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Multiple nontrivial solutions for a nonhomogeneous Schrödinger–Poisson system in $\mathbb{R}^3$

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Abstract. In this paper, we study the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) + g(x), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3. \end{cases}$$

Under appropriate assumptions on V, f and g, using the Mountain Pass Theorem and the Ekeland's variational principle, we establish two existence theorems to ensure that the above system has at least two different solutions. Recent results from the literature are extended and improved.

Keywords: Schrödinger–Poisson system, Mountain Pass Theorem, Ekeland's variational principle, multiple solutions.

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1 Introduction and main results

In this paper, we consider the following nonlinear Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) + g(x), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $V \in C\left(\mathbb{R}^3, \mathbb{R}\right)$, $f \in C\left(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}\right)$ and the conditions on g will be given later.

System (1.1) is also called Schrödinger–Maxwell system, arises in an interesting physical context. In fact, according to a classical model, the interaction of a charge particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger's and Poisson's equations. For more information on the physical relevance of the Schrödinger–Poisson system, we refer the readers to the papers [3,23] and the references therein.

If $g(x) = 0$, system (1.1) becomes the well known Schrödinger–Poisson system, which has been extensively investigated in the last years by the aid of the modern variational methods and critical point theory. Moreover, since the pioneering work of Benci and Fortunato [5],

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there is huge literature on the studies of the existence and behavior of solutions of the system (1.1) with $g(x) = 0$, see for example [1,2,7,9,10,14,16–18,21,24–26,30–32,34] and the references therein.

Compared to the homogeneous case (i.e., $g(x) = 0$), there are few papers concerning the case where $g(x) \neq 0$, see for example [8,12,15,22,28,35]. Particularly, in [8] the authors obtained the existence of two nontrivial solutions for system (1.1) by using Ekeland's variational principle and the Mountain Pass Theorem when $g \in L^2(\mathbb{R}^3), g \not\equiv 0$, and f and V satisfy the following assumptions, respectively:

(V_0) $V(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$, where V_0 is a constant. Moreover, for every $M > 0$, $\text{meas}\{x \in \mathbb{R}^3 : V(x) \leq M\} < \infty$, where (and in the sequel) $\text{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^3$.

(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R})$, and there exist constants $a > 0$ and $p \in (2, 6)$ such that

$$|f(x,u)| \leq a\left(1 + |u|^{p-1}\right), \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R},$$

where $6 = 2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent;

(f_2) $\lim_{u\to 0} \frac{f(x,u)}{u} = 0$ uniformly for $x \in \mathbb{R}^3$;

(f_3) there exists $\mu > 4$ such that

$$\mu F(x,u) \leq f(x,u)u, \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.2}$$

where (and in the sequel) $F(x,t) = \int_0^t f(x,s)ds$;

(f_4)

$$\inf_{x\in\mathbb{R}^3, |u|=1} F(x,u) > 0.$$

Specifically, the authors established the following theorem in [8].

Theorem 1.1 ([8]). *Suppose that $g \in L^2(\mathbb{R}^3)$, $g \not\equiv 0$. Let (V_0) and (f_1)–(f_4) hold, then there exists a constant $m_0 > 0$ such that problem (1.1) admits at least two different solutions when $\|g\|_{L^2} \leq m_0$.*

It is worth pointing out that the combination of (f_3)–(f_4) implies that the rang of p in condition (f_1) should be $4 < p < 6$. In fact, for any $x \in \mathbb{R}^3$, $u \in \mathbb{R}$, define

$$h(t) = F(x, t^{-1}u)t^\mu, \qquad \forall t \in [1, +\infty).$$

Then, for $|u| \geq 1$ and $t \in [1, |u|]$, it follows from (1.2) that

$$h'(t) = \left[\mu F(x, t^{-1}u) - f(x, t^{-1}u)t^{-1}u\right] t^{\mu-1} \leq 0.$$

Therefore, $h(1) \geq h(|u|)$. Hence, (f_4) implies that

$$F(x,u) \geq F\Big(x, \frac{u}{|u|}\Big)|u|^\mu \geq c|u|^\mu, \qquad \forall x \in \mathbb{R}^3 \text{ and } |u| \geq 1, \tag{1.3}$$

where, $c = \inf_{x\in\mathbb{R}^3, |u|=1} F(x,u) > 0$. If $p \leq 4$, by (f_1) we have

$$|F(x,u)| \leq \int_0^1 |f(x,tu)u|dt \leq a\int_0^1 (1 + |tu|^{p-1})|u|dt \leq a(|t| + |t|^p), \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R},$$

which implies that

$$\limsup_{t\to+\infty} \frac{F(x,t)}{t^4} \le a \quad \text{uniformly in } x \in \mathbb{R}^3.$$

This contradicts (1.3). Thus, $4 < p < 6$.

Inspired by the above facts, in the present paper we shall consider the nonhomogeneous Schrödinger–Poisson system, and we are interested in looking for multiple solutions for the problem (1.1). Under much more relaxed assumptions on the nonlinearity f and the potential function V, using some special proof techniques especially the verification of the boundedness of Palais–Smale sequence, new results on the existence of multiple nontrivial solutions for the system (1.1) are obtained, which extend and sharply improve some recent results in the literature. In order to state the main results of this paper, we make the following assumptions.

(V) $V \in C\left(\mathbb{R}^3, \mathbb{R}\right)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \ge V_0 > 0$, where V_0 is a constant. Moreover, there exists $r_0 > 0$ such that

$$\lim_{|y|\to\infty} \operatorname{meas}\{x \in \mathbb{R}^3 : |x-y| \le r_0,\ V(x) \le M\} = 0, \qquad \forall M > 0.$$

(H_1) $f \in C\left(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}\right)$, and there exist constants $c_1, c_2 > 0$ and $p \in (4,6)$ such that

$$|f(x,t)| \le c_1|t| + c_2|t|^{p-1}, \qquad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

(H_2) $\lim_{t\to0} \frac{f(x,t)}{t} < \mu^*$ uniformly for $x \in \mathbb{R}^3$ where

$$\mu^* = \inf\left\{ \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx : u \in H^1(\mathbb{R}^3),\ \int_{\mathbb{R}^3} u^2 dx = 1 \right\}.$$

(H_3) $\lim_{t\to\infty} \frac{F(x,t)}{t^4} = \infty$ uniformly in $x \in \mathbb{R}^3$.

(H_4) There exist $c_3 > 0$ and $L > 0$ such that

$$4F(x,t) \le f(x,t)t + c_3t^2, \quad \text{for a.e. } x \in \mathbb{R}^3 \quad \text{and} \quad \forall\ |t| \ge L.$$

(H_4') There exists $L > 0$ such that

$$4F(x,t) \le f(x,t)t, \quad \text{for a.e. } x \in \mathbb{R}^3 \quad \text{and} \quad \forall\ |t| \ge L.$$

(H_5) $g \in L^{p'}(\mathbb{R}^3), g \not\equiv 0$, where $\frac{1}{p'} + \frac{1}{p} = 1$, p is defined by (H_1).

Now, we are ready to state the main results of this paper as follows.

Theorem 1.2. *Assume that (V) and (H_1)–(H_5) hold. Then, there exists $m_0 > 0$ such that for any $g \in L^{p'}(\mathbb{R}^3)$ with $\|g\|_{p'} \le m_0$, the system (1.1) possesses at least two different nontrivial solutions, one is negative energy solution, and the other is positive energy solution.*

The other aim of this paper is to study the existence of at least two different nontrivial solutions for problem (1.1) involving a concave–convex nonlinearity. We also consider the effect of the parameter λ and the perturbation term g on the existence of solutions.

Theorem 1.3. *Let $g \in L^2(\mathbb{R}^3)$, $g \not\equiv 0$. Assume that (V) and*

(H_6) *$f(x,u) = \lambda h_1(x)|u|^{\sigma-2}u + h_2(x)|u|^{p-2}u$ with $1 < \sigma < 2, 4 < p < 6$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$, in which $h_1 \in L^{\sigma_0}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $h_2 \in L^\infty(\mathbb{R}^3)$ with $\sigma_0 = 2/(2-\sigma)$. Moreover, there exists a nonempty bounded domain $\Omega \subset \mathbb{R}^3$ such that $h_2 > 0$ in Ω.*

Then there exist $\lambda_0, m_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, the system (1.1) possesses at least two different nontrivial solutions whenever $\|g\|_2 \le m_0$, one is negative energy solution, and the other is positive energy solution.

Obviously, the condition $(H_4^{'})$ implies the condition (H_4), so we have the following corollary.

Corollary 1.4. *If we replace (H_4) with $(H_4^{'})$ in Theorem 1.2, then the conclusion of Theorem 1.2 remains valid.*

Remark 1.5. Since the problem (1.1) is defined in the whole space $\mathbb{R}^3$, the main difficulty of this problem is the lack of compactness for Sobolev embedding theorem. To overcome this difficulty, the condition (V), which was firstly introduced by Bartsch et al. [4], is always assumed to preserve the compactness of the embedding of the working space. Furthermore, condition (V) is weaker than condition (V_0), and there are functions $V(x)$ satisfying (V) but not satisfying (V_0), see for example Remark 2 in [33].

Remark 1.6.

(1) Theorem 1.2 sharply improves Theorem 1.1. If fact, from Remark 3. in [33], we know that the condition (H_1) is much weaker than the combination of (f_1) and (f_2), and conditions (H_3)–(H_4) are much weaker than (f_3)–(f_4).

(2) The condition (H_2) which gives the behaviour of $f(x,u)/u$ for u near to the origin, is very essential for obtain the positive energy solution in Theorem 1.2. Moreover, it seems to be nearly optimal for obtain a such existence result.

(3) As a function f satisfying the assumptions (H_1)–(H_4), one can take

$$f(x,u) = \begin{cases} u^3(4\ln|u|+1), & |u| \ge 1, \\ -(2\nu-1)u^2 + 2\nu u, & |u| \le 1, \end{cases}$$

where $0 < \nu < \frac{\mu^*}{2}$ (μ^* is given by (H_2)). A straightforward computation deduces that

$$F(x,u) = \begin{cases} u^4\ln|u| + \frac{\nu+1}{3}, & |u| \ge 1, \\ -\frac{2\nu-1}{3}u^3 + \nu u^2, & |u| \le 1, \end{cases}$$

and

$$f(x,u)u - 4F(x,u) = u^4 - \frac{4}{3}(\nu+1), \quad \forall x \in \mathbb{R}^3, \qquad |u| \ge 1.$$

Hence, it is easy to check that f satisfies the assumptions (H_1)–(H_4). However, it does not satisfy the assumptions of Theorem 1.1. In fact, we have $\lim_{t\to 0} \frac{f(x,t)}{t} = 2\nu > 0$ uniformly for $x \in \mathbb{R}^3$, which implies that f does not satisfy the condition (f_2). Moreover, for any $\mu > 4$, we have

$$f(x,u)u - \mu F(x,u) = -(\mu-4)u^4\ln|u| + u^4 - \frac{\mu}{3}(\nu+1) \to -\infty, \quad \text{as } |u| \to \infty,$$

which shows that the condition (f_3) is not satisfying for our choice.

Remark 1.7. The assumptions of Theorems 1.2 and 1.3 can be used to deal with the existence of nontrivial solutions for the following nonhomogeneous Kirchhoff-type equations

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u)+g(x), & \text{in } \mathbb{R}^3,\\ u(x)\to 0 & \text{as } |x|\to\infty,\end{cases}$$

where $a>0, b\geq 0$ are constants. So, the conclusions of Theorems 1.2 and 1.3 still hold for the above problem.

The paper is organized as follows. In Section 2, we present some preliminary results. Section 3 is devoted to the proof of Theorems 1.2 and 1.3.

2 Preliminaries

In the following, we will introduce the variational setting for Problem (1.1). In the sequel, we denote by $\|\cdot\|_p$ the usual norm of the space $L^p(\mathbb{R}^3)$, c_i, C_i or C stand for different positive constants.

As usual, for $1\leq p<+\infty$, we let

$$\|u\|_p:=\left(\int_{\mathbb{R}^3}|u|^pdx\right)^{\frac{1}{p}}, \qquad u\in L^p(\mathbb{R}^3),$$

and

$$\|u\|_\infty:=\operatorname*{ess\,sup}_{x\in\mathbb{R}^3}|u(x)|, \qquad u\in L^\infty(\mathbb{R}^3).$$

Let

$$H^1(\mathbb{R}^3)=\left\{u\in L^2(\mathbb{R}^3):\nabla u\in L^2(\mathbb{R}^3)\right\},$$

with the inner product and norm

$$\langle u,v\rangle_{H^1}=\int_{\mathbb{R}^3}(\nabla u\nabla v+uv)\,dx, \qquad \|u\|_{H^1}=\langle u,u\rangle_{H^1}^{\frac{1}{2}}.$$

Define our working space

$$E=\left\{u\in H^1(\mathbb{R}^3):\int_{\mathbb{R}^3}V(x)|u|^2dx<+\infty\right\}.$$

Then E is a Hilbert space equipped with the inner product and norm

$$\langle u,v\rangle=\int_{\mathbb{R}^3}(\nabla u\nabla v+V(x)uv)\,dx, \qquad \|u\|=\langle u,u\rangle^{\frac{1}{2}}.$$

Let $\mathcal{D}^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_{\mathcal{D}^{1,2}}=\int_{\mathbb{R}^3}|\nabla u|^2dx.$$

Then, the embedding $\mathcal{D}^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ is continuous (see for instance [29]). Since the embedding $H^1(\mathbb{R}^3)\hookrightarrow L^s(\mathbb{R}^3)$ $(2\leq s\leq 6)$ is continuous, then the embedding $E\hookrightarrow L^s(\mathbb{R}^3)$ $(2\leq s\leq 6)$ is continuous under the condition (V), that is, there exist $\eta_s>0$ such that

$$\|u\|_s\leq\eta_s\|u\|, \quad \forall u\in E, \quad s\in[2,6]. \tag{2.1}$$

Moreover, we have the following compactness results from [4, Lemma 3.1.].

Lemma 2.1 ([4]). *Under the assumption* (V), *the embedding* $E \hookrightarrow L^s\left(\mathbb{R}^N\right)$ *is compact for* $s \in [2, 6)$.

Recall that $\mu \in \mathbb{R}$ is called an eigenvalue of the operator $-\Delta + V(x)$ provided there exists a nontrivial weak solution u_0 of the equation:

$$-\Delta u + V(x)u = \mu u, \qquad x \in \mathbb{R}^3,$$

i.e., for any $\varphi \in E$,

$$\int_{\mathbb{R}^3} (\nabla u_0 \nabla \varphi + V(x) u_0 \varphi)\, dx = \mu \int_{\mathbb{R}^3} u_0 \varphi dx.$$

Lemma 2.2. *Assume that* (V) *holds. Then* μ^* *is an eigenvalue of the operator* $-\Delta + V(x)$ *and there exists a corresponding eigenfunction* φ_1 *with* $\varphi_1 > 0$ *for all* $x \in \mathbb{R}^3$.

Proof. The proof of this lemma is almost the same to the one of Lemma 2.3 in [13]. So we omit it here. □

For every $u \in H^1(\mathbb{R}^3)$, by the Lax–Milgram theorem, we know that there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta \phi_u = u^2, \quad \text{in } \mathbb{R}^3. \tag{2.2}$$

Furthermore, ϕ_u has the following integral expression

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \geq 0. \tag{2.3}$$

From (2.1), for any $u \in E$, using the Hölder inequality we obtain

$$\|\phi_u\|^2_{\mathcal{D}^{1,2}} = \int_{\mathbb{R}^3} \phi_u u^2 dx \leq \|\phi_u\|_6 \|u\|^2_{12/5} \leq C\|\phi_u\|_{\mathcal{D}^{1,2}} \|u\|^2_{12/5}. \tag{2.4}$$

Therefore

$$\|\phi_u\|_{\mathcal{D}^{1,2}} \leq C\|u\|^2_{12/5}. \tag{2.5}$$

By (2.4), (2.5) and the Sobolev inequality, we obtain

$$\frac{1}{4\pi} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dydx = \int_{\mathbb{R}^3} \phi_u u^2 dx \leq C_1 \|u\|^4. \tag{2.6}$$

Moreover, ϕ_u has the following properties (for a proof, see [6,21]).

Lemma 2.3. *For* $u \in E$ *we have*

(*i*) $\phi_{tu} = t^2 \phi_u$, *for all* $t \geq 0$;

(*ii*) *If* $u_n \rightharpoonup u$ *in* E, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^3)$ *and*

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} \phi_u u^2 dx.$$

Now, we define the energy functional $J : E \to \mathbb{R}$ associated with problem (1.1) by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)|u|^2\right) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u) dx - \int_{\mathbb{R}^3} g(x) u dx. \tag{2.7}$$

Therefore, combining (2.5), (2.6), (H_1)–(H_2) and Lemma 2.1, J is well defined and $J \in C^1(E, \mathbb{R})$ with

$$\begin{aligned}\langle J'(u), v\rangle = &\int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)\, dx + \int_{\mathbb{R}^3} \phi_u uv dx \\ &- \int_{\mathbb{R}^3} f(x,u)v dx - \int_{\mathbb{R}^3} g(x)v dx, \qquad \forall\, v \in E.\end{aligned} \tag{2.8}$$

Moreover, if $u \in E$ is a critical point of J, then the pair (u, ϕ_u) is a solution of system (1.1).

Recall that a sequence $\{u_n\} \subset E$ is said to be a Palais–Smale sequence at the level $c \in \mathbb{R}$ ($(PS)_c$-sequence for short) if $J(u_n) \to c$ and $J'(u_n) \to 0$, J is said to satisfy the Palais–Smale condition at the level c ($(PS)_c$-condition for short) if any $(PS)_c$-sequence has a convergent subsequence.

In order to prove the existence of positive energy solution for problem (1.1), we shall use the following Mountain Pass Theorem (cf. [20,29]).

Proposition 2.4 ([20,29])**.** *Let E be a Banach space, $J \in C^1(E, \mathbb{R})$ satisfies the (PS)-condition for any $c > 0$, $J(0) = 0$, and*

(i) there exist $\rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$;

(ii) there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.

Then J has at least a critical value $c \geq \alpha$.

On the other hand, the following Ekeland's variational principle is the main tool to obtain the negative energy solution for problem (1.1)

Proposition 2.5 ([19, Theorem 4.1])**.** *Let M be a complete metric space with metric d and let $J : M \mapsto (-\infty, +\infty]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\varepsilon > 0$ be given and $u \in M$ be such that*

$$J(u) \leq \inf_M J + \varepsilon.$$

Then, there exists $v \in M$ such that

$$J(v) \leq J(u), \qquad d(u,v) \leq 1,$$

and for each $w \in M$, one has

$$J(v) \leq J(w) + \varepsilon d(v,w).$$

We also need the following auxiliary result, see [27].

Lemma 2.6. *Assume that $p_1, p_2 > 1$, $r, q \geq 1$ and $\Omega \subseteq \mathbb{R}^N$. Let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying*

$$|f(x,t)| \leq a_1|t|^{(p_1-1)/r} + a_2|t|^{(p_2-1)/r}, \qquad \forall (x,t) \in \Omega \times \mathbb{R},$$

where, $a_1, a_2 \geq 0$. If $u_n \to u_0$ in $L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$, and $u_n \to u_0$ a.e. $x \in \Omega$, then for any $v \in L^{p_1 q}(\Omega) \cap L^{p_2 q}(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega |f(x,u_n) - f(x,u_0)|^r |v|^q dx \to 0.$$

3 Proof of main results

In this section we shall prove Theorems 1.2 and 1.3. We first prove some lemmas, which are crucial to prove our main results.

Lemma 3.1. *Assume that the assumptions (V), (H_1), (H_2) and (H_5) hold. Then, there exist ρ, α and $m_0 > 0$ such that $J(u) \geq \alpha$ whenever $\|u\| = \rho$ and $\|g\|_{p'} < m_0$.*

Proof. By (H_1) and (H_2), there exist $\varepsilon_0 > 0$ and $c_4 > 0$ such that

$$F(x,u) \leq \frac{\mu^* - \varepsilon_0}{2}|u|^2 + c_4|u|^p, \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.1}$$

Combining (2.1), (2.3), (2.7) and (3.1), we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx - \int_{\mathbb{R}^3} g(x)u dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} F(x,u)dx - \int_{\mathbb{R}^3} g(x)u dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\mu^* - \varepsilon_0}{2}\int_{\mathbb{R}^3} |u|^2 - c_4 \int_{\mathbb{R}^3} |u|^p dx - \|g\|_{p'}\|u\|_p \\ &\geq \frac{\varepsilon_0}{2\mu^*}\|u\|^2 - c_4\eta_p^p\|u\|^p - \eta_p\|g\|_{p'}\|u\|. \end{aligned} \tag{3.2}$$

Taking

$$\rho = \left[\frac{\varepsilon_0}{4\mu^*(c_4\eta_p^p + \eta_p)}\right]^{\frac{1}{p-2}},$$

$m_0 = \rho^{p-1}$ in (3.2), we then get

$$J(u) \geq \frac{\varepsilon_0}{4\mu^*}\rho^2 = \alpha > 0, \qquad \forall \, \|u\| = \rho.$$

The proof is completed. □

Lemma 3.2. *Assume that the assumptions (V), (H_1), (H_3) and (H_5) hold. Then there exists $e \in E$ with $\|e\| > \rho$ such that $J(e) \leq 0$, where ρ is given in Lemma 3.1.*

Proof. By (H_1) and (H_3) we have, for any $M > 0$, there exists $C_M > 0$ such that

$$F(x,u) \geq M|u|^4 - C_M|u|^2, \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.3}$$

Consequently, it follows from (2.6), (2.7) and (3.3) that

$$\begin{aligned} J(t\varphi_1) &= \frac{t^2}{2}\|\varphi_1\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{t\varphi_1}(t\varphi_1)^2 dx - \int_{\mathbb{R}^3} F(x,t\varphi_1)dx - t\int_{\mathbb{R}^3} g(x)\varphi_1(x)dx \\ &\leq \frac{t^2}{2}\|\varphi_1\|^2 + \frac{t^4}{4}C_1\|\varphi_1\|^4 - t^4 M\int_{\mathbb{R}^3} |\varphi_1|^4 dx \\ &\quad + t^2 C_M \int_{\mathbb{R}^3} |\varphi_1|^2 dx - t\int_{\mathbb{R}^3} g(x)\varphi_1(x)dx \\ &\leq \frac{t^2}{2}(1+2C_M)\|\varphi_1\|^2 - \frac{t^4}{4}\left(4M\|\varphi_1\|_4^4 - C_1\|\varphi_1\|^4\right) - t\int_{\mathbb{R}^3} g(x)\varphi_1(x)dx. \end{aligned} \tag{3.4}$$

Therefore, choosing $M > 0$ such that $4M\|\varphi_1\|_4^4 - C_1\|\varphi_1\|^4 > 0$, then, it follows from (3.4) that $J(t\varphi_1) \to -\infty$ as $t \to +\infty$. Hence, there exists $t_1 > 0$ so large that $\|t_1\varphi_1\| > \rho$ and $J(t_1\varphi_1) < 0$. Thus, the lemma is proved by taking $e = t_1\varphi_1$. □

Lemma 3.3. *Assume that (V), (H_1)–(H_5) hold. Then J satisfies the (PS)-condition on E.*

Proof. Let $\{u_n\} \subset E$ be such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0. \tag{3.5}$$

We first show that $\{u_n\}$ is bounded in E. Otherwise, set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and $\|v_n\|_p \leq \eta_p \|v_n\| = \eta_p$ (see 2.1). It follows from (H_1) that

$$\begin{aligned} |F(x,u)| &= |F(x,u) - F(x,0)| \\ &= \left| \int_0^1 f(x,tu)u dt \right| \\ &\leq \int_0^1 \left(c_1|u|^2 t + c_2 |u|^p t^{p-1} \right) dt \\ &= \frac{c_1}{2}|u|^2 + \frac{c_2}{p}|u|^p, \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \end{aligned} \tag{3.6}$$

Let $\mathcal{F}(x,u_n) = f(x,u_n)u_n - 4F(x,u_n)$. Therefore, for $x \in \mathbb{R}^3$ and $|u(x)| < L$, by (3.6), we have

$$\begin{aligned} |f(x,u)u - 4F(x,u)| &\leq |f(x,u)u| + 4|F(x,u)| \\ &\leq (c_1|u|^2 + c_2|u|^p) + \left(2c_1|u|^2 + \frac{4c_2}{p}|u|^p \right) \\ &\leq \left(3c_1 + \frac{4+p}{p} c_2 L^{p-2} \right) |u|^2 \\ &= c_7 |u|^2, \end{aligned}$$

where $L > 0$ is given by (H_4). Combining the above inequality with (H_4), we conclude that there exists $c_8 > 0$ such that

$$f(x,u)u - 4F(x,u) \geq -c_8|u|^2, \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.7}$$

By (H_5), (2.7), (2.8), (3.5), (3.7) and the Hölder inequality, without loss of generality, we may assume that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} 1 + c + \|u_n\| &\geq J(u_n) - \frac{1}{4}\langle J'(u_n), u_n \rangle \\ &= \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \mathcal{F}(x,u_n)dx - \frac{3}{4}\int_{\mathbb{R}^3} g(x)u_n dx \\ &\geq \frac{1}{4}\|u_n\|^2 - \frac{c_8}{4}\int_{\mathbb{R}^3} |u_n|^2 dx - \frac{3}{4}\|g\|_{p'}\|u_n\|_p \\ &\geq \frac{1}{4}\|u_n\|^2 - \frac{c_8}{4}\|u_n\|_2^2 - \frac{3}{4}\eta_p\|g\|_{p'}\|u_n\|, \end{aligned}$$

which implies that

$$\frac{\|u_n\|_2^2}{\|u_n\|^2} \geq \frac{1}{c_8} - \frac{1}{c_8}\left[\frac{4(c+1)}{\|u_n\|^2} + \frac{3\eta_p\|g\|_{p'}}{\|u_n\|} \right].$$

Therefore, for sufficiently large n such that $\frac{4(c+1)}{\|u_n\|^2} + \frac{3\eta_p\|g\|_{p'}}{\|u_n\|} \leq \frac{1}{2}$, we then get

$$\frac{\|u_n\|_2^2}{\|u_n\|^2} \geq \frac{1}{2c_8} > 0.$$

Consequently, we conclude that

$$\|v_n\|_2 > 0. \tag{3.8}$$

Let $\Omega_n = \{x \in \mathbb{R}^3 : |u_n(x)| \le L\}$ and $A_n = \{x \in \mathbb{R}^3 : v_n(x) \neq 0\}$, then $\operatorname{meas}(A_n) > 0$. Moreover, since $\|u_n\| \to \infty$ as $n \to \infty$, we obtain

$$|u_n(x)| \to \infty \quad \text{as } n \to \infty \quad \text{for } x \in A_n.$$

Hence, $A_n \subseteq \mathbb{R}^3 \setminus \Omega_n$ for $n \in \mathbb{N}$ large enough. It follows from (H_5) and the Hölder inequality that for any $\beta \in (1,6)$, one has

$$\left| \int_{\mathbb{R}^3} \frac{g(x)u_n}{\|u_n\|^\beta} dx \right| \le \frac{\|g\|_{p'} \|u_n\|_p}{\|u_n\|^\beta} \le \eta_p \frac{\|g\|_{p'}}{\|u_n\|^{\beta-1}} \to 0, \tag{3.9}$$

since $\|u_n\| \to \infty$ as $n \to \infty$. By (H_1), (H_3), (2.1), (2.6), (3.5), (3.6), (3.8), (3.9) and Fatou's lemma, we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{J(u_n)}{\|u_n\|^4} \\
&= \lim_{n\to\infty} \left[\frac{1}{2\|u_n\|^2} + \frac{1}{4\|u_n\|^4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|^4} dx - \int_{\mathbb{R}^3} \frac{g(x)u_n}{\|u_n\|^4} dx \right] \\
&\le C_1 - \lim_{n\to\infty} \left[\int_{\Omega_n} \frac{F(x,u_n)}{u_n^4} v_n^4 dx + \int_{\mathbb{R}^3\setminus\Omega_n} \frac{F(x,u_n)}{u_n^4} v_n^4 dx \right] \\
&\le C_1 - \lim_{n\to\infty} \left[\frac{1}{\|u_n\|^2} \left(\frac{c_1}{2} + \frac{c_2}{p} L^{p-2} \right) \eta_2^2 + \int_{\mathbb{R}^3\setminus\Omega_n} \frac{F(x,u_n)}{u_n^4} v_n^4 dx \right] \\
&\le C_1 - \liminf_{n\to\infty} \int_{\mathbb{R}^3\setminus\Omega_n} \frac{F(x,u_n)}{u_n^4} v_n^4 dx \\
&\le C_1 - \int_{A_n} \liminf_{n\to\infty} \frac{F(x,u_n)}{u_n^4} v_n^4 dx \\
&= C_1 - \int_{\mathbb{R}^3} \liminf_{n\to\infty} \frac{F(x,u_n)}{u_n^4} [\chi_{A_n}(x)] v_n^4 dx \\
&\to -\infty, \quad \text{as } n \to \infty.
\end{aligned} \tag{3.10}$$

This is an obvious contradiction. Hence $\{u_n\} \subset E$ is bounded. So, up to a subsequence we may assume that $u_n \rightharpoonup u_0$ weakly in E. By Lemma 2.1, $u_n \to u_0$ strongly in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$ and $u_n(x) \to u_0(x)$ a.e. on $\mathbb{R}^3$. It follows from (2.7) and (2.8) that

$$\begin{aligned}
\|u_n - u_0\|^2 = \langle J'(u_n) - J'(u_0), u_n - u_0 \rangle &+ \int_{\mathbb{R}^3} [f(x,u_n) - f(x,u_0)](u_n - u_0) dx \\
&- \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_{u_0} u_0)(u_n - u_0) dx.
\end{aligned} \tag{3.11}$$

Obviously, $\langle J'(u_n) - J'(u_0), u_n - u_0 \rangle \to 0$ as $n \to \infty$. Let us take $r = q = 1$ in Lemma 2.6 and combine with $u_n \to u_0$ strongly in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$, to get

$$\int_{\mathbb{R}^3} [f(x,u_n) - f(x,u_0)](u_n - u_0) dx \to 0. \tag{3.12}$$

Furthermore, from Lemma 2.3 (ii), we have that $\int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_{u_0} u_0)(u_n - u_0) dx \to 0$. Consequently, $u_n \to u_0$ in E. This completes the proof. □

Proof of Theorem 1.2. The proof is divided in two steps, the first one for the negative energy solution, the second one for the positive energy solution.

Step 1. By using Ekeland's variational principle, we first show that there exists a function $u_0 \in E$ such that $J'(u_0) = 0$ and $J(u_0) < 0$. By (3.3) fixing $M > 0$ a constant $C_M > 0$ exists such that

$$F(x,u) \geq M|u|^4 - C_M|u|^2, \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

Since $g \in L^{p'}(\mathbb{R}^3)$ and $g \not\equiv 0$, we may choose a function $v \in E$ such that

$$\int_{\mathbb{R}^3} g(x)v(x)dx > 0.$$

Therefore,

$$\begin{aligned} J(tv) &= \frac{t^2}{2}\|v\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_v v^2 dx - \int_{\mathbb{R}^3} F(x,tv)dx - t\int_{\mathbb{R}^3} g(x)v(x)dx \\ &\leq \frac{t^2}{2}\|v\|^2 + C_1\frac{t^4}{4}\|v\|^4 - Mt^4\|v\|_4^4 + C_M t^2\|v\|_2^2 - t\int_{\mathbb{R}^3} g(x)v(x)dx < 0, \end{aligned}$$

for $t > 0$ small enough, which implies that

$$\inf\{J(u) : u \in \overline{B}_\rho\} < 0,$$

where $\rho > 0$ is given by Lemma 3.1, and $\overline{B}_\rho = \{u \in E : \|u\| \leq \rho\}$. On the other hand, by (3.2), one has

$$\begin{aligned} J(u) \geq& \frac{\varepsilon_0}{2\mu^*}\|u\|^2 - c_4\eta_p^p\|u\|^p - \eta_p\|g\|_{p'}\|u\| \\ \geq& -c_4\eta_p^p\|u\|^p - \eta_p\|g\|_{p'}\|u\|, \end{aligned}$$

which implies that J is bounded below in $\overline{B}_\rho$. Thus, we obtain

$$-\infty < c_0 = \inf\{J(u) \ : \ u \in \overline{B}_\rho\} < 0.$$

By Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_0 \leq J(u_n) \leq c_0 + \frac{1}{n},$$

and

$$J(u_n) \leq J(w) + \frac{1}{n}\|u_n - w\|, \qquad \forall w \in \overline{B}_\rho.$$

Then, following the idea of [11] (see pp. 534–535), we can show that $\{u_n\}$ is a bounded Palais-Smale sequence of J. Therefore, by Lemma 3.3, $\{u_n\}$ has a strongly convergent subsequence, still denoted by $\{u_n\}$ and $u_n \to u_0 \in \overline{B}_\rho$ as $n \to \infty$. Hence, we conclude that there exists $u_0 \in E$ such that $J(u_0) = \inf_{u\in \overline{B}_\rho} J(u) = c_0 < 0$ and $J'(u_0) = 0$, this completes the Step 1.

Step 2. Now, we show that there exists a function $\overline{u}_0 \in E$ such that $J(\overline{u}_0) = \overline{c}_0 > 0$ and $J'(\overline{u}_0) = 0$ by means of the Mountain Pass Theorem. Obviously, $J \in C^1(E,\mathbb{R})$ and $J(0) = 0$. By Lemmas 3.1 and 3.2, the functional J satisfies the geometric property of the mountain pass theorem whenever $\|g\|_{p'} \leq m_0$. Lemma 3.3 implies that J satisfies the (PS)-condition. Therefore, applying Proposition 2.4, we deduce that there exists $\overline{u}_0 \in E$ such that $J(\overline{u}_0) = \overline{c}_0 \geq \alpha > 0$ and $J'(\overline{u}_0) = 0$, we complete the Step 2.

Therefore, by the above two steps the proof of Theorem 1.2 is completed. □

Next, we will give the proof of Theorem 1.3. Under the assumption (H_6), we can easily find that the energy functional associated to problem (1.1)

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{\lambda}{\sigma}\int_{\mathbb{R}^3} h_1(x)|u|^\sigma dx - \frac{1}{p}\int_{\mathbb{R}^3} h_2(x)|u|^p dx - \int_{\mathbb{R}^3} g(x)u dx, \quad (3.13)$$

is of class C^1 on E and for any $v \in E$, we have

$$\begin{aligned}\langle J'(u), v\rangle = &\int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)\, dx + \int_{\mathbb{R}^3} \phi_u uv dx - \lambda \int_{\mathbb{R}^3} h_1(x)|u|^{\sigma-2}uv dx \\ &- \int_{\mathbb{R}^3} h_2(x)|u|^{p-2}uv dx - \int_{\mathbb{R}^3} g(x)v dx.\end{aligned} \quad (3.14)$$

Lemma 3.4. *Suppose that the assumptions (V) and (H_6) are satisfied. Then, there exist ρ, α and $m_0 > 0$ such that $J(u) \geq \alpha$ whenever $\|u\| = \rho$ and $\|g\|_2 < m_0$.*

Proof. By the Hölder inequality, we have

$$\int_{\mathbb{R}^3} |h_1(x)||u|^\sigma dx \leq \|h_1\|_{\sigma_0}\|u\|_2^\sigma \leq V_0^{-\frac{\sigma}{2}}\|h_1\|_{\sigma_0}\|u\|^\sigma,$$

where $\sigma_0 = 2/(2-\sigma)$. On the other hand, by (2.1), we have

$$\int_{\mathbb{R}^3} |h_2(x)||u|^p dx \leq \|h_2\|_\infty \|u\|_p^p \leq \eta_p^p \|h_2\|_\infty \|u\|^p.$$

Similarly, we have by Young's inequality,

$$\int_{\mathbb{R}^3} |g(x)||u| dx \leq \|g\|_2\|u\|_2 \leq V_0^{-\frac{1}{2}}\|g\|_2\|u\| \leq \frac{1}{4}\|u\|^2 + \frac{1}{V_0}\|g\|_2^2.$$

Therefore, it follows from (2.3) and (3.13) that

$$J(u) \geq \frac{1}{4}\|u\|^2 - \lambda\beta_1\|u\|^\sigma - \beta_2\|u\|^p - V_0^{-1}\|g\|_2^2, \quad (3.15)$$

where, $\beta_1 = \frac{1}{\sigma}V_0^{-\frac{\sigma}{2}}\|h_1\|_{\sigma_0}, \beta_2 = \frac{1}{p}\eta_p^p\|h_2\|_\infty$. Let

$$\xi(t) = \lambda\beta_1 t^{\sigma-2} + \beta_2 t^{p-2}, \qquad t > 0.$$

We claim $\xi(t_0) < \frac{1}{4}$ for some $t_0 > 0$. Note that $\xi(t) \to +\infty$ as $t \to 0^+$ or $t \to +\infty$. Then, $\xi(t)$ has a minimum at $t_0 > 0$. In order to find t_0, note

$$\xi'(t_0) = \lambda\beta_1(\sigma-2)t_0^{\sigma-3} + \beta_2(p-2)t_0^{p-3} = 0 \quad \text{and} \quad t_0 = \lambda^{1/(p-\sigma)}\left(\frac{\beta_1(2-\sigma)}{\beta_2(p-2)}\right)^{1/(p-\sigma)} > 0.$$

Thus, $\xi(t_0) = \lambda^{(p-2)/(p-\sigma)}\big(\beta_1\beta_0^{(\sigma-2)/(p-\sigma)} + \beta_2\beta_0^{(p-2)/(p-\sigma)}\big)$ with $\beta_0 = \beta_1(2-\sigma)/\beta_2(p-2)$. This shows that there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, $\xi(t_0) < \frac{1}{4}$. Hence, (3.15) implies that there exists $m_0, \alpha > 0$ such that $J(u) \geq \alpha$ whenever $\|u\| = t_0 = \rho$ and $\|g\|_2 < m_0$. □

Lemma 3.5. *Suppose that the assumptions (V) and (H_6) are satisfied. Then there exists $e \in E$ with $\|e\| > \rho$ such that $J(e) \leq 0$, where ρ is given in Lemma 3.4.*

Proof. Choose $\varphi_2 \in C_0^\infty(\Omega)$, $\varphi_2 \geq 0$, $\varphi_2 \not\equiv 0$. By (H_6), we know that $h_2 > 0$ in Ω, then

$$\begin{aligned} J(t\varphi_2) &= \frac{t^2}{2}\|\varphi_2\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3}\phi_{\varphi_2}\varphi_2^2 dx \\ &\quad - \frac{\lambda t^\sigma}{\sigma}\int_\Omega h_1(x)|\varphi_2|^\sigma dx - \frac{t^p}{p}\int_\Omega h_2(x)|\varphi_2|^p dx - t\int_\Omega g(x)\varphi_2 dx \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$ with $1 < \sigma < 2$ and $p > 4$. Thus, there exists $t_2 > 0$ large enough, such that $J(t_2\varphi_2) < 0$. Thus, we complete the proof by taking $e = t_2\varphi_2$. □

Lemma 3.6. *Assume that (V) and (H_6) hold. Then J satisfies the (PS)-condition on E.*

Proof. Let $\{u_n\} \subset E$ satisfying (3.5). We claim that $\{u_n\}$ is bounded in E. For n large enough, it follows from (2.3), (3.5), (3.13) and (3.14) that

$$\begin{aligned} 1 + c + \|u_n\| &\geq J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 dx - \lambda\left(\frac{1}{\sigma} - \frac{1}{p}\right)\int_{\mathbb{R}^3} h_1(x)|u_n|^\sigma dx \\ &\quad - \left(1 - \frac{1}{p}\right)\int_{\mathbb{R}^3} g(x)u_n dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \lambda\left(\frac{1}{\sigma} - \frac{1}{p}\right)V_0^{-\frac{\sigma}{2}}\|h_1\|_{\sigma_0}\|u_n\|^\sigma - \left(1 - \frac{1}{p}\right)V_0^{-\frac{1}{2}}\|g\|_2\|u_n\|. \end{aligned}$$

Because $1 < \sigma < 2$ and $p > 4$, we deduce that $\{u_n\}$ is bounded in E. Therefore, there exists $u \in E$ such that, up to a subsequence, we have $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ strongly in $L^s(\mathbb{R}^3)$ for $2 \leq s < 6$ and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$. Similar to the proof of Lemma 3.3 (see (3.11)), in order to prove that $u_n \to u$ strongly in E, it sufficient to show that

$$\int_{\mathbb{R}^3} f(x,u_n)(u_n - u)dx = \int_{\mathbb{R}^3}(\lambda h_1(x)|u_n|^{\sigma-2}u_n + h_2(x)|u_n|^{p-2}u_n)(u_n - u)dx \to 0.$$

Since $u_n \to u$ strongly in $L^s(\mathbb{R}^3)$ for $2 \leq s < 6$, the Hölder inequality implies that

$$\int_{\mathbb{R}^3}|h_1||u_n|^{\sigma-1}|u_n - u|dx \leq \|h_1\|_{\sigma_0}\|u_n\|_2^{\sigma-1}\|u_n - u\|_2 \to 0,$$

and

$$\int_{\mathbb{R}^3}|h_2||u_n|^{p-1}|u_n - u|dx \leq \|h_2\|_\infty\|u_n\|_p^{p-1}\|u_n - u\|_p \to 0.$$

Therefore, J satisfies the (PS)-condition. □

Proof of Theorem 1.3. Similar to the proof of Theorem 1.2, we also divide the proof into two steps.

Step 1. As the proof of Step 1 in Theorem 1.2, we first prove the existence of negative energy solution via Ekeland's variational principle (cf. Proposition 2.5). Since $g \in L^2(\mathbb{R}^3)$ and $g \not\equiv 0$, we can choose a function $v \in E$ such that

$$\int_{\mathbb{R}^3} g(x)v(x)dx > 0.$$

It follows from (3.13) that

$$\begin{aligned} J(tv) &= \frac{t^2}{2}\|v\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3}\phi_v v^2 dx - \frac{\lambda t^\sigma}{\sigma}\int_{\mathbb{R}^3} h_1(x)|v|^\sigma dx \\ &\quad - \frac{t^p}{p}\int_{\mathbb{R}^3} h_2(x)|v|^p dx - t\int_{\mathbb{R}^3} g(x)v dx \\ &\le \frac{t^2}{2}\|v\|^2 + C_1\frac{t^4}{4}\|v\|^4 - \frac{\lambda t^\sigma}{\sigma}\int_{\mathbb{R}^3} h_1(x)|v|^\sigma dx \\ &\quad - \frac{t^p}{p}\int_{\mathbb{R}^3} h_2(x)|v|^p dx - t\int_{\mathbb{R}^3} g(x)v dx < 0, \end{aligned}$$

for $t > 0$ small enough, since $1 < \sigma < 2$ and $p > 4$. Hence we deduce that $\inf\{J(u) : u \in \overline{B}_\rho\} < 0$, where $\rho > 0$ is given by Lemma 3.4. In addition, by (3.15) we have

$$\begin{aligned} J(u) &\ge \frac{1}{4}\|u\|^2 - \lambda\beta_1\|u\|^\sigma - \beta_2\|u\|^p - V_0^{-1}\|g\|_2^2 \\ &\ge -\lambda\beta_1\|u\|^\sigma - \beta_2\|u\|^p - V_0^{-1}\|g\|_2^2, \end{aligned}$$

which implies that J is bounded below in $\overline{B}_\rho$. Furthermore, we have

$$-\infty < c_0 = \inf\{J(u) : u \in \overline{B}_\rho\} < 0.$$

Therefore, the Ekleland's variational principle implies that there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_0 \le J(u_n) \le c_0 + \frac{1}{n},$$

and

$$J(u_n) \le J(w) + \frac{1}{n}\|u_n - w\|, \qquad \forall w \in \overline{B}_\rho.$$

Then, arguing as the proof Step 1. in Theorem 1.2, we conclude that there exists $u_0 \in E$ such that $J(u_0) = \inf_{u\in\overline{B}_\rho} J(u) = c_0 < 0$ and $J'(u_0) = 0$.

Step 2. Now, we apply Proposition 2.4 to obtain the positive energy solution. Evidently, $J \in C^1(E, \mathbb{R})$ and $J(0) = 0$. By Lemma 3.4 J satisfies (i) whenever $\|g\|_2 \le m_0$. Moreover, Lemma 3.5 implies that J satisfies (ii), and J satisfies the $(PS)-$condition by Lemma 3.6. Hence, Proposition 2.4 implies that there exists a function $\overline{u}_0 \in E$ such that $J(\overline{u}_0) = \overline{c}_0 \ge \alpha > 0$ and $J'(\overline{u}_0) = 0$.

The proof is completed. □

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