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Existence of multiple solutions for a class of nonhomogeneous problems with critical growth

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Abstract. In this paper, we study the existence and multiplicity of solutions for $(p_1(x), p_2(x))$-equation with critical growth. The technical approach is mainly based on the variational method combined with the genus theory.

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1 Introduction

In this article, we are concerned with the following problem

$$\begin{cases} -\Delta_{p_1(x)}u - \Delta_{p_2(x)}u - a(x)|u|^{m(x)-2}u = \lambda|u|^{q(x)-2}u + f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded smooth domain, λ is a positive parameter and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies some assumptions provided later. Moreover, $p_1, p_2, q \in C(\overline{\Omega})$ and $m(x) = \max(p_1(x), p_2(x))$ for all $x \in \overline{\Omega}$, such that

$$\begin{aligned} 1 < p_i^- = \min_{x\in\overline{\Omega}} p_i(x) \leq p_i^+ = \max_{x\in\overline{\Omega}} p_i(x) < N, \qquad i = 1,2, \\ m^+ = \max_{x\in\overline{\Omega}} m(x) < q^- = \min_{x\in\overline{\Omega}} q(x) \leq q(x) \leq m^*(x) \qquad \forall x \in \overline{\Omega}, \end{aligned} \tag{mq_1}$$

where $m^*(x) = \frac{Nm(x)}{N-m(x)}$ for all $x \in \overline{\Omega}$ and the set

$$B = \{x \in \Omega : q(x) = m^*(x)\} \text{ is nonempty.} \tag{mq_2}$$

In recent years, the study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years. A more and more important number of surveys and books dealing with this type of problems and their corresponding functional spaces setting have been published (see [1–4, 12, 16–20]). We also have to mention

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the books [13] and [21] as important references in this field. This great interest may be justified by their various physical applications. In fact, there are applications concerning elastic mechanics [25], electrorheological fluids [23, 24], image restoration [9], dielectric breakdown, electrical resistivity and polycrystal plasticity [6, 7] and continuum mechanics [5].

It is well known that although most of the materials can be accurately modeled with the help of the classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$, where p is a fixed constant, but there are some nonhomogeneous materials, for which this is not adequate, e.g. the rheological fluids mentioned above, which are characterized by their ability to drastically change their mechanical properties under the influence of an exterior electromagnetic field. Thus it is necessary for the exponent p to be variable, hence the need for spaces with variable exponents. This leads, on the one hand,to many interesting applications, and, on the other hand,to the study of much more mathematically complicated problems.

In [19], Mihăilescu considered the problem

$$\begin{cases} -\Delta_{p_1(x)}u - \Delta_{p_2(x)}u = \pm(-\lambda|u|^{m(x)-2}u + |u|^{q(x)-2}u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $m(x) = \max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$ or $m(x) < q(x) < \frac{Nm(x)}{N-m(x)}$ for any $x \in \overline{\Omega}$. In the first case, using mountain pass theorem, he established the existence of infinity many solutions. In the second case, by simple variational arguments, he proved that the problem has a solution for λ large enough. The novelty of this paper lies in the fact we consider problem (P_λ), with growth $q(x)$ which is critical in a set with positive measure. The difficulty in this case, is due to the lack of compactness of the imbedding $W_0^{1,m(x)}(\Omega) \hookrightarrow L^{m^*(x)}(\Omega)$ and the Palais–Smale condition for the corresponding energy functional could not be checked directly. To deal with this difficulty, we use a version of the concentration compactness lemma due to Lions for variable exponents [8].

Here, we are interested in the existence and multiplicity of weak solutions under the following hypotheses on $a(x)$ and f.

(a_1) $a(x) \in L^\infty(\Omega)$ and there exists $\alpha > 0$ such that

$$\int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} - a(x)\frac{|u|^{m(x)}}{m(x)} \right) dx \geq \alpha \int_\Omega \frac{|u|^{m(x)}}{m(x)} dx, \qquad \forall\, u \in W_0^{1,m(x)}(\Omega);$$

(a_2) $m(x) = m^+$ for all $x \in \Gamma^+ := \{x \in \Omega : a(x) > 0\}$;

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, odd with respect to t such that

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{m(x)-1}} = 0, \quad \text{uniformly in } x \in \overline{\Omega},$$
$$\lim_{|t|\to +\infty} \frac{f(x,t)}{|t|^{q(x)-1}} = 0, \quad \text{uniformly in } x \in \overline{\Omega};$$

(f_2) $F(x,t) \leq \frac{1}{m^+} f(x,t)t$, $\forall t \in \mathbb{R}$ and a.e. $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)ds$.

Example 1.1. In this example we just exhibit a function $a(x)$ satisfying assumption (a_1). Let $\Omega = B(0,2) := \{x \in \mathbb{R}^N : |x| < 2\}$ $(N \geq 3)$, $p_1(x) = 2 - \frac{1}{4}x_1 - \frac{1}{16}|x|^2$ and $p_2(x) = 2 - \frac{1}{4}x_2 - \frac{1}{16}|x|^2$ for all $x = (x_1, x_2, \ldots, x_N) \in \overline{\Omega}$, where $|x|^2 = \sum_{i=1}^N x_i^2$. Then $p_i \in C(\overline{\Omega})$

and $1 < p_i^- \leq p_i^+ < N$, $i = 1,2$. Let $e_1 = (1,0,\dots,0)$. Then for any $x \in \Omega$, the function $h_1 : t \mapsto p_1(x + te_1)$ is monotone in $I_x = \{t : x + te_1 \in \Omega\}$. In fact, we have

$$h_1(t) = 2 - \frac{1}{4}(x_1 + t) - \frac{1}{16}(x_1 + t)^2 - \frac{1}{16}\sum_{i=2}^{N} x_i^2,$$

thus $h_1'(t) = -\frac{1}{8}(2 + x_1 + t)$. Since $|x_1 + t| \leq |x + te_1| < 2$, $h_1'(t) < 0$ for all $t \in I_x$, and hence h_1 is decreasing in I_x. Similarly, for $e_2 = (0,1,\dots,0)$, the function $h_2 : t \mapsto p_2(x + te_2)$ is monotone in $I_x = \{t : x + te_2 \in \Omega\}$. Therefore p_i, $i = 1,2$, satisfying conditions of [15, Theorem 3.3], thus the modular Poincaré inequality holds:

$$\int_\Omega |\nabla u|^{p_i(x)} dx \geq C_i \int_\Omega |u|^{p_i(x)} dx \qquad (C_i > 0,\ i = 1,2),$$

which is equivalent to

$$\int_\Omega \frac{|\nabla u|^{p_i(x)}}{p_i(x)} dx \geq C_i' \int_\Omega \frac{|u|^{p_i(x)}}{p_i(x)} dx \qquad (C_i' > 0,\ i = 1,2),$$

since $1 < p_i^- \leq p_i(x) \leq p_i^+ < N$.

Now, let $V \in L^\infty(\Omega)$ such that $1 + V(x) > c > 0$. Note that p_i, $i = 1,2$, are log-Hölder continuous functions. Indeed, let $x, y \in \Omega$ such that $|x - y| \leq \frac{1}{2}$, then

$$\begin{aligned}
|p_1(x) - p_1(y)| &\leq \frac{1}{4}|x_1 - y_1| + \frac{1}{16}\left||x|^2 - |y|^2\right| \\
&\leq \frac{1}{4}|x - y| + \frac{1}{16}\left||x| - |y|\right|(|x| + |y|) \\
&\leq \frac{1}{4}|x - y| + \frac{1}{16}|x - y|(|x| + |y|) \\
&\leq \frac{1}{2}|x - y| \\
&\leq \frac{1}{2\log\left(\frac{1}{|x-y|}\right)}.
\end{aligned}$$

In the same way, we get $|p_2(x) - p_2(y)| \leq \frac{1}{2\log\left(\frac{1}{|x-y|}\right)}$. Applying [11, Theorem 2.2], we can find $\alpha_1, \alpha_2 > 0$ such that

$$\int_\Omega \frac{|\nabla u|^{p_1(x)}}{p_1(x)} dx \geq \alpha_1 \int_\Omega (1 + V(x)) \frac{|u|^{p_1(x)}}{p_1(x)} dx$$

and

$$\int_\Omega \frac{|\nabla u|^{p_2(x)}}{p_2(x)} dx \geq \alpha_2 \int_\Omega (1 + V(x)) \frac{|u|^{p_2(x)}}{p_2(x)} dx.$$

Observe that $|u|^{p_1(x)} + |u|^{p_2(x)} \geq |u|^{m(x)}$, thus $\frac{|u|^{p_1(x)}}{p_1(x)} + \frac{|u|^{p_2(x)}}{p_2(x)} \geq \frac{|u|^{m(x)}}{m(x)}$. It follows that

$$\begin{aligned}
\int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} \right) dx &\geq \min(\alpha_1, \alpha_2) \int_\Omega (1 + V(x)) \left(\frac{|u|^{p_1(x)}}{p_1(x)} + \frac{|u|^{p_2(x)}}{p_2(x)} \right) dx \\
&\geq \min(\alpha_1, \alpha_2) \int_\Omega (1 + V(x)) \frac{|u|^{m(x)}}{m(x)} dx.
\end{aligned}$$

Therefore

$$\int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} \right) dx - \int_\Omega \min(\alpha_1, \alpha_2) V(x) \frac{|u|^{m(x)}}{m(x)} dx \geq \min(\alpha_1, \alpha_2) \int \frac{|u|^{m(x)}}{m(x)} dx,$$

and hence the function $a(x) := \min(\alpha_1, \alpha_2) V(x)$ satisfies condition (a_1).

2 Preliminaries

Here, we state some interesting properties of the variable exponent Lebesgue and Sobolev spaces that will be useful to discuss problem (P_λ). Every where below we consider $\Omega \subset \mathbb{R}^N$ to be a bounded domain with smooth boundary and $p(x) \in C_+(\overline{\Omega})$, where

$$C_+(\Omega) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

Define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

This space endowed with the Luxemburg norm,

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \tau > 0 : \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}$$

is a separable and reflexive Banach space. Denoting by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$; for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have the following Hölder type inequality

$$\int_\Omega |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)}.$$

Now, we introduce the modular of the Lebesgue–Sobolev space $L^{p(x)}(\Omega)$ as the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx, \qquad \forall u \in L^{p(x)}(\Omega).$$

In the following proposition, we give some relations between the Luxemburg norm and the modular.

Proposition 2.1 ([14]). *If $u, u_n \in L^{p(x)}(\Omega)$, then following properties hold true:*

(1) $\|u\|_{L^{p(x)}(\Omega)} \leq 1 \Rightarrow \|u\|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho_{p(x)}(u) \leq \|u\|^{p^-}_{L^{p(x)}(\Omega)}$;

(2) $\|u\|_{L^{p(x)}(\Omega)} \geq 1 \Rightarrow \|u\|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho_{p(x)}(u) \leq \|u\|^{p^+}_{L^{p(x)}(\Omega)}$;

(3) $\lim_{n\to\infty} \|u_n\|_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n) = 0$;

(4) $\lim_{n\to\infty} \|u_n\|_{L^{p(x)}(\Omega)} = \infty \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n) = \infty$.

Next, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)},$$

where by $\|\nabla u\|_{L^{p(x)}(\Omega)}$ we understand $\| \, |\nabla u| \, \|_{L^{p(x)}(\Omega)}$. We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the above norm. The space $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

Proposition 2.2 ([14]).

(1) *If $r \in C_+(\overline{\Omega})$ and $r(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$$

is compact and continuous.

(2) *There is a constant $C > 0$ such that*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)} \quad \textit{for all } u \in W_0^{1,p(x)}(\Omega).$$

3 Main result

From now on, we consider $m(x) = \max(p_1(x), p_2(x))$ for all $x \in \overline{\Omega}$. By (2) of Proposition 2.2, we know that $\|u\| := \|\nabla u\|_{L^{m(x)}(\Omega)}$ and $\|u\|_{W^{1,m(x)}(\Omega)}$ are equivalent norms in $W_0^{1,m(x)}(\Omega)$. In the following, we will use $\|u\|$ instead of $\|u\|_{W^{1,m(x)}(\Omega)}$ on $W_0^{1,m(x)}(\Omega)$.

Definition 3.1. We say that $u \in W_0^{1,m(x)}(\Omega)$ is a weak solution of problem (P_λ) if

$$\begin{aligned}\int_\Omega \left(|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2} \right) \nabla u \nabla v dx - \int_\Omega a(x)|u|^{m(x)-2} uv dx \\ = \lambda \int_\Omega |u|^{q(x)-2} uv dx + \int_\Omega f(x,u) v dx,\end{aligned}$$

for all $v \in W_0^{1,m(x)}(\Omega)$.

Our main result is the following theorem.

Theorem 3.2. *Assume that (a_1)–(a_2), (f_1)–(f_2) and (mq_1)–(mq_2) hold. Then, there exits a sequence $\{\lambda_k\} \subset (0, +\infty)$ with $\lambda_k > \lambda_{k+1}$, such that for any $\lambda \in (\lambda_{k+1}, \lambda_k]$, problem ($P_\lambda$) has at least k pairs of nontrivial solutions.*

4 Proof of main result

We will start by recalling an important abstract theorem involving genus theory, which will be used in the proof of Theorem 3.2.

Theorem 4.1 ([22]). *Let E be an infinite dimensional Banach space with $E = V \bigoplus X$, where V is finite dimensional and let $I \in C^1(E, \mathbb{R})$ be a even function with $I(0) = 0$ and satisfying*

(i) There are constants $\beta, \varrho > 0$ such that $I(u) \geq \beta$ for all $u \in \partial B_\varrho \cap X$;

(ii) There is $\tau > 0$ such that I satisfies the $(PS)_c$ condition, for $0 < c < \tau$;

(iii) For each finite dimensional subspace $\widetilde{E} \subset E$, there is $R = R(\widetilde{E}) > 0$ such that $I(u) \leq 0$ for all $u \in \widetilde{E} \backslash B_R(0)$.

Suppose that V is k dimensional and $V = \operatorname{span}\{e_1, \dots, e_k\}$. For $n \geq k$, inductively choose $e_{n+1} \notin E_n := \operatorname{span}\{e_1, \dots, e_n\}$. Let $R_n = R(E_n)$ and $D_n = B_{R_n} \cap E_n$. Define

$$G_n = \{h \in C(D_n, E) : h \text{ is odd and } h(u) = u,\ \forall u \in \partial B_{R_n} \cap E_n\}$$

and

$$\Gamma_j = \left\{h\left(\overline{D_n \backslash Y}\right) : h \in G_n,\ n \geq j,\ Y \in \Sigma, \text{ and } \gamma(Y) \leq n - j\right\},$$

where

$$\Sigma = \{Y \subset E \backslash \{0\} : Y \text{ is closed in } E \text{ and } Y = -Y\}$$

and $\gamma(Y)$ is the genus of $Y \in \Sigma$. For each $j \in \mathbb{N}$, let

$$c_j = \inf_{K \in \Gamma_j} \max_{u \in K} I(u).$$

Then $0 < \beta \leq c_j \leq c_{j+1}$ for $j > k$, and if $j > k$ and $c_j < \tau$, we have that c_j is the critical value of I. Moreover, if $c_j = c_{j+1} = \cdots = c_{j+l} = c < \tau$ for $j > k$, then $\gamma(K_c) \geq l + 1$, where

$$K_c = \{u \in E : I(u) = c \text{ and } I'(u) = 0\}.$$

In the sequel, we derive some results related to the above theorem and the Palais–Smale compactness condition.

Since we will rely on the critical point theory, we define the energy functional corresponding to problem (P_λ) as $I_\lambda : W_0^{1,m(x)}(\Omega) \mapsto \mathbb{R}$,

$$I_\lambda(u) = \int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)}\right) dx - \int_\Omega a(x) \frac{|u|^{m(x)}}{m(x)} - \lambda \int_\Omega \frac{|u|^{q(x)}}{q(x)} dx - \int_\Omega F(x, u) dx.$$

Clearly, I_λ is C^1 functional and the critical points of it are weak solutions of problem (P_λ).

Lemma 4.2. *Assume that (a_1), (f_1) and (mq_1) hold. Then for each $\lambda > 0$, I_λ satisfies condition (i) given in Theorem 4.1.*

Proof. Let $\delta > 0$. By (a_1), we have

$$\begin{aligned}
&\int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} - a(x)\frac{|u|^{m(x)}}{m(x)} \right) dx \\
&= \frac{1}{1+\delta} \int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} - a(x)\frac{|u|^{m(x)}}{m(x)} \right) dx \\
&\quad + \frac{\delta}{1+\delta} \int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} - a(x)\frac{|u|^{m(x)}}{m(x)} \right) dx \\
&= \frac{1}{1+\delta} \int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} - a(x)\frac{|u|^{m(x)}}{m(x)} \right) dx \\
&\quad + \frac{\delta}{1+\delta} \int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} \right) dx - \frac{\delta}{1+\delta} \int_\Omega a(x)\frac{|u|^{m(x)}}{m(x)} dx \\
&\geq \frac{\alpha}{1+\delta} \int_\Omega \frac{|u|^{m(x)}}{m(x)} dx + \frac{\delta}{1+\delta} \int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} \right) dx - \frac{\delta \|a\|_\infty}{(1+\delta)m^-} \int_\Omega |u|^{m(x)} dx \\
&\geq \frac{\delta}{1+\delta} \int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} \right) dx + \frac{1}{1+\delta} \left(\frac{\alpha}{m^+} - \frac{\delta \|a\|_\infty}{m^-} \right) \int_\Omega |u|^{m(x)} dx.
\end{aligned}$$

We can choose $\delta > 0$ such that $C_0 := \frac{1}{1+\delta}\left(\frac{\alpha}{m^+} - \frac{\delta\|a\|_\infty}{m^-}\right) > 0$. So

$$\begin{aligned}
\int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} - a(x)\frac{|u|^{m(x)}}{m(x)} \right) dx \geq \frac{\delta}{1+\delta} \int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} \right) dx \\
+ C_0 \int_\Omega |u|^{m(x)} dx.
\end{aligned} \tag{4.1}$$

By the seconde part of (f_1), there is $\eta_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon |t|^{q(x)-1} \text{ for all } |t| \geq \eta_\varepsilon \text{ and for all } x \in \overline{\Omega}.$$

Thanks to the continuity of f, there is $A_\varepsilon > 0$, such that

$$|f(x,t)| \leq A_\varepsilon \text{ for all } |t| \leq \eta_\varepsilon \text{ and for all } x \in \overline{\Omega}.$$

Therefore

$$|f(x,t)| \leq \varepsilon |t|^{q(x)-1} + A_\varepsilon \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{4.2}$$

On the other hand, by the first part of (f_1), for each $\varepsilon > 0$ there exists $0 < \delta_\varepsilon < 1$ such that

$$|f(x,t)| \leq \varepsilon |t|^{m(x)-1} \quad \text{for all } |t| \leq \delta_\varepsilon \text{ and for all } x \in \overline{\Omega}. \tag{4.3}$$

For $|t| \geq \delta_\varepsilon$, it follows from (4.2) that

$$\frac{|f(x,t)|}{|t|^{q(x)-1}} \leq \varepsilon + \frac{A_\varepsilon}{\delta_\varepsilon^{q(x)-1}} \leq \varepsilon + \frac{A_\varepsilon}{\delta_\varepsilon^{q^+-1}} = C_\varepsilon,$$

that is

$$|f(x,t)| \leq C_\varepsilon |t|^{q(x)-1} \quad \text{for all } |t| \geq \delta_\varepsilon \text{ and for all } x \in \overline{\Omega}. \tag{4.4}$$

Combining (4.3)–(4.4), we obtain

$$|f(x,t)| \leq \varepsilon |t|^{m(x)-1} + C_\varepsilon |t|^{q(x)-1} \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

By integrating this last inequality, we get

$$|F(x,t)| \leq \frac{\varepsilon}{m(x)} |t|^{m(x)} + \frac{C_\varepsilon}{q(x)} |t|^{q(x)} \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{4.5}$$

Therefore

$$\begin{aligned} I_\lambda(u) &\geq \frac{\delta}{1+\delta} \int_\Omega \left(\frac{|\nabla u|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u|^{p_2(x)}}{p_2(x)} \right) dx + \left(C_0 - \frac{\varepsilon}{m^-} \right) \int_\Omega |u|^{m(x)} dx - \frac{\lambda + C_\varepsilon}{q^-} \int_\Omega |u|^{q(x)} dx \\ &\geq \frac{\delta}{(1+\delta)\max(p_1^+, p_2^+)} \int_\Omega \left(|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)} \right) dx + \left(C_0 - \frac{\varepsilon}{m^-} \right) \int_\Omega |u|^{m(x)} dx \\ &\quad - \frac{\lambda + C_\varepsilon}{q^-} \int_\Omega |u|^{q(x)} dx. \end{aligned}$$

Hence for ε sufficiently small,

$$I_\lambda(u) \geq \frac{\delta}{(1+\delta)\max(p_1^+, p_2^+)} \int_\Omega \left(|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)} \right) dx - \frac{\lambda + C_\varepsilon}{q^-} \int_\Omega |u|^{q(x)} dx.$$

Using the fact that

$$|\nabla u|^{p_1(x)} + |\nabla u(x)|^{p_2(x)} \geq |\nabla u(x)|^{m(x)} \text{ for all } x \in \Omega, \tag{4.6}$$

it follows that

$$I_\lambda(u) \geq \frac{\delta}{(1+\delta)\max(p_1^+, p_2^+)} \int_\Omega |\nabla u|^{m(x)} dx - \frac{\lambda + C_\varepsilon}{q^-} \int_\Omega |u|^{q(x)} dx.$$

By the continuous embedding $W_0^{1,m(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, there exists $C_1 > 0$ such that

$$\|u\|_{L^{q(x)}(\Omega)} \leq C_1 \|u\|.$$

Consequently, by Proposition 2.1, for $\|u\| = \varrho$, with $0 < \varrho < 1$,

$$I_\lambda(u) \geq \frac{\delta}{(1+\delta)\max(p_1^+, p_2^+)} \|u\|^{m^+} - \frac{(\lambda + C_\varepsilon) C_2}{q^-} \|u\|^{q^-}.$$

Since $m^+ < q^-$, there exists $\beta > 0$ such that $I_\lambda(u) \geq \beta$ for $\|u\| = \varrho$, where ϱ is chosen sufficiently small. □

Lemma 4.3. *Assume that (a_1), (f_1) and (mq_1) hold. Then I_λ satisfies condition (iii) given in Theorem 4.1.*

Proof. Let E to be a finite dimensional subspace of $W_0^{1,m(x)}(\Omega)$. By (4.2),

$$|f(x,t)| \leq \varepsilon |t|^{q(x)-1} + A_\varepsilon \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Integrating this inequality, we entail

$$\begin{aligned}\frac{|F(x,t)|}{|t|^{q(x)}} &\leq \frac{\varepsilon}{q(x)} + \frac{A_\varepsilon}{|t|^{q(x)-1}} \\ &\leq \frac{\varepsilon}{q^-} + \frac{A_\varepsilon}{|t|^{q^- -1}} \quad \text{as } |t| \to +\infty,\end{aligned}$$

hence for $\varepsilon > 0$, there is $\delta_\varepsilon > 0$, such that

$$|F(x,t)| \leq \varepsilon |t|^{q(x)} \quad \text{for all } |t| \geq \delta_\varepsilon \text{ and for all } x \in \overline{\Omega}. \tag{4.7}$$

By continuity, there is $M_\varepsilon > 0$ such that

$$|F(x,t)| \leq M_\varepsilon \quad \text{for all } |t| \leq \delta_\varepsilon \text{ and for all } x \in \overline{\Omega}. \tag{4.8}$$

This and (4.7) imply that

$$F(x,t) \geq -M_\varepsilon - \varepsilon |t|^{q(x)} \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{4.9}$$

Thus

$$\begin{aligned}I_\lambda(u) \leq{}& \frac{1}{\min(p_1^-,p_2^-)} \int_\Omega \left(|\nabla u|^{p_1(x)} dx + |\nabla u|^{p_2(x)} \right) dx + \frac{\|a\|_\infty}{m^-} \int_\Omega |u|^{m(x)} dx \\ &+ \left(\varepsilon - \frac{\lambda}{q^+} \right) \int_\Omega |u|^{q(x)} dx + M_\varepsilon |\Omega|.\end{aligned}$$

By choosing $\varepsilon = \frac{\lambda}{2q^+}$, we obtain

$$\begin{aligned}I_\lambda(u) \leq{}& \frac{1}{\min(p_1^-,p_2^-)} \int_\Omega \left(|\nabla u|^{p_1(x)} dx + |\nabla u|^{p_2(x)} \right) dx + \frac{\|a\|_\infty}{m^-} \int_\Omega |u|^{m(x)} dx \\ &- \frac{\lambda}{2q^+} \int_\Omega |u|^{q(x)} dx + M_\varepsilon |\Omega| \\ \leq{}& \frac{1}{\min(p_1^-,p_2^-)} \left(2|\Omega| + 2\int_\Omega |\nabla u|^{m(x)} dx + \|a\|_\infty \int_\Omega |u|^{m(x)} dx \right) \\ &- \frac{\lambda}{2q^+} \int_\Omega |u|^{q(x)} dx + M_\varepsilon |\Omega|.\end{aligned}$$

Since $\dim E < \infty$, the norms $\|\cdot\|$ and $\|\cdot\|_{L^{q(x)}(\Omega)}$ are equivalent in E. According to Proposition 2.1, for $\min\left(\|u\|, \|u\|_{L^{m(x)}(\Omega)}, \|u\|_{L^{q(x)}(\Omega)}\right) > 1$,

$$\begin{aligned}I_\lambda(u) \leq{}& \frac{1}{\min(p_1^-,p_2^-)} \left(2|\Omega| + 2\|u\|^{m+} + \|a\|_\infty \|u\|^{m+}_{L^{m(x)}(\Omega)} \right) - \frac{\lambda}{2q^+} \|u\|^{q^-}_{L^{q(x)}(\Omega} + M_\varepsilon |\Omega| \\ \leq{}& \frac{1}{\min(p_1^-,p_2^-)} \left(2|\Omega| + 2\|u\|^{m+} + C' \|a\|_\infty \|u\|^{m+} \right) - \frac{\lambda C_3}{2q^+} \|u\|^{q^-} + M_\varepsilon |\Omega|.\end{aligned}$$

Using the fact that $m^+ < q^-$, we conclude that $I_\lambda(u) < 0$ for $\|u\| \geq R > 1$, where R is chosen large enough. $\square$

4.1 Palais–Smale condition

Lemma 4.4. *Assume that (a_1), (f_1)–(f_2) and (mq_1) hold. Then any (PS) sequence of I_λ is bounded in $W_0^{1,m(x)}(\Omega)$.*

Proof. Let $\{u_n\} \subset W_0^{1,m(x)}(\Omega)$ such that $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$. By (f_2), for n large enough,

$$\begin{aligned} c+1+\|u_n\| &\geq I_\lambda(u_n) - \frac{1}{m^+}\langle I'_\lambda(u_n), u_n\rangle \\ &= \int_\Omega \left(\frac{1}{p_1(x)} - \frac{1}{m^+}\right)|\nabla u_n|^{p_1(x)}dx + \int_\Omega \left(\frac{1}{p_2(x)} - \frac{1}{m^+}\right)|\nabla u_n|^{p_2(x)}dx \\ &\quad + \int_\Omega \left(\frac{1}{m^+} - \frac{1}{m(x)}\right) a(x)|u_n|^{m(x)}dx + \lambda\int_\Omega \left(\frac{1}{m^+} - \frac{1}{q(x)}\right)|u_n|^{q(x)}dx \\ &\quad + \int_\Omega \left(\frac{1}{m^+} f(x,u_n)u_n - F(x,u_n)\right)dx \\ &\geq \int_\Omega \left(\frac{1}{p_1(x)} - \frac{1}{m^+}\right)|\nabla u_n|^{p_1(x)}dx + \int_\Omega \left(\frac{1}{p_2(x)} - \frac{1}{m^+}\right)|\nabla u_n|^{p_2(x)}dx \\ &\quad + \int_\Omega \left(\frac{1}{m^+} - \frac{1}{m(x)}\right) a(x)|u_n|^{m(x)}dx + \lambda\int_\Omega \left(\frac{1}{m^+} - \frac{1}{q(x)}\right)|u_n|^{q(x)}dx. \end{aligned}$$

Therefore

$$\begin{aligned} \lambda\left(\frac{1}{m^+} - \frac{1}{q^-}\right)\int_\Omega |u_n|^{q(x)}dx &\leq \lambda\int_\Omega \left(\frac{1}{m^+} - \frac{1}{q(x)}\right)|u_n|^{q(x)}dx \\ &\leq c+1+\|u_n\| + \int_\Omega \left(\frac{1}{m(x)} - \frac{1}{m^+}\right) a(x)|u_n|^{m(x)}dx \\ &\leq c+1+\|u_n\| + \|a\|_\infty \left(\frac{1}{m^-} - \frac{1}{m^+}\right)\int_\Omega |u_n|^{m(x)}dx. \end{aligned}$$

On the other hand, by (mq_1), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|t|^{m(x)} \leq \varepsilon|t|^{q(x)} + C_\varepsilon \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

It follows that

$$\begin{aligned} \lambda\left(\frac{1}{m^+} - \frac{1}{q^-}\right)\int_\Omega |u_n|^{q(x)}dx &\leq c+1+\|u_n\| + \varepsilon\|a\|_\infty \left(\frac{1}{m^-} - \frac{1}{m^+}\right)\int_\Omega |u_n|^{q(x)}dx \\ &\quad + \|a\|_\infty \left(\frac{1}{m^-} - \frac{1}{m^+}\right) C_\varepsilon|\Omega|, \end{aligned}$$

hence

$$\begin{aligned} &\left(\lambda\left(\frac{1}{m^+} - \frac{1}{q^-}\right) - \varepsilon\|a\|_\infty\left(\frac{1}{m^-} - \frac{1}{m^+}\right)\right)\int_\Omega |u_n|^{q(x)}dx \\ &\qquad \leq c+1+\|u_n\| + \|a\|_\infty\left(\frac{1}{m^-} - \frac{1}{m^+}\right) C_\varepsilon|\Omega|. \end{aligned}$$

Choosing $\varepsilon = \frac{\lambda}{2\|a\|_\infty}\left(\frac{1}{m^+} - \frac{1}{q^-}\right) / \left(\frac{1}{m^-} - \frac{1}{m^+}\right)$, we obtain

$$\frac{\lambda}{2}\left(\frac{1}{m^+} - \frac{1}{q^-}\right)\int_\Omega |u_n|^{q(x)}dx \leq c+1+\|u_n\| + \|a\|_\infty\left(\frac{1}{m^-} - \frac{1}{m^+}\right) C_\varepsilon|\Omega|.$$

Thus,

$$\int_\Omega |u_n|^{q(x)}dx \le C_4(1+\|u_n\|). \tag{4.10}$$

By (4.7) and (4.8), for $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x,t)| \le \varepsilon |t|^{q(x)} + C_\varepsilon \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{4.11}$$

(4.1), (4.6), (4.10) and (4.11) imply that for n large enough

$$\begin{aligned}\frac{\delta}{(1+\delta)\max(p_1^+,p_2^+)}\int_\Omega |\nabla u_n|^{m(x)}dx &\le \frac{\delta}{1+\delta}\int_\Omega \left(\frac{|\nabla u_n|^{p_1(x)}}{p_1(x)} + \frac{|\nabla u_n|^{p_2(x)}}{p_2(x)}\right)dx \\ &\le I_\lambda(u_n) + \frac{\lambda}{q^-}\int_\Omega |u_n|^{q(x)}dx + \int_\Omega F(x,u_n)dx \\ &\le C_5 + \left(\frac{\lambda}{q^-}+\varepsilon\right)\int_\Omega |u_n|^{q(x)}dx + C_\varepsilon|\Omega| \\ &\le \left(\frac{\lambda}{q^-}+\varepsilon\right)C_4(1+\|u_n\|) + C_5 + C_\varepsilon|\Omega|.\end{aligned}$$

Hence

$$\int_\Omega |\nabla u_n|^{m(x)}dx \le C_6(1+\|u_n\|)$$

and so

$$\min\left(\|u_n\|^{m-}, \|u_n\|^{m+}\right) \le C(1+\|u_n\|).$$

Consequently $\{u_n\}$ is bounded in $W_0^{1,m(x)}(\Omega)$. □

In view of Lemma 4.4, $\{u_n\}$ is bounded in $W_0^{1,m(x)}(\Omega)$. So, up to subsequence, we may assume that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W_0^{1,m(x)}(\Omega), \\ u_n &\rightharpoonup u \quad \text{in } L^{q(x)}(\Omega), \\ u_n &\to u \quad \text{in } L^{r(x)}(\Omega),\ r \in C_+(\overline{\Omega}),\ r(x) < m^*(x)\ \forall x \in \overline{\Omega}. \end{aligned}$$

Taking in to account (mq_2), from the concentration compactness lemma of Lions [8], there exist tow nonnegative measures $\mu, \nu \in \mathcal{M}(\Omega)$, a countable set $\mathcal{J}$, points $\{x_j\}_{j\in\mathcal{J}}$ in Ω and sequences $\{\mu_j\}_{j\in\mathcal{J}}, \{\nu_j\}_{j\in\mathcal{J}} \subset [0,+\infty)$, such that

$$\begin{gathered} |\nabla u_n|^{m(x)} \rightharpoonup \mu \ge |\nabla u|^{m(x)} + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j} \quad \text{in } \Omega \\ |u_n|^{q(x)} \rightharpoonup \nu \ge |u|^{q(x)} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} \quad \text{in } \Omega \\ S\nu_j^{\frac{1}{m^*(x_j)}} \le \mu_j^{\frac{1}{m(x_j)}} \quad \text{for all } j \in \mathcal{J}, \end{gathered} \tag{4.12}$$

where

$$S = \inf_{\phi\in C_0^\infty(\Omega)} \frac{\|\nabla\phi\|_{L^{m(x)}(\Omega)}}{\|\phi\|_{L^{q(x)}(\Omega)}}.$$

Let $\phi \in C_0^\infty\left(\mathbb{R}^N\right)$ such that

$$0 \le \phi \le 1, \qquad \phi \equiv 1 \quad \text{in } B_1(0), \qquad \phi = 0 \quad \text{in } \mathbb{R}^N \backslash B_2(0).$$

For $\varepsilon > 0$ and $j \in \mathcal{J}$ denote

$$\phi_\varepsilon^j(x) = \phi\left(\frac{x - x_j}{\varepsilon}\right), \quad \text{for all } x \in \mathbb{R}^N.$$

We claim that

$$\begin{aligned}&\int_\Omega \left(|u\nabla\phi_\varepsilon^j|^{p_1(x)} + |u\nabla\phi_\varepsilon^j|^{p_2(x)}\right) dx \\ &\qquad \le C\left(|B_{2\varepsilon}(x_j)| + \|u\|^{m^+}_{L^{m^*(x)}(B_{2\varepsilon}(x_j))} + \|u\|^{m^-}_{L^{m^*(x)}(B_{2\varepsilon}(x_j))}\right),\end{aligned} \tag{4.13}$$

for all $u \in W_0^{1,m(x)}(\Omega)$, where $C > 0$ is independent of ε and $j \in \mathcal{J}$. Indeed,

$$\begin{aligned}&\int_\Omega \left(|u\nabla\phi_\varepsilon^j|^{p_1(x)} + |u\nabla\phi_\varepsilon^j|^{p_2(x)}\right) dx \\ &\le 2|B_{2\varepsilon}(x_j)| + 2\int_{|u\nabla\phi_\varepsilon^j|\ge 1} \left(|u\nabla\phi_\varepsilon^j|^{m(x)}\right) dx \\ &\le 2|B_{2\varepsilon}(x_j)| + 2\int_\Omega \left(|u\nabla\phi_\varepsilon^j|^{m(x)}\right) dx \\ &\le C_7\left(|B_{2\varepsilon}(x_j)| + \int_\Omega |u\nabla\phi_\varepsilon^j|^{m(x)} dx\right) \\ &= C_7\left(|B_{2\varepsilon}(x_j)| + \int_{B_{2\varepsilon}(x_j)} |u|^{m(x)} \left|\frac{1}{\varepsilon}\nabla\phi\left(\frac{x - x_j}{\varepsilon}\right)\right|^{m(x)} dx\right).\end{aligned} \tag{4.14}$$

Using Hölder's inequality,

$$\begin{aligned}&\int_{B_{2\varepsilon}(x_j)} |u|^{m(x)} \left|\frac{1}{\varepsilon}\nabla\phi\left(\frac{x - x_j}{\varepsilon}\right)\right|^{m(x)} dx \\ &\qquad \le C_8 \left\| |u|^{m(x)} \right\|_{L^{\frac{m^*(x)}{m(x)}}(B_{2\varepsilon}(x_j))} \left\| \left|\frac{1}{\varepsilon}\nabla\phi\left(\frac{x - x_j}{\varepsilon}\right)\right|^{m(x)} \right\|_{L^{\frac{m^*(x)}{m^*(x)-m(x)}}(B_{2\varepsilon}(x_j))}.\end{aligned} \tag{4.15}$$

Furthermore,

$$\begin{aligned}\int_{B_{2\varepsilon}(x_j)} \left|\frac{1}{\varepsilon}\nabla\phi\left(\frac{x - x_j}{\varepsilon}\right)\right|^{\frac{m(x)m^*(x)}{m^*(x)-m(x)}} dx &= \int_{B_{2\varepsilon}(x_j)} \left|\frac{1}{\varepsilon}\nabla\phi\left(\frac{x - x_j}{\varepsilon}\right)\right|^N dx \\ &= \int_{B_2(0)} |\nabla\phi(y)|^N dy = C_9.\end{aligned} \tag{4.16}$$

Gathering (4.14)–(4.16) and taking into account Proposition 2.1, we deduce

$$\begin{aligned}\int_\Omega \left(|u\nabla\phi_\varepsilon^j|^{p_1(x)} + |u\nabla\phi_\varepsilon^j|^{p_2(x)}\right) dx &\le C_7\left(|B_{2\varepsilon}(x_j)| + C_{10}\left\||u|^{m(x)}\right\|_{L^{\frac{m^*(x)}{m(x)}}(B_{2\varepsilon}(x_j))}\right) \\ &\le C\left(|B_{2\varepsilon}(x_j)| + \|u\|^{m^+}_{L^{m^*(x)}(B_{2\varepsilon}(x_j))} + \|u\|^{m^-}_{L^{m^*(x)}(B_{2\varepsilon}(x_j))}\right),\end{aligned}$$

and the claim follows.

Lemma 4.5. *Assume that (a_1), (f_1)–(f_2) and mq_1 hold. If $\{u_n\}$ is a (PS) sequence of I_λ, then for each $j \in \mathcal{J}$,*

$$\nu_j = 0 \quad \text{or} \quad \nu_j \ge \frac{S^N}{\lambda^{\frac{N}{m(x_j)}}}.$$

Proof. Let ϕ_ε^j as above. By Lemma 4.4, we see that for each $j \in \mathcal{J}$, $\{u_n\phi_\varepsilon^j\}$ is bounded in $W_0^{1,m(x)}(\Omega)$. Since $I'_\lambda(u_n) \to 0$, $\langle I'_\lambda(u_n), u_n\phi_\varepsilon^j\rangle = o_n(1)$. From (4.6),

$$\begin{aligned}\int_\Omega |\nabla u_n|^{m(x)}\phi_\varepsilon^j dx &\le \int_\Omega \left(|\nabla u_n|^{p_1(x)} + |\nabla u_n|^{p_2(x)}\right)\phi_\varepsilon^j dx\\ &= -\int_\Omega |\nabla u_n|^{p_1(x)-2}u_n\nabla u_n\nabla\phi_\varepsilon^j dx - \int_\Omega |\nabla u_n|^{p_2(x)-2}u_n\nabla u_n\nabla\phi_\varepsilon^j dx\\ &\quad + \lambda\int_\Omega |u_n|^{q(x)}\phi_\varepsilon^j dx + \int_\Omega a(x)|u_n|^{m(x)}\phi_\varepsilon^j dx\\ &\quad + \int_\Omega f(x,u_n)u_n\phi_\varepsilon^j dx + o_n(1). \end{aligned} \tag{4.17}$$

For any $\tau > 0$, by Young's inequality, there exist $C_\tau, C'_\tau > 0$ such that

$$\begin{aligned}\int_\Omega \left||\nabla u_n|^{p_1(x)-2}u_n\nabla u_n\nabla\phi_\varepsilon^j\right| dx &\le \tau\int_\Omega |\nabla u_n|^{p_1(x)}dx + C_\tau\int_\Omega |u_n\nabla\phi_\varepsilon^j|^{p_1(x)}dx,\\ &\le \tau\left(|B_{2\varepsilon}(x_j)| + \int_\Omega |\nabla u_n|^{m(x)}dx\right) + C_\tau\int_\Omega |u_n\nabla\phi_\varepsilon^j|^{p_1(x)}dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \left||\nabla u_n|^{p_2(x)-2}u_n\nabla u_n\nabla\phi_\varepsilon^j\right| dx &\le \tau\int_\Omega |\nabla u_n|^{p_2(x)}dx + C'_\tau\int_\Omega |u_n\nabla\phi_\varepsilon^j|^{p_2(x)}dx\\ &\le \tau\left(|B_{2\varepsilon}(x_j)| + \int_\Omega |\nabla u_n|^{m(x)}dx\right) + C'_\tau\int_\Omega |u_n\nabla\phi_\varepsilon^j|^{p_2(x)}dx.\end{aligned}$$

Using the fact $\{u_n\}$ is bounded in $W_0^{1,m(x)}(\Omega)$, it follows that

$$\limsup_{n\to+\infty}\int_\Omega \left||\nabla u_n|^{p_1(x)-2}u_n\nabla u_n\nabla\phi_\varepsilon^j\right| dx \le \tau C_{11} + C_\tau\int_\Omega |u\nabla\phi_\varepsilon^j|^{p_1(x)}dx$$
$$\limsup_{n\to+\infty}\int_\Omega \left||\nabla u_n|^{p_2(x)-2}u_n\nabla u_n\nabla\phi_\varepsilon^j\right| dx \le \tau C_{11} + C'_\tau\int_\Omega |u\nabla\phi_\varepsilon^j|^{p_2(x)}dx.$$

Now, bearing in mind (4.13), we obtain

$$\begin{aligned}&\limsup_{n\to+\infty}\left[\int_\Omega \left||\nabla u_n|^{p_1(x)-2}u_n\nabla u_n\nabla\phi_\varepsilon^j\right| dx + \int_\Omega \left||\nabla u_n|^{p_2(x)-2}u_n\nabla u_n\nabla\phi_\varepsilon^j\right| dx\right]\\ &\le \tau C_{12} + \max(C_\tau, C'_\tau)\left(\int_\Omega |u\nabla\phi_\varepsilon^j|^{p_1(x)}dx + \int_\Omega |u\nabla\phi_\varepsilon^j|^{p_2(x)}dx\right)\\ &\le \tau C_{12} + \max(C_\tau, C'_\tau)C\left(|B_{2\varepsilon}(x_j)| + \|u\|^{m^+}_{L^{m^*(x)}(B_{2\varepsilon}(x_j))} + \|u\|^{m^-}_{L^{m^*(x)}(B_{2\varepsilon}(x_j))}\right)\end{aligned} \tag{4.18}$$

On the other hand, by the compactness lemma of Strauss [10] and Sobolev embedding,

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n)u_n\phi_\varepsilon^j dx = \int_\Omega f(x,u)u\phi_\varepsilon^j dx \tag{4.19}$$

and

$$\lim_{n\to+\infty}\int_\Omega a(x)|u_n|^{m(x)}\phi_\varepsilon^j dx = \int_\Omega a(x)|u|^{m(x)}\phi_\varepsilon^j dx. \tag{4.20}$$

From (4.12) and (4.17)–(4.20),

$$\begin{aligned}\int_\Omega \phi_\varepsilon^j d\mu &\le \tau C_{12} + \max(C_\tau, C'_\tau)C\left(|B_{2\varepsilon}(x_j)| + \|u\|^{m^+}_{L^{m^*(x)}(B_{2\varepsilon}(x_j))} + \|u\|^{m^-}_{L^{m^*(x)}(B_{2\varepsilon}(x_j))}\right)\\ &\quad + \lambda\int_\Omega \phi_\varepsilon^j d\nu - \int_{B_{2\varepsilon}(x_j)} a(x)|u|^{m(x)}\phi_\varepsilon^j dx\\ &\quad + \int_{B_{2\varepsilon}(x_j)} f(x,u)u\phi_\varepsilon^j dx.\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\mu_j \leq \tau C_{12} + \lambda \nu_j,$$

Now, letting $\tau \to 0$, we derive $\mu_j \leq \lambda \nu_j$. Therefore

$$S \nu_j^{\frac{1}{m^*(x_j)}} \leq \mu_j^{\frac{1}{m(x_j)}} \leq (\lambda \nu_j)^{\frac{1}{m(x_j)}}$$

and hence the result follows. □

Lemma 4.6. *Assume that* (a_1)–(a_2), (f_1)–(f_2) *and* (mq_1) *hold. If* $\lambda < 1$, *then* I_λ *satisfies* $(PS)_c$ *condition for* $c < \lambda^{1-\frac{N}{m^+}} (\frac{1}{m^+} - \frac{1}{q^-}) S^N$.

Proof. Let $\{u_n\} \subset W_0^{1,m(x)}(\Omega)$ such that

$$I_\lambda(u_n) \to c \quad \text{and} \quad I'_\lambda(u_n) \to 0.$$

Then

$$\begin{aligned}
c &= I_\lambda(u_n) - \frac{1}{m^+} \langle I'_\lambda(u_n), u_n \rangle + o_n(1) \\
&= \int_\Omega \left(\frac{1}{p_1(x)} - \frac{1}{m^+} \right) |\nabla u_n|^{p_1(x)} dx + \int_\Omega \left(\frac{1}{p_2(x)} - \frac{1}{m^+} \right) |\nabla u_n|^{p_2(x)} dx \\
&\quad + \int_\Omega \left(\frac{1}{m^+} - \frac{1}{m(x)} \right) a(x) |u_n|^{m(x)} dx + \lambda \int_\Omega \left(\frac{1}{m^+} - \frac{1}{q(x)} \right) |u_n|^{q(x)} dx \\
&\quad + \int_\Omega \left(\frac{1}{m^+} f(x, u_n) u_n - F(x, u_n) \right) dx + o_n(1).
\end{aligned}$$

By (a_2) and (f_2), we obtain

$$c \geq \lambda \int_\Omega \left(\frac{1}{m^+} - \frac{1}{q^-} \right) |u_n|^{q(x)} dx + o_n(1).$$

Using (4.12), it follows that

$$\begin{aligned}
c &\geq \lambda \lim_{n \to +\infty} \int_\Omega \left(\frac{1}{m^+} - \frac{1}{q^-} \right) |u_n|^{q(x)} dx \\
&\geq \lambda \left(\frac{1}{m^+} - \frac{1}{q^-} \right) \left(\int_\Omega |u|^{q(x)} dx + \sum_{j \in \mathcal{J}} \nu_j \right) \\
&\geq \lambda \left(\frac{1}{m^+} - \frac{1}{q^-} \right) \nu_j \quad \text{for all } j \in \mathcal{J}.
\end{aligned}$$

If $\nu_j > 0$ for some $j \in \mathcal{J}$, by Lemma 4.5, we get

$$c \geq \lambda \left(\frac{1}{m^+} - \frac{1}{q^-} \right) \frac{S^N}{\lambda^{\frac{N}{m(x_j)}}}.$$

Since $\lambda < 1$,

$$c \geq \lambda \left(\frac{1}{m^+} - \frac{1}{q^-} \right) \frac{S^N}{\lambda^{\frac{N}{m^+}}} = \lambda^{1-\frac{N}{m^+}} \left(\frac{1}{m^+} - \frac{1}{q^-} \right) S^N,$$

which is impossible, and so $\nu_j = 0$ for all $j \in \mathcal{J}$. Hence

$$\lim_{n\to+\infty} \int_\Omega |u_n|^{q(x)}dx = \int_\Omega |u|^{q(x)}dx.$$

This implies $\lim_{n\to+\infty} \int_\Omega |u_n - u|^{q(x)}dx = 0$, thanks to Proposition 2.1, we deduce

$$u_n \to u \text{ in } L^{q(x)}(\Omega).$$

By standard arguments, we see that

$$\lim_{n\to+\infty} \int_\Omega \left(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right)(u_n - u)dx = 0,$$
$$\lim_{n\to+\infty} \int_\Omega \left(|u_n|^{m(x)-2}u_n - |u|^{m(x)-2}u\right)(u_n - u)dx = 0$$

and

$$\lim_{n\to+\infty} \int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx = 0.$$

On the other hand, we have

$$\begin{aligned}
o_n(1) &= \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \\
&= \int_\Omega \left(|\nabla u_n|^{p_1(x)-2}\nabla u_n - |\nabla u|^{p_1(x)-2}\nabla u\right)\nabla(u_n - u)dx \\
&\quad + \int_\Omega \left(|\nabla u_n|^{p_1(x)-2}\nabla u_n - |\nabla u|^{p_2(x)-2}\nabla u\right)\nabla(u_n - u)dx \\
&\quad - \int_\Omega a(x)\left(|u_n|^{m(x)-2}u_n - |u|^{m(x)-2}u\right)(u_n - u)dx \\
&\quad - \lambda\int_\Omega \left(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right)(u_n - u)dx - \int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{n\to+\infty} \Bigg[\int_\Omega &\left(|\nabla u_n|^{p_1(x)-2}\nabla u_n - |\nabla u|^{p_1(x)-2}\nabla u\right)\nabla(u_n - u)dx \\
&+ \int_\Omega \left(|\nabla u_n|^{p_1(x)-2}\nabla u_n - |\nabla u|^{p_2(x)-2}\nabla u\right)\nabla(u_n - u)dx\Bigg] = 0.
\end{aligned}$$

Since $(|\nabla u_n|^{p_i(x)-2}\nabla u_n - |\nabla u|^{p_i(x)-2}\nabla u)\nabla(u_n - u) \geq 0$ in Ω, $i = 1,2$,

$$\lim_{n\to+\infty} \int_\Omega \left(|\nabla u_n|^{p_i(x)-2}\nabla u_n - |\nabla u|^{p_i(x)-2}\nabla u\right)\nabla(u_n - u)dx = 0, \qquad i = 1,2. \tag{4.21}$$

Recalling the following well known inequality in $\mathbb{R}^N$,

$$\left[(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta)\right]^{\frac{p}{2}}(|\xi|^p + |\eta|^p)^{\frac{2-p}{2}} \geq (p-1)|\xi - \eta|^p \quad \text{if } 1 < p < 2, \tag{4.22}$$
$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq 2^{-p}|\xi - \eta|^p \quad \text{if } p \geq 2. \tag{4.23}$$

Divide Ω in two parts as follows:

$$\Omega_i^+ = \{x \in \Omega : p_i(x) \geq 2\} \quad \text{and} \quad \Omega_i^- = \{x \in \Omega : p_i(x) < 2\}, \qquad i = 1,2.$$

By (4.21) and (4.23), it yields

$$\lim_{n\to+\infty} \int_{\Omega_i^+} |\nabla(u_n - u)|^{p_i(x)}dx = 0, \qquad i = 1,2. \tag{4.24}$$

On the other hand, by (4.22) and Hölder's inequality,

$$\begin{aligned}\int_{\Omega_i^-} |\nabla(u_n-u)|^{p_i(x)}dx &\le \frac{1}{p_i^- -1}\int_{\Omega_i^-}(p_i(x)-1)|\nabla u_n - u|^{p_i(x)}dx \\ &\le \frac{1}{p_i^- -1}\int_{\Omega_i^-}\left[\left(|\nabla u_n|^{p_i(x)-2}\nabla u_n - |\nabla u|^{p_i(x)-2}\nabla u\right)\nabla(u_n-u)\right]^{\frac{p_i(x)}{2}} \\ &\quad\times\left(|\nabla u_n|^{p_i(x)}+|\nabla u|^{p_i(x)}\right)^{\frac{2-p_i(x)}{2}}dx \\ &\le C_{p_i}\left\|\left[\left(|\nabla u_n|^{p_i(x)-2}\nabla u_n - |\nabla u|^{p_i(x)-2}\nabla u\right)\nabla(u_n-u)\right]^{\frac{p_i(x)}{2}}\right\|_{L^{\frac{2}{p_i(x)}}(\Omega_i^-)} \\ &\quad\times\left\|\left(|\nabla u_n|^{p_i(x)}+|\nabla u|^{p_i(x)}\right)^{\frac{2-p_i(x)}{2}}\right\|_{L^{\frac{2}{2-p_i(x)}}(\Omega_i^-)}.\end{aligned}$$

Since $\{u_n\}$ is bounded in $W_0^{1,p_i(x)}(\Omega)$, Proposition 2.1 implies

$$\int_{\Omega_i^-}|\nabla(u_n-u)|^{p_i(x)}dx \le C'_{p_i}\left\|\left[\left(|\nabla u_n|^{p_i(x)-2}\nabla u_n - |\nabla u|^{p_i(x)-2}\nabla u\right)\nabla(u_n-u)\right]^{\frac{p_i(x)}{2}}\right\|_{L^{\frac{2}{p_i(x)}}(\Omega_i^-)}.$$

Using again Proposition 2.1 and (4.21), it follows that

$$\lim_{n\to+\infty}\int_{\Omega_i^-}|\nabla(u_n-u)|^{p_i(x)}dx = 0, \qquad i=1,2. \tag{4.25}$$

(4.24) and (4.25) imply

$$\lim_{n\to+\infty}\int_{\Omega}\left(|\nabla(u_n-u)|^{p_1(x)}+|\nabla(u_n-u)|^{p_2(x)}\right)dx = 0.$$

Using inequality (4.6), we get

$$\int_{\Omega}|\nabla(u_n-u)|^{m(x)}dx = 0,$$

and hence $u_n \to u$ in $W_0^{1,m(x)}(\Omega)$. □

Lemma 4.7. *Under assumptions of Theorem 3.2, there exists a sequence $\{M_n\}\subset(0,+\infty)$ independent of λ, with $M_n \le M_{n+1}$, such that for any $\lambda>0$,*

$$c_n^\lambda = \inf_{K\in\Gamma_n}\max_{u\in K} I_\lambda(u) < M_n.$$

Proof. The proof is similar to [26, Lemma 5] and so, we will omit it. □

4.2 Proof of Theorem 3.2

By choosing for each $k\ge1$, λ_k sufficiently small, we construct a sequence (λ_k), with $\lambda_k>\lambda_{k+1}$ such that $M_k < \lambda_k^{1-\frac{N}{m^+}}\left(\frac{1}{m^+}-\frac{1}{q^-}\right)S^N$. Therefore, for $\lambda\in(\lambda_{k+1},\lambda_k]$,

$$0 < c_1^\lambda \le c_2^\lambda \le \cdots \le c_k^\lambda < M_k < \lambda^{1-\frac{N}{m^+}}\left(\frac{1}{m^+}-\frac{1}{q^-}\right)S^N.$$

Thanks to Theorem 4.1, the levels $c_1^\lambda \le c_2^\lambda \le \cdots \le c_k^\lambda$ are critical values of I_λ. So, if

$$c_1^\lambda < c_2^\lambda < \cdots < c_k^\lambda,$$

I_λ has at least k critical points. Now, if $c_j^\lambda = c_{j+1}^\lambda$ for some $j = 1, \dots, k-1$, again Theorem 4.1 implies that $K_{c_j^\lambda}$ is an infinite set [22, Chap. 7] and hence in this case, problem (P_λ) has infinitely many solutions. Conclusion, problem (P_λ) has at least k pair solutions.

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