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# On functional differential equations associated to controlled structures with propagation

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**Abstract.** The method of integration along the characteristics has turned to be quite fruitful for qualitative analysis of physical and engineering systems described by large classes of partial differential equations of hyperbolic type in the plane (time and one space dimension) with real characteristics. In this paper there is presented an overview of such models under the aforementioned approach. We mention in this abstract the models describing transport phenomena (e.g. circulating fuel nuclear reactors and tubular reactors of the biotechnology) and propagation phenomena (e.g. electrical transmission lines such as waveguides or water, steam and gas pipes).
In the first case (transport phenomena) there exists a single forward (progressive) wave due to the fact that there exists a single family of characteristics which are increasing. In the second case (propagation) there are to be met both forward (progressive) and backward (reflected) waves and two families of characteristics. In the nonlinear case the systems of conservation laws belong to both categories of systems.
Integration along the characteristics allows association of some systems of functional (differential) equations; a one-to-one (injective) correspondence between the solutions of the two mathematical objects (the boundary value problem for the partial differential equations and the functional equations) is established such that all properties obtained for one of them is projected back on the other. In this way continuous and discontinuous classical solutions can be analyzed from the point of view of the well-posedness in the sense of Hadamard (existence, uniqueness and data/parameter dependence), existence of some invariant sets and stability.
The various functional equations thus introduced are mathematical objects interesting for themselves such as the neutral functional differential equations which appear in lossless and distortionless wave propagation when differential equations are to be met in the boundary conditions.

**Keywords:** hyperbolic equations, functional differential equations, characteristics, stability

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# 1 Overview

An already common fact is that some connection may be established between the solutions of some classes of partial differential equations (in most cases of hyperbolic type) and solutions of some functional differential equations (in most cases – equations with deviated arguments). The very first paper on equations with deviated argument, belonging to Johan Bernoulli [3], displays the time delay equation

$$y'(t) = y(t-1) \tag{1.1}$$

with reference to the vibrating string equation. As pointed out later, the deduction of this equation might have been mistaken, but the idea turned nevertheless to be sound.

Closer to our time, the paper of Abolinia and Myshkis [1] considers a rather general framework of BVP (Boundary Value Problems) for hyperbolic PDE (Partial Differential Equations) in the plane ("time" and one space variable) where integration along the characteristics allows association of the solutions of some Volterra like integral equations to the solutions of the BVP. This paper relies on some previous papers of the 50s (previous century) and is continued by two others [20,21]. Somehow independently, K. L. Cooke [7,8] developed a similar technique for a simpler case but having wide applications in physics and engineering (mechanical, thermal, hydraulic, electrical). The approach of Cooke represents also a rather natural way of introducing the equations with deviated argument of retarded, neutral or advanced type.

Other papers following the same line are those of D. L. Russell [27] and the most recent one belonging to Karafyllis and Krstić [19].

Our point of view is that the interest for this approach is due to the possibility of establishing a one-to-one correspondence between the solutions of the two mathematical objects. At its turn this correspondence allows any result concerning one mathematical object to be projected on the other one; in this way one may use the results of each field to state and solve problems for the other one. Along several decades we published several surveys on this subject - we cite but the most recent of them [25]. Our interest for the problem had been driven mainly by the increasing number and diversity of the applications.

In the following we shall rely on the one-to-one correspondence stated by Cooke [7] and rigorously and completely proved in [25]. Some applications described by transport (convection) equations and by hyperbolic propagation equations will be considered: circulating nuclear reactor, biotechnology reactor, combined heat electricity generation. All these applications will be discussed from the point of view of the so called *augmented validation*. We need to comment here this concept. In general, any mathematical model is elaborated starting from some laws of the physical reality. These laws are however subject to several transformations aiming usually to simplify the basic modeling structure. The resulting "mathematical monster" fails to be accepted as a rigorous consequence of the modeling process but rather has to pass "validation", being judged as a mathematical object only.

The first element of this validation is the well-posedness in the sense of J. Hadamard: existence, uniqueness and continuous data dependence. The significance of well-posedness is quite obvious but we send the reader to the beautiful explanation of R. Courant [9]. We added to well-posedness two other properties.

The first one is existence of some invariant sets, especially positiveness of some variables such as temperatures, pressures, concentrations. The "real" variables are nonnegative hence their mathematical representations should have the same property but deduced from the mathematical model only: nonnegative initial conditions should imply nonnegative variables at any future moment. This property may turn useful in other problem solving.

The other property we would like to mention here arises from the so called "Stability postulate" stated by N. G. Četaev in [4, 5]. According to it, only those steady states are physically observable and measurable which are at least stable in the sense of Liapunov if not more i.e. asymptotically or even exponentially stable. Consequently we add *inherent stability of some steady states* as another property to be checked within the framework of the aforementioned augmented validation.

Since the aim of this paper is to obtain augmented validation for some applications described by initial boundary value problems for transport (convection) and propagation equations, the rest of the paper is structured as follows. The next section will reproduce the basic result of Cooke [7], completely proved in [25] which will allow association of a Cauchy problem for some functional equations. This theorem establishes a one-to-one correspondence between the solutions of the initial boundary value problems and the corresponding functional equations. This association will be performed for the considered applications arising from physics and engineering. For the resulting functional equations the aforementioned augmented validation procedure will be applied and the results projected back on the basic models described by PDE. Additional problems (e.g. those connected to control will be also discussed.

## 2 The theorem of K. L. Cooke and its consequences

Consider the following BVP with initial and derivative boundary conditions

$$\begin{aligned}
&\frac{\partial u^+}{\partial t} + \tau^+(\lambda,t)\frac{\partial u^+}{\partial \lambda} = \Phi^+(\lambda,t)\\
&\frac{\partial u^-}{\partial t} + \tau^-(\lambda,t)\frac{\partial u^-}{\partial \lambda} = \Phi^-(\lambda,t), \quad 0 \le \lambda \le 1,\ t \ge t_0,\\
&\sum_{k=0}^{m}\left[a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^+(0,t) + a_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^-(0,t)\right] = f_0(t)\\
&\sum_{k=0}^{m}\left[b_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^+(1,t) + b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^-(1,t)\right] = f_1(t)\\
&u^\pm(\lambda,t_0) = \omega^\pm(\lambda), \quad 0 \le \lambda \le 1
\end{aligned} \tag{2.1}$$

with $\tau^+(\lambda,t) > 0$, $\tau^-(\lambda,t) < 0$.

The differential equations

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{1}{\tau^\pm(\lambda,t)}, \quad \tau^+(\lambda,t) > 0, \quad \tau^-(\lambda,t) < 0 \tag{2.2}$$

define the two families of characteristic curves crossing some point $(\lambda,t)$ of the strip $[0,1] \times [t_0,t_1]$. Define

$$T^+(t) := t^+(1;0,t) - t, \qquad T^-(t) := t^-(0;1,t) - t \tag{2.3}$$

called *propagation times along the characteristics* or *forward and backward* propagation time respectively. Denoting

$$\Psi^+(t) := \int_0^1 \frac{\Phi^+(\sigma,t^+(\sigma;0,t))}{\tau^+(\sigma,t^+(\sigma;0,t))}\mathrm{d}\sigma, \qquad \Psi^-(t) := \int_0^1 \frac{\Phi^-(\sigma,t^-(\sigma;1,t))}{\tau^-(\sigma,t^-(\sigma;1,t))}\mathrm{d}\sigma \tag{2.4}$$

we write down the following system of differential equations

$$\begin{aligned}
&\sum_{k=0}^{m}\left[a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^+(t+T^+(t))+a_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^-(t)\right]=f_0(t)+\sum_{k=0}^{m}a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Psi^+(t)\\
&\sum_{k=0}^{m}\left[b_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^+(t)+b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^-(t+T^-(t))\right]=f_1(t)-\sum_{k=0}^{m}b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Psi^-(t).
\end{aligned}\tag{2.5}$$

We can state now the following theorem.

**Theorem 2.1.** *Consider the system* (2.1) *and let* $u^\pm(\lambda,t)$ *be a (possibly) discontinuous classical solution of it. Then* $y^\pm(\lambda,t)$ *defined by*

$$y^+(t):=u^+(1,t),\qquad y^-(t):=u^-(0,t)\tag{2.6}$$

*is a solution of* (2.5) *with the initial conditions defined by*

$$\begin{aligned}
y_0^+(t^+(1;\lambda,t_0))&=\omega^+(\lambda)+\int_\lambda^1\frac{\Phi^+(\sigma,t^+(\sigma;\lambda,t_0))}{\tau^+(\sigma,t^+(\sigma;\lambda,t_0))}\mathrm{d}\sigma\\
y_0^-(t^-(0;\lambda,t_0))&=\omega^-(\lambda)-\int_0^\lambda\frac{\Phi^-(\sigma,t^-(\sigma;\lambda,t_0))}{\tau^-(\sigma,t^-(\sigma;\lambda,t_0))}\mathrm{d}\sigma
\end{aligned}\tag{2.7}$$

*where* $0\le\lambda\le1 \Leftrightarrow t_0\le t^+(1;\lambda,t_0)\le t_0+T^+(t_0)$ *and* $0\le\lambda\le1 \Leftrightarrow t_0\le t^-(0;\lambda,t_0)\le t_0+T^-(t_0)$.

*Conversely, let* $y^\pm(t)$ *be some solution of* (2.5) *with some initial conditions defined on* $t_0\le t\le t_0+T^\pm(t_0)$*. Then the functions*

$$\begin{aligned}
u^+(\lambda,t)&=y^+(t^+(1;\lambda,t))-\int_\lambda^1\frac{\Phi^+(\sigma,t^+(\sigma;\lambda,t))}{\tau^+(\sigma,t^+(\sigma;\lambda,t))}\mathrm{d}\sigma\\
u^-(\lambda,t)&=y^-(t^-(0;\lambda,t))+\int_0^\lambda\frac{\Phi^-(\sigma,t^-(\sigma;\lambda,t))}{\tau^-(\sigma,t^-(\sigma;\lambda,t))}d\sigma
\end{aligned}\tag{2.8}$$

*define a (possibly discontinuous) classical solution of* (2.1) *with the initial conditions* $\omega^\pm(\lambda)$ *obtained from* (2.8) *at* $t=t_0$*, satisfying the PDE and the boundary conditions.*

The complete proof of this theorem is given in [25] and will not be reproduced here. Some comments however will be useful in the following sections of the paper. The first one concerns the form of (2.1) where the forward and backward waves – the Riemann invariants – are decoupled in the PDE and coupled through the boundary conditions only. This structure is crucial for the described approach and is far from being valid in most applications. It is not mistaken to call the approach – *integration of the Riemann invariants along the characteristics.*

Another comment concerns system (2.5). Following [7], define the integers

$$\begin{aligned}
L^+&=\max\{k:a_k^+(t)\neq0\}, &\quad L^-&=\max\{k:b_k^-(t)\neq0\}\\
K^+&=\max\{k:b_k^+(t)\neq0\}, &\quad K^-&=\max\{k:a_k^-(t)\neq0\}\\
M&=L^++L^--(K^++K^-).
\end{aligned}\tag{2.9}$$

According to the sign of $M$ system (2.5) belongs to one of the three classes of systems with deviated argument: if $M>0$ it is of delayed type; if $M<0$ it is of advanced type; if $M=0$ it is of neutral type. This assertion follows in a straightforward way from the definitions of [2] and is consistent with the so called classification of G. A. Kamenskii [14].

As it will appear in the sequel (and as follows from our experience), most systems with deviated arguments associated to the boundary value problems for hyperbolic partial differential equations are of neutral type. This means that their solutions are not smoothed in time (unlike those of the systems of delayed type). The aforementioned aspect accounts for the propagation of the initial discontinuities – a consequence of the unmatched initial and boundary conditions.

For simplicity only classical (continuous or discontinuous) solutions of the boundary value problems will be considered while the approach of [1] allows also consideration of the generalized solutions.

## 3 The model of the tubular plug-flow bioreactor (the convection equations)

According to [6, 11–13] the model of a plug-flow tubular bioreactor, where the simplest autocatalytic reaction takes place, reads

$$\begin{aligned} &\frac{\partial X}{\partial t} + q\frac{\partial X}{\partial z} + k_d X = \mu(S)X \\ &\frac{\partial S}{\partial t} + q\frac{\partial S}{\partial z} = -k_1\mu(S)X \qquad (z,t) \in [0,l] \times [0,T] \end{aligned} \tag{3.1}$$

with the boundary conditions

$$X(0,t) = X_{\text{in}}(t), \quad S(0,t) = S_{\text{in}}(t), \qquad X_d(0,t) = 0, \tag{3.2}$$

and some initial conditions $X_0(z)$, $S_0(z)$ given on $[0,l]$. We do not insist on the significance of the state variables $X(z,t)$, $S(z,t)$ but just mention that they are substance concentrations hence they ought be at least nonnegative. The term $\mu(S)$ – the kinetic function – is a continuous increasing function, also $\mu(S) > 0$ for $S > 0$.

Let $X_0(z) > 0$, $S_0(z) > 0$, $X_{\text{in}}(t) > 0$, $S_{\text{in}}(t) > 0$. If (3.1)–(3.2) has a continuous solution then it is also positive on $[0,T)$ provided $T > 0$ is small enough. We can then write on $(0,\ l) \times (0,\ T)$:

$$\begin{aligned} &\frac{1}{X}\frac{\partial X}{\partial t} + q\frac{1}{X}\frac{\partial X}{\partial z} + k_d = \mu(S) \\ &\frac{1}{\mu(S)}\frac{\partial S}{\partial t} + q\frac{1}{\mu(S)}\frac{\partial S}{\partial z} = -k_1 X. \end{aligned} \tag{3.3}$$

Introducing the new functions:

$$\xi(X) := \ln X, \qquad \sigma(S) := \ln \mu(S), \tag{3.4}$$

which are well defined locally since $X(z,t) > 0$, $S(z,t) > 0$, $\mu(\cdot) > 0$, on the aforementioned sufficiently small rectangle $(0,\ l) \times (0,\ T)$, the following modified boundary value problem is obtained:

$$\begin{aligned}
&\frac{\partial \xi}{\partial t} + q\frac{\partial \xi}{\partial z} + k_d = \mathrm{e}^{\sigma} \\
&\frac{\partial \sigma}{\partial t} + q\frac{\partial \sigma}{\partial z} = -k_1 \mathrm{e}^{\xi} \\
&\xi(0,t) = \ln X_{\mathrm{in}}(t) := \xi_{\mathrm{in}}(t), \quad \sigma(0,t) = \ln \mu(S_{\mathrm{in}}(t)) := \sigma_{\mathrm{in}}(t) \\
&\xi(z,0) = \ln X^0(z) := \xi^0(z), \quad \sigma(z,0) = \ln \mu(S^0(z)) := \sigma^0(z).
\end{aligned} \tag{3.5}$$

Denoting $\phi(x) := \mathrm{e}^x$, our mathematical object will be (in a slightly more general setting)

$$\begin{aligned}
&\frac{\partial \xi}{\partial t} + q\frac{\partial \xi}{\partial z} + k_d = \phi(\sigma) \\
&\frac{\partial \sigma}{\partial t} + q\frac{\partial \sigma}{\partial z} = -k_1 \phi(\xi) \\
&\xi(0,t) = \xi_{\mathrm{in}}(t), \quad \sigma(0,t) = \sigma_{\mathrm{in}}(t), \quad t > 0 \\
&\xi(z,0) = \xi^0(z), \quad \sigma(z,0) = \sigma^0(z), \quad 0 \le z \le L.
\end{aligned} \tag{3.6}$$

Consider some characteristic line crossing some point $(z,t)$ that is $t(\zeta; z, t) = t + (\zeta - z)/q$. We consider some solution of (3.6) along the characteristic that is

$$\tilde{\xi}(\zeta) := \xi(\zeta; t + (\zeta - z)/q), \qquad \tilde{\sigma}(\zeta) := \sigma(\zeta; t + (\zeta - z)/q) \tag{3.7}$$

and integrate along the characteristic from $\zeta = z$ to $\zeta = L$ to obtain

$$\begin{aligned}
\xi(z,t) &= \xi(L, t + (L-z)/q) - \frac{1}{q}\int_z^L (-k_d + \phi(\sigma(\lambda, t + (\lambda - z)/q)))\mathrm{d}\lambda \\
\sigma(z,t) &= \sigma(L, t + (L-z)/q) + \frac{k_1}{q}\int_z^L \phi(\xi(\lambda, t + (\lambda - z)/q))\mathrm{d}\lambda.
\end{aligned} \tag{3.8}$$

Now, there are two kinds of characteristics. Those with $t > z/q$ can be extended "to the left" up to $\zeta = 0$. If the characteristic in (3.8) is such, one can take there $z = 0$ to obtain

$$\begin{aligned}
\xi(0,t) = \xi_{\mathrm{in}}(t) &= \xi(L, t + L/q) - \frac{1}{q}\int_0^L (-k_d + \phi(\sigma(\lambda, t + \lambda/q)))\mathrm{d}\lambda \\
\sigma(z,t) = \sigma_{\mathrm{in}}(t) &= \sigma(L, t + L/q) + \frac{k_1}{q}\int_0^L \phi(\xi(\lambda, t + \lambda/q))\mathrm{d}\lambda.
\end{aligned} \tag{3.9}$$

It is quite obvious that the functions $(\xi(\lambda, t + \lambda/q), \sigma(\lambda, t + \lambda/q))$ verify the nonlinear continuous time difference system that can be deduced from (3.9) namely

$$\begin{aligned}
&\tilde{\xi}_t(L/q) - \int_0^{L/q} (-k_d + \phi(\tilde{\sigma}_t(\vartheta)))\mathrm{d}\vartheta = \xi_{\mathrm{in}}(t) \\
&\tilde{\sigma}_t(L/q) + k_1 \int_0^{L/q} \phi(\tilde{\xi}_t(\vartheta)))\mathrm{d}\vartheta = \sigma_{\mathrm{in}}(t), \quad t > L/q
\end{aligned} \tag{3.10}$$

where, as usual, $\tilde{\xi}_t(\cdot) = \tilde{\xi}(t + \cdot)$ etc.

The initial conditions on $(0, L/q)$ for (3.10) are to be obtained by integrating along the other kind of characteristics, with $t < z/q$: they can be extended "to the left" only up to $\hat{\zeta}$ defined by

$$t(\hat{\zeta}; z, t) = t + (\hat{\zeta})/q = 0 \;\Rightarrow\; \hat{\zeta} = z - qt > 0.$$

We deduce from (3.8) that for such characteristics

$$\begin{aligned}\xi(z-qt,0) = \xi^0(z-qt) &= \xi(L, t+(L-z)/q) \\ &\quad - \frac{1}{q}\int_{z-qt}^{L}(-k_d + \phi(\sigma(\lambda, t+(\lambda-z)/q)))\mathrm{d}\lambda \\ \sigma(z-qt,0) = \sigma^0(z-qt) &= \sigma(L, t+(L-z)/q) \\ &\quad + \frac{k_1}{q}\int_{z-qt}^{L}\phi(\sigma(\lambda, t+(\lambda-z)/q))\mathrm{d}\lambda.\end{aligned} \tag{3.11}$$

Following the same procedure as in [25] for the initial conditions, (3.11) is rewritten as

$$\begin{aligned}&\xi(L,(L-z)/q) - \frac{1}{q}\int_{z}^{L}(-k_d + \phi(\sigma(\lambda,(\lambda-z)/q)))\mathrm{d}\lambda = \xi^0(z) \\ &\sigma(L,(L-z)/q) + \frac{k_1}{q}\int_{z}^{L}\phi(\xi(\lambda,(\lambda-z)/q))\mathrm{d}\lambda = \sigma^0(z).\end{aligned} \tag{3.12}$$

Let as previously

$$\xi(\lambda,(\lambda-z)/q) := \tilde{\xi}_0((\lambda-z)/q), \quad \sigma(\lambda,(\lambda-z)/q) := \tilde{\sigma}_0((\lambda-z)/q).$$

Therefore

$$\begin{aligned}&\tilde{\xi}_0((L-z)/q) - \frac{1}{q}\int_{z}^{L}(-k_d + \phi(\tilde{\sigma}_0((\lambda-z)/q)))\mathrm{d}\lambda = \xi^0(z) \\ &\tilde{\sigma}_0((L-z)/q) + \frac{k_1}{q}\int_{z}^{L}\phi(\tilde{\xi}_0((\lambda-z)/q))\mathrm{d}\lambda = \sigma^0(z)\end{aligned} \tag{3.13}$$

what shows that $(\tilde{\xi}_0(\cdot)\ ,\ \tilde{\sigma}_0(\cdot))$ will result from the nonlinear system of integral equations

$$\begin{aligned}&\tilde{\xi}_0(\theta) - \int_0^{\theta}(-k_d + \phi(\tilde{\sigma}_0(\vartheta))\mathrm{d}\lambda = \xi^0(L-q\theta) \\ &\tilde{\sigma}_0(\theta) + k_1\int_0^{\theta}\phi(\tilde{\xi}_0(\vartheta))\mathrm{d}\lambda = \sigma^0(L-q\theta), \quad 0 < \theta < L/q.\end{aligned} \tag{3.14}$$

In this way we have defined the initial conditions for (3.11) and the functions $(\tilde{\xi}_t(\cdot), \tilde{\sigma}_t(\cdot))$ can be constructed for $t > 0$, starting from some solution of (3.6). These functions can have discontinuities at $t = m(L/q)$ with $m > 0$ – a positive integer. Conversely, if $(\tilde{\xi}_t(\cdot), \tilde{\sigma}_t(\cdot))$ is some solution of (3.13) with the initial conditions defined by (3.14) then we can define

$$\xi(\lambda, t+\lambda/q) := \tilde{\xi}(t+\lambda/q), \qquad \sigma(\lambda, t+\lambda/q) := \tilde{\sigma}(t+\lambda/q) \tag{3.15}$$

and consider the representation formulae (3.8). It can be verified by direct manipulation that $(\xi(z,t), \sigma(z,t))$ thus defined are solutions of the PDE (3.6); by taking $z = 0$ in (3.8) we obtain (3.9) hence the boundary conditions are fulfilled. If we take $t = 0$ in (3.8) we find (3.12) hence the initial conditions are fulfilled. We have thus obtained and proved the following theorem.

**Theorem 3.1.** *Consider the initial/boundary value problem for PDE* (3.6) *and let* $(\xi(z,t), \sigma(z,t))$ *be some classical solution of it. Then the functions*

$$\tilde{\xi}(t) := \xi(\lambda, t+\lambda/q), \qquad \tilde{\sigma}(t) := \sigma(\lambda, t+\lambda/q)$$

*are solutions of the continuous time difference system* (3.10) *with the initial conditions* (3.14). *Conversely, let* $(\tilde{\xi}(t), \tilde{\sigma}(t))$ *be a solution of* (3.10) *with the initial conditions* (3.14) *where* $(\xi^0(\cdot), \sigma^0(\cdot))$ *are given on* $[0, L]$ *and* $(\xi_{\text{in}}(t), \sigma_{\text{in}}(t))$ *are given for* $t > 0$. *Define* $(\xi(z,t), \sigma(z,t))$ *using the representation formulae* (3.15) *and* (3.8). *These functions define a (possibly discontinuous) classical solution of* (3.6) *where the initial and boundary value data are taken from* (3.10) *and* (3.14).

We shall comment here on the usefulness of Theorem 3.1. If existence and uniqueness of (3.10) holds, this is true for (3.6) also and the same holds for the continuity with respect to data and parameters. Recalling that $\xi = \ln X$, $\sigma = \ln \mu(S)$, positiveness of the concentrations $X(z,t)$, $S(z,t)$ follows from existence and uniqueness of the solutions for (3.6). We shall not discuss here the stability issues since they are connected to the feedback control structures at the boundaries.

## 4 The model of the circulating fuel nuclear reactor

This kind of nuclear reactor was quite popular as a research reactor. In our days it got a renewed attention within the framework of the research on low power mobile reactors while the mathematical model maintains some challenges and unsolved problems.

We shall consider here the basic model of [15] under a slightly transformed form e.g. with rated variables and parameters

$$\begin{aligned}
&\frac{\mathrm{d}n}{\mathrm{d}t} = (\rho - \beta)n + \sum_1^m \beta_i \bar{c}_i(t), \quad \beta = \sum_1^m \beta_i \\
&\frac{\partial c_i}{\partial t} + \frac{\partial c_i}{\partial \eta} + \sigma_i c_i = \sigma_i \varphi(\eta) n(t); \quad t > 0,\ 0 \leq \eta \leq h \\
&c_i(0,t) = c_i(h,t), \quad \bar{c}_i(t) = \int_0^h \varphi(\eta) c_i(\eta, t) \mathrm{d}\eta, \quad i = \overline{1,m}.
\end{aligned} \tag{4.1}$$

**A.** The mathematical model of the circulating delayed neutrons with densities $c_i(\eta, t)$ is a system of linear convection equations subjects to periodic boundary conditions. The "outputs" $\bar{c}_i(t)$ are applied to the equation of the basic – thermal – neutrons whose density is applied – in a distributed way – to the aforementioned convection equations. This kind of internal feedback suggests possible stability problems.

Some remarks about the parameters will be useful: all parameters are strictly positive, $\varphi : (0, h) \mapsto (0, 1)$ will be extended by periodicity to $\mathbb{R}$; $\rho$ – the reactivity – will introduce another feedback, the external one, that may include the control loops.

The association of the functional equations requires some tedious but straightforward manipulation relying again on integration along the characteristics defined by

$$t(\lambda; \eta, t) = t + \lambda - \eta. \tag{4.2}$$

Defining $q_i(t) := c_i(h,t)$, the functions $n(t)$, $q_i(t)$ associated to a solution $n(t)$, $c_i(\eta, t)$ of (4.1) will satisfy the following differential and difference equations

$$\begin{aligned}\frac{\mathrm{d}n}{\mathrm{d}t} &= (\rho-\beta)n(t) + \sum_1^m \beta_i\sigma_i \int_{-h}^0 \mathrm{e}^{\theta\sigma_i}\left(\int_{-\theta}^h \varphi(\eta)\varphi(\eta+\theta)\mathrm{d}\eta\right) n(t+\theta)\mathrm{d}\theta \\ &\quad + \sum_1^m \beta_i \int_{-h}^0 \mathrm{e}^{\theta\sigma_i}\varphi(-\theta)q_i(t+\theta)\mathrm{d}\theta \\ q_i(t) &= \mathrm{e}^{-h\sigma_i}q_i(t-h) + \sigma_i\int_{-h}^0 \mathrm{e}^{\theta\sigma_i}\varphi(\theta)n(t+\theta)\mathrm{d}\theta\end{aligned} \tag{4.3}$$

for $t > h$, with the initial conditions defined on $(0, h)$ by

$$\begin{aligned}\frac{\mathrm{d}n}{\mathrm{d}t} &= (\rho-\beta)n(t) + \sum_1^m \beta_i\sigma_i \int_0^t \left(\int_0^h \varphi(\eta)\varphi(\eta+\theta-t)\mathrm{d}\eta\right) \mathrm{e}^{-(t-\theta)\sigma_i}n(\theta)\mathrm{d}\theta \\ &\quad + \sum_1^m \beta_i \mathrm{e}^{-t\sigma_i}\int_0^h \varphi(t+\theta)q_i^0(\theta)\mathrm{d}\theta, \quad n(0)=n_0 \\ q_i(t) &= \mathrm{e}^{-t\sigma_i}q_i^0(h-t) + \sigma_i\int_0^t \mathrm{e}^{-(t-\theta)\sigma_i}\varphi(\theta-t)n(\theta)\mathrm{d}\theta\end{aligned} \tag{4.4}$$

where $q_i^0(\eta) := c_i(\eta, 0)$. The following result holds.

**Theorem 4.1.** *Consider the system* (4.1) *and let* $(n(t), c_i(\eta, t))$ *be some classical solution corresponding to the initial conditions* $(n_0\ ,\ q_i^0(\eta))$*; define* $q_i(t) = c_i(h, t)$*. Then* $(n(t), q_i(t+\cdot))$ *is a solution of* (4.3) *for* $t > h$ *with the initial conditions defined on* $(0, h)$ *by* (4.4)*. Conversely, let* $(n(t), q_i(t+\cdot))$ *be some (possibly discontinuous) solution of* (4.3) *with the initial conditions defined by* (4.4) *on* $(0, h)$ *with* $(n_0, q_i^0(\eta))$ *being given. Then* $(n(t), c_i(\eta, t))$ *where*

$$c_i(\eta,t) = \mathrm{e}^{(h-\eta)\sigma_i}\left[q_i(t+h-\eta) - \sigma_i\int_{-h+\eta}^0 \mathrm{e}^{\theta\sigma_i}\varphi(\theta)n(t+h+\theta-\eta)\mathrm{d}\theta\right] \tag{4.5}$$

*is a (possibly discontinuous) classical solution of* (4.1).

**B.** Another step of model studies is given by emphasizing positiveness of $n(t)$ and $c_i(\eta, t)$ for $t > 0$ provided $n_0 \geq 0$ and $c_i(\eta, 0) = q_i^0(\eta) \geq 0$ for $0 \leq \eta \leq h$. Assume this is true and let $(n(t), c_i(\eta, t))$ be some solution of (4.1) corresponding to these initial conditions. Associate $(n(t),\ q_i(t) = c_i(h, t)$ which satisfy (4.3) and (4.4). Since $\varphi(\eta) \geq 0$, the "free" term of the integro-differential equation of (4.4) is positive, as is the kernel of it. Since $n_0 \geq 0$, $n(t) > 0$ on some interval $[0, \hat{t})$. If $n(\hat{t}) = 0$ nevertheless $\mathrm{d}n/\mathrm{d}t|_{t=\hat{t}} > 0$ hence $n(t)$ cannot be negative on $[0, h)$. Therefore $q_i(t) > 0$ on $[0, h)$. Consider further system (4.3) which is a system of coupled delay-differential and difference equations. The construction of the solution by steps shows that $n(t) > 0$, $q_i(t) > 0$ for all $t > 0$.

In order to show that $c_i(\eta, t) > 0$, $0 \leq \eta \leq h$, consider the representation formula (4.5) in two cases. First, if $t > h$ then (4.5) is rewritten as

$$c_i(\eta,t) = \mathrm{e}^{-\eta\sigma_i}q_i(t-\eta) + \sigma_i\int_{-h}^{-h+\eta} \mathrm{e}^{(h-\eta+\theta)\sigma_i}\varphi(\theta)n(t+h-\eta+\theta)\mathrm{d}\theta \tag{4.6}$$

and clearly $c_i(\eta, t) > 0$. The same holds for $\eta < t \leq h$ but the case $0 \leq t \leq \eta$ has to be discussed separately.

Consider again (4.5) where $q_i(t+h-\eta)$ will be replaced by its expression from (4.4):

$$c_i(\eta,t) = \mathrm{e}^{-t\sigma_i}q_i(\eta-t) + \sigma_i\int_0^t \mathrm{e}^{-(t-\theta)\sigma_i}\varphi(\theta-t+\eta)n(\theta)\mathrm{d}\theta > 0. \tag{4.7}$$

We have thus proved the following theorem.

**Theorem 4.2.** *Consider the system* (4.1) *with* $n_0 \geq 0$, $q_i^0(\eta) = c_i(\eta, 0) \geq 0$, $0 \leq \eta \leq h$. *Then the solution of* (4.3)–(4.4) *satisfies* $n(t) > 0$, $q_i(t) > 0$ *and, consequently,* $c_i(\eta, t) > 0$ *for all* $t > 0$, $0 \leq \eta \leq h$.

**C.** As known, one of the basic engineering problems for nuclear reactors is the control of the power level for the thermal neutrons. In order to state the control problem we shall consider the steady states for (4.1) by letting the time derivatives to be 0:

$$\begin{aligned} &(\rho - \beta)n_\infty + \sum_1^m \beta_i \bar{\hat{c}}_i = 0, \quad \bar{\hat{c}}_i = \int_0^h \varphi(\eta)\hat{c}_i(\eta)\mathrm{d}\eta \\ &\frac{\mathrm{d}\hat{c}_i}{\mathrm{d}\eta} + \sigma_i \hat{c}_i = \sigma_i \varphi(\eta) n_\infty, \quad \hat{c}_i(0) = \hat{c}_i(h);\ i = \overline{1, m} \end{aligned} \tag{4.8}$$

where $n_\infty$ accounts for some imposed power level. A simple manipulation will give

$$\begin{aligned} \hat{c}_i(\eta) &= \left[\frac{\mathrm{e}^{-h\sigma_i}}{1 - \mathrm{e}^{-h\sigma_i}} \left(\int_0^h \mathrm{e}^{\theta\sigma_i}\varphi(\theta)\mathrm{d}\theta\right) + \int_0^\eta \mathrm{e}^{\theta\sigma_i}\varphi(\theta)\mathrm{d}\theta\right] \sigma_i \mathrm{e}^{-\eta\sigma_i} n_\infty \\ \bar{\hat{c}}_i &= \sigma_i n_\infty \int_0^h \mathrm{e}^{-\lambda\sigma_i} \left(\int_0^h \varphi(\eta)\varphi(\eta - \lambda)\mathrm{d}\eta\right) \mathrm{d}\lambda = \xi_i n_\infty \end{aligned} \tag{4.9}$$

where it can be shown that $0 < \xi_i < 1$. Introducing the second equality of (4.9) in (4.8) it follows that

$$\rho(n_\infty) = \sum_1^m \beta_i (1 - \xi_i) > 0. \tag{4.10}$$

Since $n_\infty$ multiplies all steady state equations, we can take $n_\infty = 1$ and define the deviations with respect to the steady state $(1, \hat{c}_i(\eta))$. If the deviations are introduced

$$\begin{aligned} &\zeta(t) := n(t) - 1; && y_i(\eta, t) := c_i(\eta, t) - \hat{c}_i(\eta) \\ &\bar{y}_i(t) := \frac{1}{\xi_i}(\bar{c}_i(t) - \xi_i); && \nu := \rho - \sum_1^m \beta_i(1 - \xi_i) \end{aligned} \tag{4.11}$$

the system in deviations is easily found

$$\begin{aligned} &\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \nu(1 + \zeta) - \sum_1^m \beta_i \xi_i (\zeta - \bar{y}_i); \quad \zeta(0) = n_0 - 1 \\ &\frac{\partial y_i}{\partial t} + \frac{\partial y_i}{\partial \eta} + \sigma_i y_i = \sigma_i \varphi(\eta)\zeta(t); \quad \bar{y}_i(t) = \frac{1}{\xi_i}\int_0^h \varphi(\eta) y_i(\eta, t)\mathrm{d}\eta \\ &y_i(0, t) = y_i(h, t); \quad y_i(\eta, 0) = y_i^0(\eta) = q_i^0(\eta) - \hat{c}_i(\eta). \end{aligned} \tag{4.12}$$

Being linear, the subsystem described by the PDE looks like his correspondent from (4.1). Therefore we can associate the functional equations

$$\begin{aligned} \frac{\mathrm{d}\zeta}{\mathrm{d}t} &= \nu(1 + \zeta) - \sum_1^m \beta_i \xi_i \left[\zeta(t) - \frac{\sigma_i}{\xi_i}\int_{-h}^0 \left(\int_{-\theta}^h \varphi(\lambda)\varphi(\lambda + \theta)\mathrm{d}\lambda\right) \mathrm{e}^{\theta\sigma_i}\zeta(t + \theta)\mathrm{d}\theta\right] \\ &\quad + \sum_1^m \beta_i \int_{-h}^0 \mathrm{e}^{\theta\sigma_i}\varphi(\theta) r_i(t + \theta)\mathrm{d}\theta \\ r_i(t) &= \mathrm{e}^{-h\sigma_i} r_i(t - h) + \sigma_i \int_{-h}^0 \mathrm{e}^{\theta\sigma_i}\varphi(\theta)\zeta(t + \theta)\mathrm{d}\theta \end{aligned} \tag{4.13}$$

for $t > h$ and their initial conditions for $0 < t < h$

$$\begin{aligned} \frac{d\zeta}{dt} &= \nu(1+\zeta) - \sum_1^m \beta_i \xi_i \left[ \zeta(t) - \frac{\sigma_i}{\xi_i} \int_0^t \left( \int_0^h \varphi(\lambda)\varphi(\lambda+\theta-t)d\lambda \right) e^{-(t-\theta)\sigma_i} \zeta(\theta)d\theta \right] \\ &\quad + \sum_1^m \beta_i e^{-t\sigma_i} \int_0^h \varphi(t+\theta) y_i^0(\theta)d\theta \\ r_i(t) &= e^{-t\sigma_i} y_i^0(h-t) + \sigma_i \int_0^t e^{-(t-\theta)\sigma_i} \varphi(\theta-t)\zeta(\theta)d\theta. \end{aligned} \tag{4.14}$$

Taking into account the result of Theorem 4.2 we deduce that (4.12) has the invariant set

$$1+\zeta > 0; \quad \hat{c}_i(\eta) + y_i(\eta,\cdot) > 0, \quad 0 \le \eta \le h. \tag{4.15}$$

Following [15], system (4.13)–(4.14) can be given the form of an integro-differential equation by using the Cauchy formula for the difference equation. The manipulation is quite tedious here also, leading to

$$\begin{aligned} \frac{d\zeta}{dt} &= \nu(1+\zeta) - \sum_1^m \beta_i \xi_i \left[ \zeta(t) - \frac{\sigma_i}{\xi_i} \int_0^t \left( \int_0^h \varphi(\lambda)\varphi(\lambda+\theta-t)d\lambda \right) e^{-(t-\theta)\sigma_i} \zeta(\theta)d\theta \right] \\ &\quad + \sum_1^m \beta_i e^{-t\sigma_i} \int_0^h \varphi(t+\theta) y_i^0(\theta)d\theta, \qquad t > 0 \end{aligned} \tag{4.16}$$

which is exactly the integro-differential equation from the initial conditions (4.14) but valid this time for all $t > 0$ In (4.16), also in all equations preceding it, the variable $\nu$ represents the reactivity deviation from the corresponding steady state value. It is in fact an output of the external dynamics which integrates the automatic control circuits. In the linear case the generating system reads

$$\dot{x} = Ax + b\zeta; \qquad \nu = c^* x + \alpha\zeta \tag{4.17}$$

where $x \in \mathbb{R}^n$, $A$ is a $n \times n$ matrix, $b$, $c$ are $n$-vectors and $\alpha$ is a real number.

**D.** Everything is ready now for the stability analysis of the zero solution of the basic system. However system (4.16)–(4.17) does not have the zero solution because of the last term of (4.16) accounting for the initial condition of the distributed part. As a consequence, the stability problem is not solved in fact in a rigorous way in [15]. Instead, equation (4.16) is replaced by

$$\frac{d\zeta}{dt} = \nu(1+\zeta) - \sum_1^m \beta_i \xi_i \left[ \zeta(t) - \frac{\sigma_i}{\xi_i} \int_{-\infty}^t \left( \int_0^h \varphi(\lambda)\varphi(\lambda+\theta-t)d\lambda \right) e^{-(t-\theta)\sigma_i} \zeta(\theta)d\theta \right] \tag{4.18}$$

which may be combined in a "tractable" way with (4.17); tractable means here that a Liapunov functional can be inferred and a Welton type stability criterion can be obtained. The Welton type criteria are quite common in nuclear reactor dynamics: under the assumption that the external dynamics is linear e.g. described by (4.17), its transfer function defined by

$$H(s) = \alpha + c^*(sI - A)^{-1}b\,, \quad s \in \mathbb{C} \tag{4.19}$$

should be positive real. For the sake of completeness we give below the theorem stating the aforementioned criterion of Welton type.

**Theorem 4.3.** *Consider the system defined by* (4.17)–(4.18) *under the assumptions of Theorems 4.1 and 4.2. Assume also that: i)* $\det(sI - A) \neq 0$ *for* $\Re e(s) > 0$ *and the possible eigenvalues on* $\imath\mathbb{R}$ *have simple elementary divisors; ii) the triplet* $(A, b, c)$ *has the limit stability property i.e. there exists some* $\Delta > 0$ *sufficiently small such that* $A - \varepsilon bc^*$ *is a Hurwitz matrix for all* $\varepsilon \in (0, \Delta]$*; iii)* $\alpha \neq 0$*; iv) the frequency domain inequality* $\Re e\, H(\imath\omega) \geq 0$*, where* $H(s)$ *is given by* (4.19)*, holds for all* $\omega \geq 0$*. Then the zero solution of* (4.17)–(4.18) *is asymptotically stable in the large with respect to the invariant set* $1 + \zeta > 0$.

The proof relies on the Liapunov functional $V : \mathbb{R}^n \times \mathcal{B} \mapsto \mathbb{R}_+$ where $\mathcal{B}$ is some Banach space of functions defined on $(-\infty, 0]$. This functional reads

$$V(x,\phi) = x^*Px + \int_0^{\phi(0)} \frac{\lambda}{1+\lambda}\mathrm{d}\lambda + \sum_1^m \beta_i\xi_i \left[\int_{-\infty}^0 \left(\frac{\phi^2(\lambda)}{1+\phi(\lambda)} - \frac{\phi(\lambda)}{1+\phi(\lambda)}\int_{-\infty}^{\lambda} \kappa_i(\lambda-\sigma)\phi(\sigma)\mathrm{d}\sigma\right)\mathrm{d}\lambda\right]. \tag{4.20}$$

In (4.20) we denoted

$$\kappa_i(t) := \frac{\sigma_i}{\xi_i}\left(\int_0^h \varphi(\eta)\varphi(\eta - t)\mathrm{d}\eta\right)\mathrm{e}^{-t\sigma_i} > 0 \quad \Rightarrow \quad \int_0^\infty \kappa_i(t)\mathrm{d}t = 1.$$

The matrix $P \geq 0$ is chosen from the Yakubovich–Kalman–Popov lemma which states that if ii) and iv) from Theorem 4.3 hold then $P \geq 0$, the vector $w$ and the scalar $\gamma$ can be found such that

$$x^*P(Ax + b\zeta) + (Ax + b\zeta)^*Px - \zeta(c^*x + \alpha\zeta) = -|\gamma\zeta + w^*x|^2. \tag{4.21}$$

Also definite positiveness of $V$ follows from the positiveness of the expressions under the integrals – a consequence of Hardy–Littlewood–Pólya rearrangement inequality No. 368. The derivative is nonpositive definite and asymptotic stability follows from the invariance principle of Barbašin–Krasovskii–LaSalle.

**E.** The still open problem is how to use this result for the global asymptotic stability of the zero solution of (4.16)-(4.17); if this is proved then the same result for (4.13), (4.17) follows quite easily.

We shall sketch in the following how the aforementioned way might be realized. The replacement of (4.16) by (4.18) is possible provided

$$\sum_1^m \beta_i \mathrm{e}^{-t\sigma_i}\left[\sigma_i \int_{-\infty}^0 \left(\int_0^h \varphi(\lambda)\varphi(\lambda+\theta-t)\mathrm{d}\lambda\right)\mathrm{e}^{\theta\sigma_i}\zeta(\theta)\mathrm{d}\theta - \int_0^h \varphi(t+\lambda)y_i^0(\lambda)\mathrm{d}\lambda\right] = 0 \tag{4.22}$$

which clearly defines some subspace of $\mathcal{B}$. This subspace is not void provided the following equalities are fulfilled

$$\sigma_i \int_{-\infty}^0 \left(\int_0^h \varphi(\lambda)\varphi(\lambda+\theta-t)\mathrm{d}\lambda\right)\mathrm{e}^{\theta\sigma_i}\zeta(\theta)\mathrm{d}\theta = \int_0^h \varphi(t+\lambda)y_i^0(\lambda)\mathrm{d}\lambda \tag{4.23}$$

which suggest the choice of $\zeta_i(\theta)$ from some first kind Fredholm equations. Application of the Fubini theorem together with the periodicity of $\varphi$ will send to the following periodic convolution equations

$$\sigma_i \int_0^h \mathrm{e}^{-\theta\sigma_i}\varphi(\lambda-\theta)\psi(\theta)\mathrm{d}\theta = (1 - \mathrm{e}^{-h\sigma_i})y_i^0(\lambda); \qquad 0 \leq \lambda \leq h. \tag{4.24}$$

## 5 Some models occurring in combined heat electricity generation

The model which will be considered here is obtained by merging the standard models for the dynamics of the steam turbines applied to combined heat electricity generation [16] and some pioneering papers in distributed parameters systems [17,18,28]. A first result of this merging of models is to be viewed in [23]. Later it appeared that the equations of the isentropic flow, which are basic in the aforementioned papers, are in fact a system of conservation laws. Consequently the following model is obtained

$$\begin{aligned}
&T_a\dot{s} = \alpha\pi_1 + (1-\alpha)\pi_2 - \nu_r; \quad T_1\dot{\pi}_1 = \mu_1 - \pi_1 \\
&T_p\dot{\pi}_s = \pi_1 - \beta_1\mu_2\pi_s - (1-\beta_1)\xi_w(0,t); \quad T_2\dot{\pi}_2 = \mu_2\pi_s - \pi_2 \\
&T_c\frac{\partial\xi_p}{\partial t} + \frac{\partial\xi_w}{\partial\lambda} = 0 \\
&\psi_c^2 T_c\frac{\partial\xi_w}{\partial t} + \frac{\partial}{\partial\lambda}\left(\xi_p + \psi_c^2\frac{\xi_w^2}{\xi_p}\right) = 0; \quad t>0, \quad 0<\lambda<1 \\
&\xi_w(0,t) = \alpha_p(\pi_s(t) - \beta_2\xi_p(0,t)); \quad \xi_w(1,t) = \psi_s\xi_p(1,t).
\end{aligned} \tag{5.1}$$

We have here a nonlinear system of conservation laws with non-standard boundary conditions – the differential equations describing the steam turbine. Worth mentioning that the variables are in fact rated to some steady state constant values.

This PDE had been linearized by neglecting the term $\psi_c^2(\partial/\partial\lambda)(\xi_w^2/\xi_p)$ – the resulting system being tackled in the subsequent research e.g. [23, 24]. More rigorous would be to linearize the equations around some steady state to obtain

$$\begin{aligned}
&T_a\dot{s} = \alpha\pi_1 + (1-\alpha)\pi_2 - \nu_r; \quad T_1\dot{\pi}_1 = \mu_1 - \pi_1 \\
&T_p\dot{\pi}_s = \pi_1 - \beta_1\mu_2\pi_s - (1-\beta_1)\xi_w(0,t); \quad T_2\dot{\pi}_2 = \mu_2\pi_s - \pi_2 \\
&T_c\frac{\partial\xi_p}{\partial t} + \frac{\partial\xi_w}{\partial\lambda} = 0; \quad t>0, \quad 0<\lambda<1 \\
&\psi_c^2 T_c\frac{\partial\xi_w}{\partial t} + \frac{\partial}{\partial\lambda}\left((1-\delta_0\psi_c^2)\xi_p + 2\delta_0\psi_c^2\xi_w\right) = 0 \\
&\xi_w(0,t) = \alpha_p(\pi_s(t) - \beta_2\xi_p(0,t)); \quad \xi_w(1,t) = \psi_s\xi_p(1,t)
\end{aligned} \tag{5.2}$$

where $\delta_0 = 1$ accounts for the new type of linearization while $\delta_0 = 0$ corresponds to the first one. Since the case $\delta_0 = 0$ was already analyzed, we shall consider here the case $\delta_0 = 1$. We skip some straightforward but tedious manipulation of pointing out the Riemann invariants and integrating along the characteristics to give directly the associated system of functional differential equations

$$\begin{aligned}
&T_a\dot{s} = \alpha\pi_1 + (1-\alpha)\pi_2 - \nu_r; \quad T_1\dot{\pi}_1 = \mu_1 - \pi_1 \\
&T_p\dot{\pi}_s = \pi_1 - \left(\beta_1\mu_2 + \frac{(1-\beta_1)(1+\psi_c)\alpha_p}{1+\psi_c+\beta_2\alpha_p\psi_c}\right)\pi_s + \frac{(1-\beta_1)\alpha_p}{1+\psi_c+\beta_2\alpha_p\psi_c}\eta^-(t-\psi_c T_c(1-\psi_c)^{-1}) \\
&T_2\dot{\pi}_2 = \mu_2\pi_s - \pi_2 \\
&\eta^+(t) = \frac{1-\psi_c-\beta_2\alpha_p\psi_c}{1+\psi_c+\beta_2\alpha_p\psi_c\eta^-(t-\psi_c T_c(1-\psi_c)^{-1}) + \frac{2\alpha_p\psi_c}{1+\psi_c+\beta_2\alpha_p\psi_c}\pi_s(t)} \\
&\eta^-(t) = \frac{1+\psi_c-\psi_s\psi_c}{1-\psi_c+\psi_s\psi_c}\eta^+(t-\psi_c T_c(1+\psi_c)^{-1})
\end{aligned} \tag{5.3}$$

together with the representation formulae

$$\begin{aligned}\xi_p(\lambda,t) &= 0.5(\eta^+(t-\lambda\psi_c T_c(1+\psi_c)^{-1})+\eta^-(t-(1-\lambda)\psi_c T_c(1-\psi_c)^{-1}))\\ \xi_w(\lambda,t) &= (2\psi_c)^{-1}((1+\psi_c)\eta^+(t-\lambda\psi_c T_c(1+\psi_c)^{-1})\\ &\quad -(1-\psi_c)\eta^-(t-(1-\lambda)\psi_c T_c(1-\psi_c)^{-1})).\end{aligned} \tag{5.4}$$

Using these representation formulae and the construction by steps of the solutions of (5.3), the following theorem on invariant sets is obtained.

**Theorem 5.1.** *Let system (5.2) have all coefficients and control signals strictly positive, also* $0<\psi_s<1$, $0<\psi_c<(1+\beta_2\alpha_p)^{-1}<1$. *Then if*

$$\begin{aligned}&\pi_1(0)\geq 0,\quad \pi_s(0)\geq 0,\quad \pi_2(0)\geq 0;\\ &\xi_p(\lambda,0)\geq 0,\quad \xi_w(\lambda,0)+(1/\psi_c-1)\xi_p(\lambda,0)\geq 0\end{aligned} \tag{5.5}$$

*then either*

$$\pi_1(t)\equiv 0\,,\quad \pi_s(t)\equiv 0\,,\quad \pi_2(t)\equiv 0\,;\quad \xi_p(\lambda,t)\equiv 0\,,\quad \xi_w(\lambda,t)+(1/\psi_c-1)\xi_p(\lambda,t)\equiv 0 \tag{5.6}$$

*for all* $t>0$, $0\leq\lambda\leq 1$, *or the aforementioned variables are strictly positive on the same domain.*

The next stage of the analysis is to use system (5.3) for the feedback control design as in [24]. Following the same line as in [24] we take $T_1=T_2\approx 0$ since the structure of the equations allows application of the singular perturbations and eliminate the variable $\eta^-(t)$ from the system to obtain

$$\begin{aligned}T_a\dot{s} &= \alpha\mu_1+(1-\alpha)\mu_2\pi_s-\nu_r\\ T_p\dot{\pi}_s &= \mu_1-\left(\beta_1\mu_2+\frac{(1-\beta_1)(1+\psi_c)\alpha_p}{1+(1+\beta_2\alpha_p)\psi_c}\right)\pi_s\\ &\quad+\frac{(1-\beta_1)\alpha_p}{1+(1+\beta_2\alpha_p)\psi_c}\cdot\frac{1+\psi_c(1-\psi_s)}{1-\psi_c(1-\psi_s)}\eta^+(t-2\psi_c(1-\psi_c^2)^{-1}T_c)\\ \eta^+(t) &= \frac{1+(1-\beta_2\alpha_p)\psi_c}{1+(1+\beta_2\alpha_p)\psi_c}\cdot\frac{1+\psi_c(1-\psi_s)}{1-\psi_c(1-\psi_s)}\eta^+(t-2\psi_c(1-\psi_c^2)^{-1}T_c)+\frac{2\alpha_p\psi_c}{1+(1+\beta_2\alpha_p)\psi_c}\pi_s(t).\end{aligned} \tag{5.7}$$

We introduce the difference operator

$$\mathcal{D}\phi := \phi(0)-\frac{1+(1-\beta_2\alpha_p)\psi_c}{1+(1+\beta_2\alpha_p)\psi_c}\cdot\frac{1+\psi_c(1-\psi_s)}{1-\psi_c(1-\psi_s)}\phi(-2\psi_c(1-\psi_c^2)^{-1}T_c) \tag{5.8}$$

which is exponentially stable and rewrite system (5.7) by taking into account that $(\mathcal{D}\eta^+)(t)=\pi_s(t)$

$$\begin{aligned}T_a\dot{s} &= \alpha\mu_1+(1-\alpha)\mu_2(\mathcal{D}\eta^+)(t)-\nu_r\\ T_p\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{D}\eta^+)(t) &= \mu_1-\left(\beta_1\mu_2+\frac{(1-\beta_1)(1+\psi_c)\alpha_p}{1+(1+\beta_2\alpha_p)\psi_c}\right)(\mathcal{D}\eta^+)(t)\\ &\quad+\frac{(1-\beta_1)\alpha_p}{1+(1+\beta_2\alpha_p)\psi_c}\cdot\frac{1+\psi_c(1-\psi_s)}{1-\psi_c(1-\psi_s)}\eta^+(t-2\psi_c(1-\psi_c^2)^{-1}T_c)\\ \pi_s(t) &= (\mathcal{D}\eta^+)(t)\\ \eta^+(t) &= \frac{1+(1-\beta_2\alpha_p)\psi_c}{1+(1+\beta_2\alpha_p)\psi_c}\cdot\frac{1+\psi_c(1-\psi_s)}{1-\psi_c(1-\psi_s)}\eta^+(t-2\psi_c(1-\psi_c^2)^{-1}T_c)\\ &\quad+\frac{2\alpha_p\psi_c}{1+(1+\beta_2\alpha_p)\psi_c}\pi_s(t).\end{aligned} \tag{5.9}$$

Following [24,26], there is constructed a Liapunov functional of the form

$$\begin{aligned} V(s,\phi) = \frac{1}{2}T_a\left(s + \gamma_0\frac{T_p}{T_a}\left(\mathcal{D}\phi + \frac{\gamma_1}{T_p}\int_{-2\psi_c T_c'}^{0}\phi(\theta)\mathrm{d}\theta\right)\right)^2 \\ + \frac{1}{2}\gamma_2 T_p(\mathcal{D}\phi)^2 + \gamma_3\int_{-2\psi_c T_c'}^{0}\phi^2(\theta)\mathrm{d}\theta \end{aligned} \tag{5.10}$$

where $T_c' = T_c(1-\psi_c^2)^{-1}$ and $\gamma_i > 0$, $i = 0,1,2,3$ are free constants that may be chosen to obtain adequate properties of the functional and of the system. This Liapunov functional is treated as a control Liapunov function(al) – c.l.f. – allowing synthesis of the feedback control. Due to the fact that $\pi_s > 0$ is an invariant set, the feedback control can be chosen linear saturated and the steady state of the closed loop system will result globally asymptotically stable as in the simpler case of [24]. Moreover, this property is extended to system (5.3) provided $T_1$, $T_2$ are small enough, by applying a proof based on singular perturbations.

## 6 The case of the system of conservation laws

We turn back to the basic system (5.1) of the previous section. For this system the research is in progress. Unlike other papers where the equations are finally linearized, we accomplished the first step in tackling the nonlinear case by separating the Riemann invariants as follows

$$\xi^{\pm}(\xi_w,\xi_p) = \pm\psi_c\xi_w/\xi_p + \ln\xi_p. \tag{6.1}$$

The Riemann invariants are subject to the following equations

$$\psi_c T_c\frac{\partial\xi^{\pm}}{\partial t} + (\xi^+ - \xi^- \pm 1)\frac{\partial\xi^{\pm}}{\partial\lambda} = 0. \tag{6.2}$$

It follows that hyperbolicity holds for $-1 < \xi^+ - \xi^- < 1$ and we may apply integration along the characteristics (locally) following [20,21] and associate again some system of functional differential equations as in the linearized case.

## 7 Some conclusions

We have examined throughout this paper three engineering applications leading to non-standard boundary value problems for hyperbolic partial differential equations (transport and propagation). The integration of the Riemann invariants along the characteristics allowed association of some functional differential equations to the basic systems. Using these associated mathematical objects it was possible to develop the basic theory – well-posedness in the sense of J. Hadamard – at least for the (possibly discontinuous) classical solutions; also some invariant sets accounting for positiveness of some physical variables have been pointed out.

At the same time the analysis of the control problems (feedback control synthesis and stabilization pointed out some (still) unsolved problems; their solving would ensure a rigorous basis for the engineering development. A special mention should be made for the application describing combined heat electricity generation: here there are two interesting results which require further research. The first one concerns a new approach in linearization which incorporates the older one. A comparison would require some computer simulation: in our case – of the PDE – a most common method is the method of lines for which a most recent well structured programmable approach has been elaborated [10]. The second result is the start of the analysis in the nonlinear case, relying on the systems of conservation laws. We have considered this case for hydraulic system [22]. It is hoped that further research will provide new results in both applications.

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