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On periods of non-constant solutions to functional differential equations

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Abstract. We show that periods of solutions to Lipschitz functional differential equations cannot be too small. The problem on such periods is closely related to the unique solvability of the periodic value problem for linear functional differential equations. Sharp bounds for periods of non-constant solutions to functional differential equations with Lipschitz nonlinearities are obtained.

Keywords: periodic problem, periods, functional differential equations, unique solvability.

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1 Introduction

Consider a problem on periodic solutions of the differential equation with deviating argument

$$x^{(n)}(t) = f(x(\tau(t))), \qquad t \in \mathbb{R}, \tag{1.1}$$

where $x(t) \in \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^m$ is a Lipschitz function, $\tau : \mathbb{R} \to \mathbb{R}$ is a measurable function.

If $\tau(t) \equiv t$, the sharp lower estimate

$$T \geqslant 2\pi/L^{1/n} \tag{1.2}$$

for periods T of non-constant periodic solutions to (1.1) is obtained in [28] for $n = 1$ and [16] for $n \geqslant 1$ for Lipschitz f in the Euclidian norm, and in [30] for even n and Lipschitz functions f satisfying the condition

$$\max_{i=1,\dots,m} |f_i(x) - f_i(\tilde{x})| \leqslant L \max_{i=1,\dots,m} |x_i - \tilde{x}_i|, \qquad x, \widetilde{x} \in \mathbb{R}^m. \tag{1.3}$$

The estimate (1.2) gives the minimal time required for an object described by a system of ordinary differential equations with the Lipschitz constant L to return to its initial state.

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For equations (1.1) with an arbitrary piece-wise continuous deviating argument τ and Lipschitz f under condition (1.3), the best constants in the lower estimates for periods T of non-constant periodic solutions are found by A. Zevin for $n = 1$ [29]

$$T \geqslant 4/L,$$

and for even n [30]

$$T \geqslant \alpha(n)/L^{1/n},$$

where $\alpha(n)$ are defined with the help of solutions to some boundary value problem for an ordinary differential equation of n-th order.

Here, for all n, we find a simple representation of the best constants in the estimate for periods of non-constant periodic solutions of some more general equations than (1.1) with Lipschitz nonlinearities. Some properties of the sequence of the best constants will be obtained. It turns out that the best constants in lower estimates of periods are the Favard constants [7, § 4.2].

If equation (1.1) has a T-periodic solution x with absolutely continuous derivatives up to the order $n-1$, then the restriction of x to the interval $[0, T]$ is a solution to the periodic boundary value problem

$$x^{(n)}(t) = f(x(\widetilde{\tau}(t))), \quad t \in [0, T], \quad x^{(i)}(0) = x^{(i)}(T), \quad i = 0, \ldots, n-1, \tag{1.4}$$

where $\widetilde{\tau}(t) = \tau(t) + k(t)T$ for some integer $k(t)$ such that $\tau(t) + k(t)T \in [0, T]$. If boundary value problem (1.4) has no non-constant solutions, then (1.1) has no T-periodic non-constant solutions either.

Therefore, we can consider the equivalent periodic boundary value problem for a system of m functional differential equations of the n-th order

$$x^{(n)}(t) = (Fx)(t), \quad t \in [0, T], \quad x^{(i)}(0) = x^{(i)}(T), \quad i = 0, \ldots, n-1, \tag{1.5}$$

where x belongs to the space $\mathbf{AC}^{n-1}([0, T], \mathbb{R}^m)$ of all functions with absolutely continuous derivatives up to order $n-1$; the equality $x^{(n)}(t) = (Fx)(t)$ holds for almost all $t \in [0, 1]$; the continuous operator F acts from the space $\mathbf{C}([0, T], \mathbb{R}^m)$ of all continuous functions into the space $\mathbf{L}_\infty([0, T], \mathbb{R}^m)$ of all measurable essentially bounded functions (with the norm $\|z\|_{\mathbf{L}_\infty} = \max_{i=1,\ldots,m} \operatorname{ess\,sup}_{t \in [0,T]} |z_i(t)|$).

We assume that there exists a positive constant $L \in \mathbb{R}$ such that the following inequality holds

$$\max_{i=1,\ldots,m} \Big(\operatorname*{ess\,sup}_{t \in [0,T]} (Fx)_i(t) - \operatorname*{ess\,inf}_{t \in [0,T]} (Fx)_i(t) \Big) \leqslant L \max_{i=1,\ldots,m} \Big(\max_{t \in [0,T]} x_i(t) - \min_{t \in [0,T]} x_i(t) \Big) \tag{1.6}$$

for all functions $x \in \mathbf{C}([0, T], \mathbb{R}^m)$.

If the operator F in (1.5) is defined by the equality $(Fx)(t) = f(x(\tau(t)))$, $t \in [0, T]$, where $\tau : [0, T] \to [0, T]$ is a measurable function (not equivalent to a constant), then condition (1.6) implies that the function $f : \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz and satisfies (1.3).

Our approach is close to the work [25], where the periodic boundary value problem is considered on the interval and a general way for obtaining the lower estimate of the periods of non-constant solutions is proposed.

Note that there are a number of papers on minimal periods of non-constant solutions for different classes of equations, in particular, [13] in Hilbert spaces, [14] in Banach spaces for equations with delay, [24, 27] in Banach spaces, [5] in Banach spaces for difference equations, [17] in Banach spaces for equations with differentiable delays, [23] in spaces ℓ_p and $\mathbf{L}_p$.

2 Main results

Define rational constants K_n, $n = 1, 2, \ldots$, by the equalities

$$K_n = \frac{(2^{n+1}-1)|B_{n+1}|}{2^{n-1}(n+1)!} \quad \text{if } n \text{ is odd}, \quad K_n = \frac{|E_n|}{4^n n!} \quad \text{if } n \text{ is even}, \tag{2.1}$$

where B_n are the Bernoulli numbers, E_n are the Euler numbers (see, for examples, [1, p. 804]).

Proposition 2.1.

a) K_n *are the Favard constants, that is, the best constants in the inequality*

$$\max_{t\in[0,1]} |x(t)| \leqslant K_n \operatorname*{ess\,sup}_{t\in[0,1]} |x^{(n)}(t)|$$

which holds for all functions $x \in \mathbf{AC}^{n-1}([0,1], \mathbb{R})$ *such that* $x^{(n)} \in \mathbf{L}_\infty([0,1], R^1)$ *and* $x^{(i)}(0) = x^{(i)}(1)$, $i = 0, \ldots, n-1$, $\int_0^1 x(t)\,dt = 0$;

b)

$$K_n(2\pi)^n = \min_{\xi\in\mathbb{R}} \int_0^{2\pi} |\phi_n(s) - \xi|\,ds = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{(n+1)(k+1)}}{(2k-1)^{n+1}},$$

where $\phi_n(t) = \dfrac{1}{\pi}\displaystyle\sum_{k=1}^{\infty} k^{-n}\cos\left(kt - \frac{n\pi}{2}\right)$;

c) $K_{n+1} = \dfrac{1}{8(n+1)}\displaystyle\sum_{k=0}^{n} K_k K_{n-k},\ n \geqslant 1,\ K_0 = 1,\ K_1 = 1/4$;

d) $\dfrac{1}{\cos(t/4)} + \tan(t/4) = 1 + \displaystyle\sum_{n=1}^{\infty} K_n t^n, \quad |t| < 2\pi$;

e) $\lim_{n\to\infty} K_n(2\pi)^n = 4/\pi$;

f) $K_1 = 1/4,\ K_2 = 1/32,\ K_3 = 1/192,\ K_4 = 5/6144,\ K_5 = 1/7680,\ K_6 = 61/2949120, \ldots$

Proof. All these assertions are well known. One can see proofs of a), b), f) in [3, 4, 6, 15, 26], proofs of items c), d), e) in, for example, [4]. □

Theorem 2.2. *If F satisfies inequality* (1.6) *and periodic problem* (1.5) *has a non-constant solution, then*

$$T \geqslant \frac{1}{(L\,K_n)^{1/n}}. \tag{2.2}$$

To prove Theorem 2.2, we need two lemmas.

Lemma 2.3. *Let F satisfy* (1.6)*. If problem* (1.5) *has a non-constant solution, there exist a measurable function* $\tau : [0, T] \to [0, T]$ *and a constant C such that at least one of non-constant components of the solution satisfies the scalar periodic boundary problem*

$$\begin{aligned} &y^{(n)}(t) = L\,y(\tau(t)) + C, && t \in [0, T], \\ &y^{(i)}(0) = y^{(i)}(T), && i = 0, \ldots, n-1. \end{aligned} \tag{2.3}$$

Proof. Suppose $y = x_j$ is a non-constant component of the solution x to (1.5) such that

$$\max_{t\in[0,T]} x_j(t) - \min_{t\in[0,T]} x_j(t) = \max_{i=1,\dots,m} \left(\max_{t\in[0,T]} x_i(t) - \min_{t\in[0,T]} x_i(t) \right). \tag{2.4}$$

From (1.6) it follows that there exist a measurable function $\tau : [0,T] \to [0,T]$ and a constant C such that

$$(Fx)_j(t) = L\,y(\tau(t)) + C$$

for almost all $t \in [0,T]$. This proves the lemma. □

Lemma 2.4. *Let $L > 0$. Periodic boundary value problem (2.3) has a unique solution for every measurable $\tau : [0,T] \to [0,T]$ and for every constant $C \in \mathbb{R}$ if*

$$L < \frac{1}{K_n\,T^n}. \tag{2.5}$$

Proof. Problem (2.3) has the Fredholm property [2]. Hence, this problem is uniquely solvable if and only if the homogeneous problem

$$y^{(n)}(t) = L\,y(\tau(t)), \quad t \in [0,T], \qquad y^{(i)}(0) = y^{(i)}(T), \quad i = 0,\dots,n-1, \tag{2.6}$$

has only the trivial solution. Let y be a nontrivial solution of (2.6). From [15, D. 32, p. 386] (or [26]) it follows that for some constant C_1 and for all constants ξ the solution y satisfies the equality

$$\begin{aligned} y(t) &= \frac{T^{n-1}}{(2\pi)^{n-1}} \int_0^T (\phi_n(2\pi s/T) - \xi) y^{(n)}(t-s)\,ds + C_1 \\ &= \frac{T^{n-1}}{(2\pi)^{n-1}} \int_0^T (\phi_n(2\pi s/T) - \xi) Ly(\tau(t-s))\,ds + C_1, \end{aligned} \tag{2.7}$$

where $t \in [0,T]$, $y(t-s) \equiv y(t-s+T)$, $\tau(t-s) \equiv \tau(t-s+T)$ if $t-s \in [-T,0)$; the functions ϕ_n are defined in Proposition 2.1.

Therefore, if

$$L < \frac{(2\pi)^{n-1}}{T^{n-1} \inf\limits_{\xi\in\mathbb{R}} \int_0^T |\phi_n(2\pi s/T) - \xi|\,ds} = \frac{(2\pi)^n}{T^n \inf\limits_{\xi\in\mathbb{R}} \int_0^{2\pi} |\phi_n(s) - \xi|\,ds} = \frac{1}{K_n T^n}, \tag{2.8}$$

then the linear operator A in the right-hand side of (2.7)

$$(Ay)(t) = \frac{T^{n-1}L}{(2\pi)^{n-1}} \int_0^T (\phi_n(2\pi s/T) - \xi) y(\tau(t-s))\,ds + C_1, \qquad t \in [0,T],$$

is a contraction mapping in $\mathbf{L}_\infty([0,T],\mathbb{R})$. In this case for each C_1, equation (2.7) has a unique solution which is a constant (we use here the equality $\int_0^T \phi_n(2\pi t/T)\,dt = 0$). From (2.6) it follows that this constant is zero. Therefore, problem (2.3) is uniquely solvable. □

Proof of Theorem 2.2. Let (1.5) have a non-constant solution. From Lemma 2.3 it follows that the non-constant component x_j (from the proof of Lemma 2.3) of the solution x to (1.5) is a solution to (2.3) with some constant C and some measurable function $\tau : [0,T] \to [0,T]$. If inequality (2.5) holds, it follows from Lemma 2.4 that this solution is unique: $x_j(t) \equiv -C/L$. Then from (1.6) it follows that each component x_i of the non-constant solution x is constant. Therefore, inequality (2.5) does not hold, and inequality (2.2) is valid. □

Now assume that an operator F in equation (1.5) acts into the space of integrable functions $\mathbf{L}_1([0,T],\mathbb{R}^m)$ with the norm

$$\|z\|_{\mathbf{L}_1} = \max_{i=1,\dots,m} \int_0^T |z_i(t)|\,dt.$$

Theorem 2.5. *Suppose an operator $F : \mathbf{C}([0,T],\mathbb{R}^m) \to \mathbf{L}_1([0,T],\mathbb{R}^m)$ is continuous.*

Let there exist positive functions $p_i \in \mathbf{L}_1([0,T],\mathbb{R})$, $i = 1,\dots,m$, such that for every $x \in \mathbf{C}([0,T],\mathbb{R}^m)$ the inequality

$$\max_{i=1,\dots,m} \left(\operatorname*{ess\,sup}_{t\in[0,T]} \frac{(Fx)_i(t)}{p_i(t)} - \operatorname*{ess\,inf}_{t\in[0,T]} \frac{(Fx)_i(t)}{p_i(t)} \right) \leqslant \max_{i=1,\dots,m} \left(\max_{t\in[0,T]} x_i(t) - \min_{t\in[0,T]} x_i(t) \right) \tag{2.9}$$

holds. If periodic problem (1.5) has a non-constant solution, then the following inequalities

$$\|p_i\|_{\mathbf{L}_1} \geqslant 4 \ \textit{if} \ n=1, \quad \|p_i\|_{\mathbf{L}_1} > \frac{4}{K_{n-1}T^{n-1}} \ \textit{if} \ n \geqslant 2. \tag{2.10}$$

are fulfilled for each $i = 1,\dots,m$.

To prove Theorem 2.5, we also need two lemmas.

Lemma 2.6. *Let F satisfy inequality (2.9). If problem (1.5) has a non-constant solution, there exist a measurable function $\tau : [0,T] \to [0,T]$ and a constant C such that one of non-constant components of the solution satisfies the scalar periodic boundary value problem*

$$\begin{aligned} &y^{(n)}(t) = p(t)(y(\tau(t)) + C), \qquad t \in [0,T], \\ &y^{(i)}(0) = y^{(i)}(T), \qquad i = 0,\dots,n-1. \end{aligned} \tag{2.11}$$

Proof. Suppose $y = x_j$ is a non-constant component of the solution x to (1.5) such that equality (2.4) holds. Then the essential diameter of the range of the function $(Fx)_j/p_j$ does not exceed the length of the range of x_j. So, there exist a measurable function $\tau : [0,T] \to [0,T]$ and a constant C such that

$$(Fx)_j(t) = p(t)(y(\tau(t)) + C) \quad \text{for almost all } t \in [0,T],$$

where $p = p_j$. This proves the lemma. □

Lemma 2.7 ([4,8–10,18–20])**.** *Let a positive number $\mathcal{P}$ be given. Problem (2.11) has a unique solution for every measurable function $\tau : [0,T] \to [0,T]$ and every non-negative function $p \in \mathbf{L}_1([0,T],\mathbb{R})$ with norm $\|p\|_{\mathbf{L}_1} = \mathcal{P}$ if and only if*

$$\mathcal{P} < 4 \quad \textit{if} \ n=1, \qquad \mathcal{P} \leqslant \frac{4}{K_{n-1}T^{n-1}} \quad \textit{if} \ n \geqslant 2. \tag{2.12}$$

For $n = 1$, $n = 2$, $n = 3$, $n = 4$ this lemma is proved in [8,18–20], for arbitrary n in [4,9,10].

Proof of Theorem 2.5. Let (1.5) have a non-constant solution. From Lemma 2.6 it follows that a non-constant component x_j (from the proof of Lemma 2.6) of the solution x to (1.5) is a solution to (2.11), where $p = p_j$, C is some constant, $\tau : [0,T] \to [0,T]$ is some measurable function. If condition (2.12) hold, from Lemma 2.7 it follows that the solution x_j is unique: $x_j(t) \equiv -C$. From (2.9) it follows that each component x_i of the non-constant solution x is constant. Therefore, inequalities (2.12) do not hold, and inequalities (2.10) are valid. □

3 The sharpness of estimates

The estimates (2.2) and (2.10) in Theorems 2.2 and 2.5 are sharp. The sharpness of (2.10) is shown in [4]. The sharpness of (2.2) for even n was shown in [30] in other terms.

Now for every $n \geqslant 1$ we obtain functions $\tau : [0,T] \to [0,T]$ such that the periodic boundary value problem

$$x^{(n)}(t) = Lx(\tau(t)), \quad t \in [0,T], \quad x^{(i)}(0) = x^{(i)}(T), \quad i = 0,\dots,n-1, \tag{3.1}$$

has a non-constant solution provided that (2.2) is an identity:$L = \frac{1}{K_n T^n}$. Find a solution to the auxiliary problem

$$x^{(n)}(t) = L\,h(t), \quad t \in [0,T], \quad x^{(i)}(0) = x^{(i)}(T), \quad i = 0,\dots,n-1, \tag{3.2}$$

where $h(t) = 1$ for $t \in [0, T/2]$ and $h(t) = -1$ for $t \in (T/2, T]$. Since $\int_0^T h(t)\,dt = 0$, this problem has a solution. It is not unique and defined by the equality

$$x(t) = C + L\int_0^T G(t,s)h(s)\,ds, \qquad t \in [0,T],$$

where C is an arbitrary constant, $G(t,s)$ is the Green function of the problem

$$\begin{aligned} x^{(n)}(t) &= f(t), && t \in [0,T], \\ x(0) &= 0, && \\ x(T) &= 0 && (\text{if } n \geqslant 2), \\ x^{(i)}(0) &= x^{(i)}(T), && i = 1,\dots,n-2 \quad (\text{if } n > 2). \end{aligned}$$

We have a simple representation for the Green function $G(t,s)$:

$$G(t,s) = \frac{T^n}{n!}\big(B_n(t/T) - B_n(0) - \mathcal{B}_n((t-s)/T) + B_n(1-s/T)\big), \qquad t,s \in [0,T],$$

where $B_n(t)$ are the Bernoulli polynomials [1, p. 804] which can be defined as unique solutions to the problems

$$\begin{aligned} B_n^{(n)}(t) &= n!, && t \in [0,T], \\ \int_0^1 B_n(t)\,dt &= 0, && \\ B_n^{(i)}(0) &= B_n^{(i)}(T), && i = 0,\dots,n-2 \quad (\text{if } n \geqslant 2), \end{aligned}$$

$\mathcal{B}_n(t) = B_n(\{t\})$ are the periodic Bernoulli functions, $\{t\}$ is the fractional part of t.

Using the equality [1, p. 805, 23.1.11]

$$\int_{t_1}^{t_2} B_n(s)\,ds = (B_{n+1}(t_2) - B_{n+1}(t_1))/(n+1), \qquad n \geqslant 1,$$

which is also valid for the functions $\mathcal{B}_n(t)$, we obtain the representation for solutions y to problem (3.2):

$$y(t) = C + \frac{2LT^n}{(n+1)!}\big(B_{n+1}(1/2) - B_{n+1}(0) + B_{n+1}(t/T) - \mathcal{B}_{n+1}(t/T - 1/2)\big), \qquad t \in [0,T],$$

for every constant $C \in \mathbb{R}$.

For even $n = 2m$, using the equalities [1, p. 805, 23.19–22, 23.1.15]

$$\begin{aligned} B_{2m+1}(1/4) &= -B_{2m+1}(3/4) = (2m+1)4^{-2m-1}E_{2m}, \\ B_{2m+1}(1/2) &= B_{2m+1}(0) = 0, \qquad (-1)^m E_{2m} > 0, \end{aligned}$$

we obtain that $y(T/4) = -y(3T/4) = (-1)^m$ for $C = 0$. Therefore, for $C = 0$ the function y is a non-constant solution to problem (3.1), where

$$\tau(t) = \begin{cases} T/4 & \text{if } t \in [0, T/2], \\ 3T/4 & \text{if } t \in (T/2, T], \end{cases} \qquad \text{for } n = 0 \bmod 4,$$

and

$$\tau(t) = \begin{cases} 3T/4 & \text{if } t \in [0, T/2], \\ T/4 & \text{if } t \in (T/2, T], \end{cases} \qquad \text{for } n = 2 \bmod 4.$$

Note that these functions τ were found in [30].

For odd $n = 2m - 1$ using the equalities [1, p. 805, 23.1.20–21, 23.1.15]

$$B_{2m} = B_{2m}(0) = B_{2m}(1), \quad B_{2m}(1/2) = (2^{1-2m} - 1)B_{2m}, \quad (-1)^{m+1}B_{2m} > 0,$$

we see that $y(0) = -y(T/2) = (-1)^m$ for $C = (-1)^m$. Therefore, for $C = (-1)^m$ the function y is a non-constant solution to problem (3.1), where

$$\tau(t) = \begin{cases} T/2 & \text{if } t \in [0, T/2], \\ 0 & \text{if } t \in (T/2, T], \end{cases} \qquad \text{for } n = 1 \bmod 4,$$

$$\tau(t) = \begin{cases} 0 & \text{if } t \in [0, T/2], \\ T/2 & \text{if } t \in (T/2, T], \end{cases} \qquad \text{for } n = 3 \bmod 4.$$

4 Example. Equations with "maxima"

Let L be a constant, $\tau, \theta : \mathbb{R} \to \mathbb{R}$ measurable functions such that $\tau(t) \leqslant \theta(t)$ for all $t \in \mathbb{R}$. From Theorem 2.2, it follows that periods T of non-constants solutions of the equation

$$x^{(n)}(t) = L \max_{s \in [\tau(t), \theta(t)]} x(s), \qquad t \in \mathbb{R},$$

satisfy the inequality

$$|L|\, T^n \geqslant \frac{1}{K_n}, \tag{4.1}$$

where the constants K_n are defined by (2.1).

Suppose $p : \mathbb{R} \to \mathbb{R}$ is a positive locally integrable T-periodic function: $p(t+T) = p(t)$, $p(t) > 0$ for all $t \in \mathbb{R}$. From Theorem 2.5, it follows that if there exists a T-periodic non-constants solution to the equation

$$x^{(n)}(t) = p(t) \max_{s \in [\tau(t), \theta(t)]} x(s), \qquad t \in \mathbb{R},$$

then

$$\int_0^T p(t)\, dt \geqslant 4 \quad \text{for } n = 1, \qquad \int_0^T p(t)\, dt\, T^{n-1} > \frac{4}{K_{n-1}} \quad \text{for } n \geqslant 2. \tag{4.2}$$

Inequalities (4.1) and (4.2) are sharp.

5 Conclusion

Now we formulate unimprovable necessary conditions for the existence of a non-constant periodic solution to (1.5) which follow from Theorems 2.2 and 2.5: if F satisfies (1.6) and there exists a non-constant solution to (1.5), then the constants $L = L_n$ satisfy the inequalities

$$L_1 \geqslant 4/T, \quad L_2 \geqslant 32/T^2, \quad L_3 \geqslant 132/T^3, \quad L_4 \geqslant 6144/(5T^4), \quad L_5 \geqslant 7680/T^5, \dots;$$

if F satisfies (2.9) and there exists a non-constant solution to (1.5), then the constants $\mathcal{P} = \mathcal{P}_n = \max_{i=1,\dots,n} \|p_i\|_{\mathbf{L}_1}$ satisfy the inequalities

$$\mathcal{P}_1 \geqslant 4, \quad \mathcal{P}_2 > 16/T, \quad \mathcal{P}_3 > 128/T^2, \quad \mathcal{P}_4 > 768/T^3, \quad \mathcal{P}_5 > 24776/(5T^4), \dots$$

It follows from Proposition 2.1 that $\lim_{n\to\infty}(K_n)^{1/n} = 1/(2\pi)$, therefore estimate (2.2) for large n is close to estimate (1.2) for equations without deviating arguments.

New results on existence and uniqueness of periodic solutions for higher order functional differential equations are obtained in [11,12,21,22]. Note that Theorems 2.2 and 2.5 cannot be derived from the results of these articles.

The short message on these results (without proofs) was published in the journal "Russian Mathematics" (Iz. VUZ), No. 12 (2013) in Russian.

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