mild solution ν to (E) is said to be *small*, if for every $\gamma \in \mathbb{R}$ one has

$$\lim_{t\to\infty} e^{\gamma t} \|\nu(t)\| = 0.$$

While small solutions can occur e.g. in the context of delay differential equations (we refer to [6, pp. 74ff., Section 3.3]), the next result rules out nontrivial small solutions in our setting.

Theorem 4.1. *Under the above assumptions* $(\mathbf{L})$ *and* $(\mathbf{N})$ *with*

$$0 \le L < \inf_{k\in\mathbb{N}} \frac{\beta_k - \alpha_k}{6K_k} \tag{4.1}$$

one has $\bigcap_{k\in\mathbb{N}} \mathcal{W}_k = J \times \{0\}$, *i.e. for every nontrivial (mild) solution* $\nu : [\tau, \infty) \to \mathcal{X}$ *to* (E) *there exists a* $k \in \mathbb{N}$ *such that* $(\tau, \nu(\tau)) \in \mathcal{W}_{k+1} \setminus \mathcal{W}_k$.

In few words, Theorem 4.1 implies that for every nontrivial mild solution ν there exists a $\gamma \in \mathbb{R}$ with $\limsup_{t\to\infty} e^{\gamma(\tau-t)} \|\nu(t)\| > 0$. This means that (E) cannot have nontrivial solutions decaying to 0 faster than exponentially. As the subsequent proof demonstrates, our Theorem 4.1 is essentially a corollary of the classical Hadamard–Perron theorem on stable manifolds. Concerning a version appropriate for our purposes we refer to [13, Theorem 2.4(a)].

Proof. Let $\tau \in J$ be arbitrary. Given $\gamma \in \mathbb{R}$ it is easy to see that the sets

$$B_{\tau,\gamma} := \left\{\phi \in C[\tau, \infty; \mathcal{X}) : \lim_{t\to\infty} e^{\gamma(\tau-t)} \|\phi(t)\| = 0\right\}$$

with the norm $\|\phi\|_{\tau,\gamma} := \sup_{\tau\le t} e^{\gamma(\tau-t)} \|\phi(t)\|$ are Banach spaces.

(I) Our assumptions $(\mathbf{L_1})$–$(\mathbf{L_2})$ on the dichotomy spectrum ensure that for every $k \in \mathbb{N}$ there exist reals $K_k \ge 1$ and an invariant projector $\mathcal{P}_k^+ : J \to L(\mathcal{X})$ so that the estimates

$$\left\|\mathcal{T}(t,s)\mathcal{P}_k^+(s)\right\| \le K_k e^{\alpha_k(t-s)}, \qquad \left\|\bar{\mathcal{T}}(s,t)\mathcal{P}_k^-(t)\right\| \le K_k e^{\beta_k(s-t)} \quad \text{for } s \le t \tag{4.2}$$

are fulfilled with the complementary projector $\mathcal{P}_k^-(t) := \mathrm{id}_{\mathcal{X}} - \mathcal{P}_k^+(t)$. For every particular growth rate $\gamma := \frac{\alpha_k+\beta_k}{2} \in (\alpha_k, \beta_k)$, $k \in \mathbb{N}$ fixed, let us define the operators

$$\begin{aligned} &S_\tau \in L(\mathcal{X}, B_{\tau,\gamma}), && S_\tau u_0 := \mathcal{T}(\cdot,\tau)\mathcal{P}_k^+(\tau)u_0, \\ &T_\tau : B_{\tau,\gamma} \to B_{\tau,\gamma}, && T_\tau(\phi) := \int_\tau^\cdot \mathcal{T}(\cdot,s)\mathcal{P}_k^+(s)\mathcal{F}(s,\phi(s))\,\mathrm{d}s \\ & && \qquad - \int_\cdot^\infty \bar{\mathcal{T}}(\cdot,s)\mathcal{P}_k^-(s)\mathcal{F}(s,\phi(s))\,\mathrm{d}s. \end{aligned}$$

They are well-studied in the literature (e.g. [1,11,13,15]) when $B_{\tau,\gamma}$ is replaced by the space of all continuous functions ϕ satisfying $\|\phi\|_{\tau,\gamma} < \infty$. Thus, it remains to show that the mappings S_τ, T_τ are well-defined.

First, for every $u_0 \in \mathcal{X}$ one has the limit relation

$$\|(S_\tau u_0)(t)\| e^{\gamma(\tau-t)} = \left\|\mathcal{T}(t,\tau)\mathcal{P}_k^+(\tau)u_0\right\| e^{\gamma(\tau-t)} \stackrel{(4.2)}{\leq} K_k e^{(\alpha_k-\gamma)(t-\tau)} \|u_0\| \xrightarrow[t\to\infty]{} 0$$

and therefore $S_\tau u_0 \in B_{\tau,\gamma}$.

Second, concerning the operator T_τ choose an arbitrary $\phi \in B_{\tau,\gamma}$. This ensures that for every $\varepsilon > 0$ there exists a $T \geq \tau$ such that

$$\max\left\{\frac{K_k L}{\gamma-\alpha_k}, \frac{K_k L}{\beta_k-\gamma}\right\} e^{\gamma(\tau-t)} \|\phi(t)\| < \frac{\varepsilon}{3} \quad \text{for all } t \geq T. \tag{4.3}$$

Because of (**N**) we arrive at the estimate

$$\begin{aligned} \|T_\tau(\phi)(t)\| &\stackrel{(4.2)}{\leq} K_k L \int_\tau^t e^{\alpha_k(t-s)} \|\phi(s)\| \,\mathrm{d}s + K_k L \int_t^\infty e^{\beta_k(t-s)} \|\phi(s)\| \,\mathrm{d}s \\ &\leq K_k L \int_\tau^T e^{\alpha_k(t-s)} e^{\gamma(s-\tau)} \,\mathrm{d}s \,\|\phi\|_{\tau,\gamma} + K_k L \int_T^t e^{\alpha_k(t-s)} \|\phi(s)\| \,\mathrm{d}s \\ &\quad + K_k L \int_t^\infty e^{\beta_k(t-s)} \|\phi(s)\| \,\mathrm{d}s \quad \text{for all } \tau \leq t. \end{aligned}$$

This, in turn, implies

$$\begin{aligned} \|T_\tau(\phi)(t)\| e^{\gamma(\tau-t)} &\leq \frac{K_k L}{\gamma-\alpha_k}\left[e^{(\alpha_k-\gamma)(t-T)} - e^{(\alpha_k-\gamma)(t-\tau)}\right] \|\phi\|_{\tau,\gamma} \\ &\quad + K_k L \int_T^t e^{\alpha_k(t-s)} \|\phi(s)\| e^{\gamma(\tau-s)} e^{\gamma(s-t)} \,\mathrm{d}s \\ &\quad + K_k L \int_t^\infty e^{\beta_k(t-s)} \|\phi(s)\| e^{\gamma(\tau-s)} e^{\gamma(s-t)} \,\mathrm{d}s \\ &\stackrel{(4.3)}{<} \frac{K_k L}{\gamma-\alpha_k} e^{(\alpha_k-\gamma)(t-T)} \|\phi\|_{\tau,\gamma} + \frac{\gamma-\alpha_k}{3}\varepsilon \int_T^t e^{\alpha_k(t-s)} e^{\gamma(s-t)} \,\mathrm{d}s \\ &\quad + \frac{\beta_k-\gamma}{3}\varepsilon \int_t^\infty e^{\beta_k(t-s)} e^{\gamma(s-t)} \,\mathrm{d}s \\ &< \frac{K_k L}{\gamma-\alpha_k} e^{(\alpha_k-\gamma)(t-T)} \|\phi\|_{\tau,\gamma} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{for all } T \leq t \end{aligned}$$

and due to $\alpha_k < \gamma$ there is a $T' \geq T$ such that $\frac{K_k L}{\gamma-\alpha_k} e^{(\alpha_k-\gamma)(t-T)} \|\phi\|_{\tau,\gamma} < \frac{\varepsilon}{3}$ holds for all $t \geq T'$. Consequently, $\|T_\tau(\phi)(t)\| e^{\gamma(\tau-t)} < \varepsilon$ for every $t \geq T'$, i.e. $T_\tau(\phi) \in B_{\tau,\gamma}$.

(II) Thanks to (I) the Lyapunov–Perron operator

$$L_\tau : B_{\tau,\gamma} \times \mathcal{X} \to B_{\tau,\gamma}, \qquad L_\tau(\phi, u_0) := S_\tau u_0 + T_\tau(\phi)$$

is well-defined. As in the proof of [13, Theorem 2.4] one establishes that (4.1) guarantees L_τ to be a uniform contraction in the first argument. From the contraction mapping theorem we deduce a unique fixed-point $\phi_\tau^*(u_0) \in B_{\tau,\gamma}$. Setting $w_k(\tau, u_0) := \mathcal{P}_k^-(\tau)\big(\phi_\tau^*(u_0)\big)(\tau)$ one obtains a function $w_k : J \times \mathcal{X} \to \mathcal{X}$ fulfilling $w_k(\tau, 0) \equiv 0$ on J and a global Lipschitz condition with constant < 1. Moreover, it holds the representation

$$\mathcal{W}_k = \left\{(\tau, \xi + w_k(\tau,\xi)) \in J \times \mathcal{X} : \xi \in R(\mathcal{P}_k^+(\tau))\right\}.$$

(III) After these preparations the actual proof is quite immediate. Indeed, let us suppose that $\nu : [\tau, \infty) \to \mathcal{X}$ is a mild solution of (E) which is contained in all $\mathcal{W}_k$, $k \in \mathbb{N}$. This implies $\nu(\tau) = \mathcal{P}_k^+(\tau)\nu(\tau) + w_k(\tau, \mathcal{P}_k^+(\tau)\nu(\tau))$ and consequently

$$\|\nu(\tau)\| \leq \left\|\mathcal{P}_k^+(\tau)\nu(\tau)\right\| + \left\|w_k(\tau, \mathcal{P}_k^+(\tau)\nu(\tau)) - w_k(\tau, 0)\right\| \leq 2\left\|\mathcal{P}_k^+(\tau)\nu(\tau)\right\| \xrightarrow[k\to\infty]{} 0,$$

because $\mathcal{P}_k^+(\tau)v(\tau) = \sum_{j=k+1}^{\infty} \mathcal{P}_k(\tau)v(\tau)$ are the remainders in the convergent infinite series $\sum_{k\in\mathbb{N}} \mathcal{P}_k(\tau)v(\tau)$ (cf. ($\mathbf{L}_3$)). Thus, $v(\tau) = 0$ yielding the claim. □

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