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Asymptotic behavior and uniqueness of entire large solutions to a quasilinear elliptic equation

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Abstract. In this paper, combining the upper and lower solution method with perturbation theory, we study the asymptotic behavior of entire large solutions to Eq. $\Delta_p u = b(x)f(u)$, $u(x) > 0$, $x \in \mathbb{R}$, where $b \in C^{\alpha}_{\rm loc}(\mathbb{R}^N)$ ($\alpha \in (0,1)$) is positive in $\mathbb{R}^N$ ($N \geq 3$), $f \in C^1[0,\infty)$ is positive on $(0,\infty)$ which satisfies a generalized Keller–Osserman condition and is rapidly varying or regularly varying with index $\mu \geq p-1$. We then discuss the uniqueness of solutions by the asymptotic behavior of entire large solutions at infinity.

Keywords: p-Laplacian, asymptotic behavior, entire large solutions.

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1 Introduction

In this article, we study the exact asymptotic behavior of entire large solutions $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N) \cap C^{1,\alpha}_{\rm loc}(\mathbb{R}^N)$ ($\alpha \in (0,1)$) to the following quasilinear elliptic equation

$$\Delta_p u = b(x)f(u), \quad u(x) > 0, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ stands for p-Laplacian operator with $1 < p < N$ ($N \geq 3$). The entire large solution means that u solve Eq. (1.1) in $\mathbb{R}^N$ and $u(x) \to \infty$ as $|x| \to \infty$, which is also called "entire blow-up solution" or "entire explosive solution" in many different contexts. The nonlinearity f satisfies the following hypotheses:

($\mathbf{f_1}$) $f \in C^1[0,\infty)$, $f(0) = 0$, $f'(t) \geq 0$ and $f(t) > 0$ for $t > 0$;

($\mathbf{f_2}$) the following generalized Keller–Osserman condition holds,

$$\int_1^{\infty} [(p/(p-1))F(t)]^{-1/p} dt < +\infty, \ F(t) = \int_0^t f(s)ds;$$

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$(\mathbf{f_3})$ there exists $C_f \geq 0$ such that

$$\lim_{t\to\infty} \frac{(F(t))^{(p-1)/p}}{f(t)\int_t^\infty (F(s))^{-1/p}ds} =: C_f \geq 0, \tag{1.2}$$

the weight b satisfies

$(\mathbf{b_1})$ $b \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ $(\alpha \in (0,1))$ is positive in $\mathbb{R}^N$;

$(\mathbf{b_2})$ there exist a positive constant λ and a function $k \in \mathcal{K}$ such that

$$0 < b_1 := \liminf_{|x|\to\infty} \frac{b(x)}{|x|^{-\lambda}k^{p-1}(|x|)} \leq b_2 := \limsup_{|x|\to\infty} \frac{b(x)}{|x|^{-\lambda}k^{p-1}(|x|)} < \infty,$$

where

$$\begin{cases} \lambda \in [p, p_2), & \text{if } p \in (1,2]; \\ \lambda \in [p_1, p_2), & \text{if } p \in (2, N), \end{cases}$$

$$\begin{cases} \int_{t_0}^{\infty} \frac{k(s)}{s} ds < \infty, & \text{if } p \in (1,2] \text{ and } \lambda = p; \\ \int_{t_0}^{\infty} \frac{k^{p-1}(s)}{s} ds < \infty, & \text{if } p \in (2,N) \text{ and } \lambda = p_1, \end{cases}$$

with

$$p_1 = ((p-2)N + p)/(p-1) \quad \text{and} \quad p_2 = (p^2(N+1) - p(N+3))/(p^2-3),$$

moreover, $\mathcal{K}$ denotes the set of Karamata functions k defined on $[t_0, \infty)$ by

$$k(t) := c \exp\left(\int_{t_0}^{t} \frac{y(s)}{s} ds\right), \qquad t > t_0 > 0$$

with $c > 0$ and $y \in C[t_0, \infty)$ such that $\lim_{t\to\infty} y(t) = 0$.

For $p = 2$, Eq. (1.1) has been extensively investigated by many authors and the link between Eq. (1.1) and geometric problem has been known for a long time, for instance, when $b \equiv 1$ in Ω, $f(u) = e^u$ and $N = 2$, Bieberbach [7] first analyzed the existence, uniqueness and asymptotic behavior of boundary blow-up solutions to Eq. (1.1) with $p = 2$ in a bounded domain $\Omega \subseteq \mathbb{R}^N$ with C^2-boundary. In the case, Eq. (1.1) plays an important role in the theory of Riemannian surfaces of constant negative curvatures and in the theory of automorphic functions. Later, Rademacher [40], using the ideas of Bieberbach, extended the results to a bounded domain in $\mathbb{R}^3$. On the other hand, when $f(u) = u^\gamma$, $\gamma = (N+2)/(N-2)$, Yamabe [45] showed the relationship between solvability of Eq. (1.1) with $p = 2$ and the existence of a conformal metric on the Euclidean space $\mathbb{R}^N$, with a prescribed scalar curvature. It is worth while to point out that, Keller [25] and Osserman [39] carried out a systematic research on Eq. (1.1) with $p = 2$ and gave, respectively, the necessary and sufficient condition for the existence of large solutions when $b \equiv 1$ in bounded domain Ω and $b \equiv 1$ in $\mathbb{R}^N$. Then Lazer and McKenna [29], Lair [26–28], Cîrstea and Rădulescu [10], further investigate the existence of large solutions to Eq. (1.1) in bounded and unbounded domains.

Motivated by certain geometric problems, for $b \equiv 1$ in bounded $\Omega \subseteq \mathbb{R}^N$ and $f(u) = u^\gamma$, $\gamma = (N+2)/(N-2)$ with $N > 2$, Loewner and Nirenberg [31] proved Eq. (1.1) with $p = 2$ has a unique positive large solution u in Ω satisfying

$$\lim_{d(x)\to 0} u(x)(d(x))^{(N-2)/2} = (N(N-2)/4)^{(N-2)/4},$$

where $d(x) := \mathrm{dist}(x, \partial\Omega)$. If f satisfies $(\mathbf{f_1})$–$(\mathbf{f_2})$ with $p = 2$ and the condition that

($\mathbf{f_{01}}$) there exist $\theta > 0$ and $t_0 \geq 1$ such that $f(\xi t) \leq \xi^{1+\theta} f(t)$ for each $\xi \in (0,1)$ and $t \geq t_0/\xi$,

Bandle and Marcus [5] further analyzed the asymptotic behavior of large solutions to Eq. (1.1) with $p = 2$ in a bounded domain $\Omega \subseteq \mathbb{R}^N$ by a appropriate comparison function.

If f satisfies ($\mathbf{f_1}$) and the condition that

($\mathbf{f_{02}}$) $\int_1^t \frac{dt}{f(t)} < \infty$, the limit $\lim_{t\to 0^+} f'(t) \int_t^\infty \frac{ds}{f(s)} := \mathcal{L}$ exists and satisfies $\mathcal{L} > 0$;

($\mathbf{f_{03}}$) there exist $\gamma > 1$, $t_0 \geq 0$ such that $t \mapsto f(t)/t^\gamma$ is increasing if $t \geq t_0$,

the weight b satisfies

($\mathbf{b_{01}}$) $b \in C^\alpha_{\text{loc}}(\Omega)$ ($\alpha \in (0,1)$) is positive in bounded domain Ω;

($\mathbf{b_{02}}$) there exists $\beta \in (0,2)$ such that $\lim_{d(x)\to 0} b(x)(d(x))^\beta = b_0 > 0$,

then García-Melián [18] derived that

(**i**) when $\mathcal{L} > 1$, every large solution u of Eq. (1.1) with $p = 2$ in bounded domain Ω satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{\Phi(A(d(x))^{2-\beta})} = 1 \quad \text{with} \quad A = \frac{b_0}{(2-\beta)((2-\beta)(\mathcal{L}-1)+1)},$$

and Φ satisfies

$$\int_{\Phi(t)}^\infty \frac{ds}{f(s)} = t, \qquad t > 0;$$

(**ii**) when $\mathcal{L} = 1$ and $tf'(\Phi(t)) \geq 1$ for small enough $t > 0$, (**i**) still holds.

When $b \equiv 1$ in a bounded domain Ω, the existence of large solutions to Eq. (1.1) was first studied by Diaz and Letelier [16] for $f(u) = u^\gamma$ ($\gamma > p-1$). Then, Matero [33] studied the existence and asymptotic behavior of large solutions to Eq. (1.1) in a bounded smooth domain with a C^2-boundary. If $b \equiv 1$ in bounded domain $\Omega \subseteq \mathbb{R}^N$ and f is a smooth, positive, and increasing function which satisfies ($\mathbf{f_2}$), Gladiali and Porru [21] showed that if $F(t)t^{-p}$ is increasing for large t, then any weak solution u to problem (1.1) satisfies

$$|u(x) - \psi(d(x))| < cd(x)\psi(d(x)) \quad \text{near } \partial\Omega$$

with

$$\int_{\psi(t)}^\infty [(p/(p-1))F(s)]^{-1/p} ds = t, \qquad t > 0. \tag{1.3}$$

Furthermore, they showed that, under the additional assumption $F(t)t^{-2p} \to \infty$ as $t \to \infty$, one obtains

$$u(x) - \psi(d(x)) \to 0 \quad \text{as } d(x) \to 0.$$

If b is non-negative and continuous on a bounded domain $\Omega \subseteq \mathbb{R}^N$ and satisfies some appropriate additional condition, Mohammed [36] established the existence and asymptotic behavior of large solutions to Eq. (1.1). Then, when b satisfies some suitable integral condition and $p \in (1, N)$ ($N \geq 2$), Covei [15] studied the existence of entire large solutions to Eq. (1.1) in $\mathbb{R}^N$. On the other hand, for the cases of $f(u) = u^\gamma$ with $\gamma > p-1$ and $f(u) = e^u$, García-Melián [19, 20] investigated, respectively, the existence, uniqueness and asymptotic behavior of boundary blow-up solutions to Eq. (1.1) in a smooth bounded domain.

In different direction, by applying Karamata regular variation theory Cîrstea and Rădulescu [11–14] opened up a unified new approach to studied the boundary behavior and uniqueness of large solutions to Eq. (1.1) with $p = 2$ in a bounded domain, which enables us to obtain some significant information about the qualitative behavior of large solutions in a general framework. Later, Mohammed [37], Zhang et al. [46], Zhang [47–49], Huang et al. [23], Huang [24], Mi et al. [34], Mi and Liu [35] apply similar techniques to further study asymptotic behavior and uniqueness of boundary blow-up solutions to (1.1) in a bounded domain $\Omega \subseteq \mathbb{R}^N$. Most recently, inspired by the above works, we [44] investigated the asymptotic behavior of entire large solutions to Eq. (1.1) with $p = 2$ in $\mathbb{R}^N$ by using Karamata regular variation theory.

For further insight on Eq. (1.1), we refer the interested reader to the papers [1–4,6,9,17,22, 30,38,43] and the references therein.

In this paper, we investigate the exact asymptotic behavior and uniqueness of entire large solutions to (1.1) in $\mathbb{R}^N$. Let f satisfy $(\mathbf{f_1})$–$(\mathbf{f_2})$, ψ be the solution of (1.3), we conclude by Lemmas 3.1 and 3.2 $(\mathbf{v})$ that

- $(\mathbf{i})$ if $(\mathbf{f_3})$ holds, then $C_f \leq 1/p$;
- $(\mathbf{ii})$ if $(\mathbf{f_3})$ holds with $C_f = 1/p$, then f is rapidly varying to infinity at infinity (please refer to Definition 2.2);
- $(\mathbf{iii})$ $(\mathbf{f_3})$ holds with $C_f \in (0, 1/p)$ if and only if $f \in RV_{(p(1+C_f)-1)/(1-pC_f)}$ (please refer to Definition 2.1);
- $(\mathbf{iv})$ if $f \in RV_{p-1}$, then $(\mathbf{f_3})$ holds with $C_f = 0$ and in the case, ψ is rapidly varying to infinity at zero (please refer to Definition 2.3).

Our results are summarized as follows.

Theorem 1.1. *Let f satisfy $(\mathbf{f_1})$–$(\mathbf{f_3})$, b satisfy $(\mathbf{b_1})$–$(\mathbf{b_2})$.*

$(\mathbf{I})$ *If $C_f \in (0, 1/p]$ in $(\mathbf{f_3})$, then any entire large solution u of problem (1.1) satisfies*

$$\begin{aligned} \xi_2^{(pC_f-1)/pC_f} &\leq \liminf_{|x|\to\infty} \frac{u(x)}{\psi\Big(\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)} \\ &\leq \limsup_{|x|\to\infty} \frac{u(x)}{\psi\Big(\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)} \leq \xi_1^{(pC_f-1)/pC_f}, \end{aligned} \tag{1.4}$$

where ψ is uniquely determined by (1.3) and

$$\xi_i = \left(\frac{b_i p^p}{(p-1)^{p-3}[\rho(\lambda,p,N)pC_f + (p-1)^2(\lambda-p)]}\right)^{1/p}, \qquad i = 1, 2$$

with

$$\rho(\lambda,p,N) := p^2(N+1) + \lambda(3-p^2) - p(N+3).$$

In particular,

$(\mathbf{i})$ *when $b_1 = b_2 = b_0$ in $(\mathbf{b_2})$*

$$\begin{aligned} &\lim_{|x|\to\infty} \frac{u(x)}{\psi\Big(\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)} \\ &= \left(\frac{b_0 p^p}{(p-1)^{p-3}[\rho(\lambda,p,N)pC_f + (p-1)^2(\lambda-p)]}\right)^{(pC_f-1)/p^2C_f}. \end{aligned}$$

(**ii**) *when* $C_f = 1/p$ *in* $(\mathbf{f_3})$

$$\lim_{|x|\to\infty} \frac{u(x)}{\psi\big(\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\big)} = 1.$$

(**II**) *If* $C_f = 0$ *in* $(\mathbf{f_3})$, *then any entire large solution* u *of problem* (1.1) *satisfies*

$$\begin{aligned} &\lim_{\varepsilon\to 0}\limsup_{|x|\to\infty} \frac{u(x)}{\psi\big(\tau_1\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\big)} \le 1; \\ &\lim_{\varepsilon\to 0}\liminf_{|x|\to\infty} \frac{u(x)}{\psi\big(\tau_2\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\big)} \ge 1, \end{aligned} \tag{1.5}$$

where $\tau_1 = \big(\xi_1^p - \varepsilon\xi_1^p/b_1\big)^{1/p}$, $\tau_2 = \big(\xi_2^p + \varepsilon\xi_2^p/b_2\big)^{1/p}$ *with* $\xi_i = \big(\frac{b_i p^p}{(p-1)^{p-1}(\lambda-p)}\big)^{1/p}$, $i = 1, 2$.

Theorem 1.2. *Let* f *satisfy* $(\mathbf{f_1})$–$(\mathbf{f_3})$ *with* $C_f \in (0, 1/p]$ *and further satisfy the condition that*

$(\mathbf{f_4})$ $t \mapsto f(t)t^{1-p}$ *is nondecreasing on* $(0,\infty)$,

b *satisfy* $(\mathbf{b_1})$–$(\mathbf{b_2})$ *with* $b_1 = b_2$, *then problem* (1.1) *possesses a unique entire large solution.*

Remark 1.3. $k \in \mathcal{K}$ is normalized slowly varying at infinity and $\lim_{t\to\infty} \frac{tk'(t)}{k(t)} = 0$.

Remark 1.4. Some basic examples of the functions which satisfy $(\mathbf{f_1})$–$(\mathbf{f_3})$ are

(**1**) Let

$$F(t) = \begin{cases} 0, & \text{if } t = 0; \\ t^p(\ln t)^{p\beta}, & \text{if } t \in (0,\infty), \end{cases}$$

where $\beta > 1$. Then a direct calculation shows that $C_f = 0$ and

$$\psi(t) = \exp\big((((p-1)/p)^{1/p}(\beta-1)t)^{1/(1-\beta)}\big),\ t > 0.$$

(**2**) Let

$$F(t) = \begin{cases} \tilde{F}(t), & \text{if } t \in [0,e]; \\ t^{p+\beta}(1 + c_0(\ln t)^{-1}), & \text{if } t \in (e,\infty), \end{cases}$$

where $c_0 \ge 0$, $\beta > 0$, $\tilde{F} \in C^1[0,e]$ is a differential continuation of the function $t \mapsto t^{p+\beta}(1 + c_0(\ln t)^{-1})$ on (e,∞), which satisfies $\tilde{F}$ and $\tilde{F}'$ are increasing on $[0,e]$ and

$$\begin{aligned} &\tilde{F}(0) = \tilde{F}'(0) = 0, \qquad \tilde{F}(e) = (1 + c_0)\exp\big(p+\beta\big), \\ &\tilde{F}'(e) = (p + \beta + (p+\beta-1)c_0)\exp\big(p+\beta-1\big). \end{aligned}$$

In the case, a simple calculation shows that $C_f = \beta/(p(p+\beta))$. Since

$$((p-1)/p)^{1/p}\int_t^{\infty} \big(s^{p+\beta}(1 + c_0(\ln s)^{-1})\big)^{-1/p} ds \sim ((p-1)/p)^{1/p}(p/\beta)t^{-\beta/p}, \qquad t \to \infty,$$

we arrive at

$$\psi(t) \sim \big(((p-1)/p)^{-1/p}(\beta/p)t\big)^{-p/\beta}, \qquad t \to 0^+.$$

(**3**)

$$F(t)=\begin{cases}0, & \text{if } t=0;\\ \exp(\beta)\exp\left(\beta(1-t^{-1})\right), & \text{if } t\in(0,1];\\ \exp(\beta t), & \text{if } t\in(1,\infty),\end{cases}$$

where $\beta\geq 2$. In the case, a straightforward calculation shows that $C_f=0$ and

$$\psi(t)=-(p/\beta)\ln\left(((p-1)/p)^{1/p}(\beta/p)t\right)\sim-(p/\beta)\ln t,\qquad t\to 0^+.$$

The paper is organized as follows. In Section 2, we give some bases of Karamata regular variation theory. In Section 3, we collect some preliminary considerations. The proof of Theorem 1.1 is given in Section 4. Finally, Section 5 is devoted to prove the uniqueness of entire large solutions.

2 Some basic facts from Karamata regular variation theory

In this section, we introduce some preliminaries of Karamata regular variation theory which come from [32,41,42].

Definition 2.1. A positive continuous function f defined on $[a,\infty)$, for some $a>0$, is called **regularly varying at infinity** with index μ, denoted by $f\in RV_\mu$, if for each $\xi>0$ and some $\mu\in\mathbb{R}$,

$$\lim_{t\to\infty}\frac{f(\xi t)}{f(t)}=\xi^\mu. \tag{2.1}$$

In particular, when $\mu=0$, f is called **slowly varying at infinity**.

Clearly, if $f\in RV_\mu$, then $L(t):=f(t)/t^\mu$ is slowly varying at infinity.

We also see that a positive continuous function h defined on $(0,a)$ for some $a>0$, is **regularly varying at zero** with index μ (written as $h\in RVZ_\mu$) if $t\to h(1/t)\in RV_{-\mu}$.

Definition 2.2. A positive continuous function f defined on $[a,\infty)$, for some $a>0$, is called rapidly varying to infinity at infinity if

$$\lim_{t\to\infty}\frac{f(t)}{t^\mu}=\infty\quad\text{for each }\mu>0.$$

Definition 2.3. A positive continuous function h defined on $(0,a]$, for some $a>0$, is called rapidly varying to infinity at zero if

$$\lim_{t\to 0^+}h(t)t^\mu=\infty\quad\text{for each }\mu>0.$$

Proposition 2.4 (Uniform convergence theorem). *If $f\in RV_\mu$, then (2.1) holds uniformly for $\xi\in[c_1,c_2]$ with $0<c_1<c_2$.*

Proposition 2.5 (Representation theorem). *A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(t)=\varphi(t)\exp\left(\int_{a_1}^t\frac{y(s)}{s}ds\right),\qquad t\geq a_1,$$

for some $a_1 \geq a$, where the functions φ and y are continuous and for $t \to \infty$, $y(t) \to 0$ and $\varphi(t) \to c_0$, with $c_0 > 0$. If $\varphi \equiv c_0$, then L is called **normalized** *slowly varying at infinity and*

$$f(t) = t^\mu \hat{L}(t), \qquad t \geq a_1,$$

is called **normalized** *regularly varying at infinity with index μ (written as $f \in NRV_\mu$).*

A function $f \in NRV_\mu$ if and only if

$$f \in C^1[a_1, \infty), \quad \text{for some} \quad a_1 > 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{tf'(t)}{f(t)} = \mu.$$

Proposition 2.6 (Asymptotic behavior). *If a function L is slowly varying at infinity, then for $t \to \infty$,*

$$\int_t^\infty s^\mu L(s)ds \sim (-\mu - 1)^{-1} t^{1+\mu} L(t), \quad \text{for } \mu < -1.$$

3 Auxiliary results

In this section, we collect some useful results.

Lemma 3.1. *Let f satisfy $(\mathbf{f_1})$–$(\mathbf{f_2})$.*

- $(\mathbf{i})$ *If f satisfies $(\mathbf{f_3})$, then $C_f \leq 1/p$.*
- $(\mathbf{ii})$ *If $(\mathbf{f_3})$ holds with $C_f \in (0, 1/p)$ if and only if $f \in RV_{(p(1+C_f)-1)/(1-pC_f)}$.*
- $(\mathbf{iii})$ *If $f \in RV_{p-1}$, then $(\mathbf{f_3})$ holds with $C_f = 0$.*
- $(\mathbf{iv})$ *If $(\mathbf{f_3})$ holds with $C_f = 1/p$, then f is rapidly varying to infinity at infinity.*

Proof. $(\mathbf{i})$ Let

$$J(t) = ((F(t))^{1/p})' \int_t^\infty (F(s))^{-1/p} ds, \qquad t > 0.$$

Integrate J from $a > 0$ to $t > a$ and integrate by parts, we obtain that

$$\int_a^t J(s)ds = (F(t))^{1/p} \int_t^\infty (F(s))^{-1/p} ds - (F(a))^{1/p} \int_a^\infty (F(s))^{-1/p} ds + t - a, \qquad t > a. \tag{3.1}$$

It follows by L'Hospital's rule that

$$0 \leq \lim_{t\to\infty} \frac{(F(t))^{1/p}}{t} \int_t^\infty (F(s))^{-1/p} ds = \lim_{t\to\infty} J(t) - 1, \tag{3.2}$$

i.e.,

$$\lim_{t\to\infty} \frac{(F(t))^{(p-1)/p}}{f(t) \int_t^\infty (F(s))^{-1/p} ds} \leq 1/p.$$

$(\mathbf{ii})$ (Necessity.) By (3.1) and L'Hospital's rule, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{F(t)}{tf(t)} &= \lim_{t\to\infty} \frac{(F(t))^{(p-1)/p}}{f(t) \int_t^\infty (F(s))^{-1/p} ds} \cdot \frac{(F(t))^{1/p}}{t} \int_t^\infty (F(s))^{-1/p} ds \\ &= \lim_{t\to\infty} \frac{(F(t))^{(p-1)/p}}{f(t) \times \int_t^\infty (F(s))^{-1/p} ds} \\ &\quad \times \left(\frac{\int_a^t J(s)ds}{t} + t^{-1} (F(a))^{1/p} \int_a^\infty (F(s))^{-1/p} ds - \frac{t-a}{t} \right) \\ &= \lim_{t\to\infty} \frac{(F(t))^{(p-1)/p} (J(t) - 1)}{f(t) \int_t^\infty (F(s))^{-1/p} ds} = (1 - pC_f)/p. \end{aligned} \tag{3.3}$$

So, $F \in NRV_{p/(1-pC_f)}$, i.e., there exist a large constant $t_0 > 0$ and a slowly varying function at infinity $\hat{L} \in C^2[t_0, \infty)$ such that

$$F(t) = t^{p/(1-pC_f)}\hat{L}(t), \qquad t \in [t_0, \infty),$$

where

$$\hat{L}(t) = c\exp\left(\int_{t_0}^{t} \frac{y(s)}{s}ds\right) \quad \text{with } c > 0,\ y \in C^1([t_0,\infty)) \text{ and } \lim_{t\to\infty} y(t) = 0.$$

Furthermore, we have

$$f(t) = t^{(p(1+C_f)-1)/(1-pC_f)}\big((p/(1-pC_f)) + y(t)\big)\hat{L}(t), \qquad t \in [t_0, \infty),$$

i.e.,

$$f \in RV_{(p(1+C_f)-1)/(1-pC_f)}.$$

(Sufficiency). Let

$$F(t) = \int_0^t f(s)ds = \int_0^1 tf(t\tau)d\tau.$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to\infty} \frac{F(t)}{tf(t)} = \lim_{t\to\infty} \int_0^1 \frac{f(t\tau)}{f(t)}d\tau = \int_0^1 \tau^{(p(1+C_f)-1)/(1-pC_f)}d\tau = (1-pC_f)/p. \tag{3.4}$$

This implies that $F \in p/(1-pC_f)$. On the other hand, by using reduction to absurdity we can see that

$$\lim_{t\to\infty} \frac{(F(t))^{1/p}}{t} = \infty. \tag{3.5}$$

Combining $(\mathbf{f_2})$ with (3.5) we can apply L'Hospital's rule to obtain

$$\lim_{t\to\infty} \frac{t(F(t))^{-1/p}}{\int_t^\infty (F(s))^{-1/p}ds} = \lim_{t\to\infty} \left(\frac{tf(t)}{pF(t)} - 1\right) = \frac{pC_f}{1-pC_f}.$$

This together with (3.4) implies (1.2) holds.

$(\mathbf{iii})$ From the similar calculation as (3.4), we arrive at

$$\lim_{t\to\infty} \frac{F(t)}{tf(t)} = 1/p. \tag{3.6}$$

On the other hand, by using L'Hospital's rule, we have

$$\lim_{t\to\infty} \frac{t(F(t))^{-1/p}}{\int_t^\infty (F(s))^{-1/p}ds} = \lim_{t\to\infty} \left(\frac{tf(t)}{pF(t)} - 1\right) = 0. \tag{3.7}$$

We conclude by (3.6)-(3.7) that $(\mathbf{f_3})$ holds with $C_f = 0$.

$(\mathbf{iv})$ When $C_f = 1/p$, from the similar calculation as (3.3), we can see that

$$\lim_{t\to\infty} \frac{F(t)}{tf(t)} = 0. \tag{3.8}$$

So, for an arbitrary $\gamma > 1$, there exists $t_0 > 0$ such that

$$\frac{f(t)}{F(t)} > (1+\gamma)t^{-1}, \qquad t \geq t_0.$$

Integrating the above inequality from t_0 to t, we obtain

$$\ln F(t) - \ln F(t_0) > (1+\gamma)(\ln t - \ln t_0), \qquad t \geq t_0,$$

i.e.,

$$\frac{F(t)}{t^\gamma} > \frac{F(t_0)t}{t_0^{1+\gamma}}, \qquad t \geq t_0.$$

Letting $t \to \infty$, the Definition 2.2 shows that F is rapidly varying at infinity. This combined with $(\mathbf{f_1})$ shows that f is also rapidly varying at infinity. □

Lemma 3.2. *Let f satisfy $(\mathbf{f_1})$–$(\mathbf{f_3})$ and ψ is the solution of problem (1.3). Then*

- $(\mathbf{i})$ $\psi'(t) = -\big((p/(p-1))F(\psi(t))\big)^{1/p}$, $|\psi'(t)|^{p-2}\psi''(t) = (p-1)^{-1}f(\psi(t))$, $\psi(t) > 0$, $t > 0$;
- $(\mathbf{ii})$ $\lim_{t\to 0^+} \psi(t) = \infty$;
- $(\mathbf{iii})$ $\lim_{t\to 0^+} \frac{\psi'(t)}{t\psi''(t)} = -pC_f$;
- $(\mathbf{iv})$ $\lim_{t\to 0^+} \frac{t\psi'(t)}{\psi(t)} = -\frac{1-pC_f}{pC_f}$, *where* $C_f \in (0, 1/p]$;
- $(\mathbf{v})$ *when $C_f = 0$ in $(\mathbf{f_3})$, ψ is rapidly varying at zero.*

Proof. $(\mathbf{i})$ By the definition of ψ and a straightforward calculation, we can show that $(\mathbf{i})$–$(\mathbf{ii})$ holds.
$(\mathbf{iii})$

$$\lim_{t\to 0^+} \frac{\psi'(t)}{t\psi''(t)} = \lim_{t\to 0^+} (1-p)\frac{((p/(p-1))F(\psi(t)))^{(p-1)/p}}{f(\psi(t))\int_{\psi(t)}^{\infty}(((p-1)/p)F(s))^{-1/p}ds} = -pC_f.$$

$(\mathbf{iv})$ By using (3.1) and L'Hospital's rule, we obtain

$$\begin{aligned}\lim_{t\to 0^+} \frac{t\psi'(t)}{\psi(t)} &= -\lim_{t\to\infty} \frac{(F(t))^{1/p}}{t}\int_t^\infty (F(s))^{-1/p}ds \\ &= -\lim_{t\to\infty} \frac{f(t)\int_t^\infty (F(s))^{-1/p}ds}{p(F(t))^{(p-1)/p}} + 1 = -\frac{1-pC_f}{pC_f}.\end{aligned} \tag{3.9}$$

$(\mathbf{v})$ It follows by the similar calculation as (3.9) that

$$\lim_{t\to 0^+} \frac{t\psi'(t)}{\psi(t)} = -\infty.$$

Hence, for an arbitrary $\gamma > 0$, there exists a small enough $t_0 > 0$ such that

$$-\frac{\psi'(t)}{\psi(t)} > (1+\gamma)t^{-1}, \qquad t \in (0, t_0].$$

Integrate it from t to t_0, we obtain that

$$\ln\big(\psi(t)\big) - \ln\big(\psi(t_0)\big) > (1+\gamma)(\ln t_0 - \ln t), \qquad t \in (0, t_0],$$

i.e.,

$$\psi(t)t^\gamma > \psi(t_0)t_0^{1+\gamma}t^{-1}, \qquad t \in (0, t_0].$$

Letting $t \to 0^+$, we see by Definition 2.3 that ψ is rapidly varying to infinity at zero. □

Lemma 3.3 ([8, Lemma 2.4])**.** *Let $k \in \mathcal{K}$, then*

$$\lim_{t\to\infty} \frac{k(t)}{\int_{t_0}^{t} \frac{k(s)}{s} ds} = 0.$$

If further $\int_{t_0}^{\infty} \frac{k(s)}{s} ds < \infty$, then

$$\lim_{t\to\infty} \frac{k(t)}{\int_{t}^{\infty} \frac{k(s)}{s} ds} = 0.$$

Lemma 3.4 (Weak comparison principle)**.** *Let Ω be a bounded domain and $G : \Omega \times \mathbb{R} \to \mathbb{R}$ be non-increasing in the second variable and continuous. Let $u, w \in W^{1,p}(\Omega)$ satisfy the respective inequalities*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx \le \int_\Omega G(x,u)\varphi dx; \tag{3.10}$$

$$\int_\Omega |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi dx \ge \int_\Omega G(x,w)\varphi dx, \tag{3.11}$$

for all non-negative $\varphi \in W_0^{1,p}(\Omega)$. Then the inequality $u \le w$ on $\partial\Omega$ implies $u \le w$ in Ω.

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Proof. Let $\varepsilon \in (0, b_1(p-1)/2)$ and

$$\tau_1 = \left(\xi_1^p - \varepsilon\xi_1^p/(b_1(p-1))\right)^{1/p}, \qquad \tau_2 = \left(\xi_2^p + \varepsilon\xi_2^p/(b_2(p-1))\right)^{1/p}.$$

It follows that

$$(1/2)^{1/p}\xi_1 < \tau_1 < \tau_2 < (3/2)^{1/p}\xi_2.$$

For any constant $R > t_0$ (t_0 is given by the definition $\mathcal{K}$), we define $\Omega_R := \{x \in \mathbb{R}^N : |x| > R\}$.

From $(\mathbf{b_1})$–$(\mathbf{b_2})$, Proposition 2.6 and Lemma 3.2 (**iii**), we see that corresponding to ε, there exist sufficiently small $\delta_\varepsilon > 0$ and large enough $R_\varepsilon > 0$ such that for any $(x,r) \in \Omega_{R_\varepsilon} \times (0, 2\delta_\varepsilon)$,

$$\begin{aligned} I_i(x,r) = &\left(\tau_i^p((p-1)/p)^p(p-1) - \tau_i^p((p-1)/p)^{p-1}((3-p)/p)\frac{\psi'(r)}{r\psi''(r)}\right) \\ &\times \left|\left(\frac{|x|^{(p-\lambda)/(p-1)}k(|x|)}{\int_{|x|}^{\infty} t^{(1-\lambda)/(p-1)}k(t)dt} - \frac{\lambda-p}{p-1}\right)\right| \\ &- \tau_i^p((p-1)/p)^{p-1}(p-1)\frac{\psi'(r)}{r\psi''(r)}\frac{|x|k'(|x|)}{k(|x|)} \\ &+ \left(\frac{\tau_i}{p}\right)^p \rho(\lambda,p,N)(p-1)^{p-2}\left|\frac{\psi'(r)}{r\psi''(r)} + pC_f\right| \le \varepsilon/2 \end{aligned} \tag{4.1}$$

and

$$|x|^{-\lambda}k(|x|)\big(b_1 - \varepsilon/(2(p-1))\big) < b(x) < |x|^{-\lambda}k(|x|)\big(b_2 + \varepsilon/(2(p-1))\big), \qquad x \in \Omega_{R_\varepsilon}. \tag{4.2}$$

Take

$$\sigma \in (0,\delta_\varepsilon) \quad \text{with } \sigma < (1/2)^{1/p}\xi_1\left(\int_{R_\varepsilon}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\right)^{(p-1)/p}$$

and let u be an arbitrary entire large solution of Eq. (1.1).
Define

$$D_-^\sigma := \Omega_{R_\varepsilon} \setminus \Omega_-^\sigma, \qquad D_+^\sigma = \Omega_{R_\varepsilon} \setminus \Omega_+^\sigma,$$

where

$$\Omega_-^\sigma := \left\{ x \in \Omega_{R_\varepsilon} : \tau_1 \left(\int_{|x|}^\infty s^{(1-\lambda)/(p-1)} k(s) ds \right)^{(p-1)/p} \le \sigma \right\}$$

and

$$\Omega_+^\sigma := \left\{ x \in \Omega_{R_\varepsilon+1} : \psi\left(\tau_2 \left(\int_{|x|}^\infty s^{(1-\lambda)/(p-1)} k(s) ds \right)^{(p-1)/p} + \sigma \right) \le u(x) \right\}. \tag{4.3}$$

We may as well assume that

$$(3/2)^{(p-1)/p} \xi_2 \left(\int_{|x|}^\infty s^{(1-\lambda)/(p-1)} k(s) ds \right)^{(p-1)/p} < \delta_\varepsilon, \qquad x \in \Omega_{R_\varepsilon}$$

and set

$$\overline{u}_\varepsilon = \psi\left(\tau_1 \left(\int_{|x|}^\infty s^{(1-\lambda)/(p-1)} k(s) ds \right)^{(p-1)/p} - \sigma \right),$$
$$\underline{u}_\varepsilon = \psi\left(\tau_2 \left(\int_{|x|}^\infty s^{(1-\lambda)/(p-1)} k(s) ds \right)^{(p-1)/p} + \sigma \right).$$

A straightforward calculation combined with (4.1) and (4.2) shows that for any D_-^σ

$$\begin{aligned} &\Delta_p \overline{u}_\varepsilon - b(x) f(\overline{u}_\varepsilon) \\ &\quad \le (-\psi'(r))^{p-2} \psi''(r) |x|^{-\lambda} k^{p-1}(|x|) \Bigg[I_1(x,r) - \left(\frac{b(x)}{|x|^{-\lambda} k^{p-1}(|x|)} - b_1 \right)(p-1) \\ &\qquad + \left(\frac{\tau_1}{p} \right)^p (p-1)^{p-2} \big(\rho(\lambda,p,N) p C_f + (p-1)^2 (\lambda - p) \big) - b_1 (p-1) \Bigg] \le 0 \end{aligned}$$

with

$$r = \tau_1 \left(\int_{|x|}^\infty t^{(1-\lambda)/(p-1)} k(t) dt \right)^{(p-1)/p} - \sigma.$$

This implies that $\overline{u}_\varepsilon$ is a supersolution of Eq. (1.1) in D_-^σ.

In a similar way, we can show that $\underline{u}_\varepsilon$ is a subsolution of Eq. (1.1) in D_σ^+.

We assert that there exists a large constant $M > 0$ independent of σ such that

$$u(x) \le \overline{u}_\varepsilon(x) + M, \qquad x \in D_-^\sigma \tag{4.4}$$

and

$$\underline{u}_\varepsilon(x) \le u(x) + M, \qquad x \in \Omega_{R_\varepsilon}. \tag{4.5}$$

In fact, we can choose a positive constant M independent of σ such that when $x \in \{x \in \mathbb{R}^N : |x| = R_\varepsilon\}$, we have

$$u(x) \le \overline{u}_\varepsilon(x) + M \tag{4.6}$$

and

$$\underline{u}_\varepsilon(x) \le u(x) + M. \tag{4.7}$$

Moreover, we also see

$$u(x) < \overline{u}_\varepsilon = \infty,\ x \in \left\{ x \in \mathbb{R}^N : \tau_1\Big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)} k(s) ds \Big)^{(p-1)/p} = \sigma \right\}.$$

This implies that, we can take a sufficiently small $\rho > 0$ such that

$$\sup_{x \in D_-^\sigma} u(x) \le \overline{u}_\varepsilon(x), \qquad x \in D_-^\sigma \setminus \tilde{D}_-^\sigma, \tag{4.8}$$

where

$$\tilde{D}_-^\sigma = \Omega_{R_\varepsilon} \setminus \tilde{\Omega}_-^\sigma$$

with

$$\tilde{\Omega}_-^\sigma = \left\{ x \in \Omega_{R_\varepsilon} : \tau_1\Big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)} k(s) \Big)^{(p-1)/p} \le \sigma(1+\rho) \right\}.$$

Combining (4.6) with (4.8), we have

$$u(x) \le \overline{u}_\varepsilon(x) + M, \qquad x \in \partial(\tilde{D}_-^\sigma).$$

On the other hand, we conclude by (4.7) and the definition of Ω_σ^+ (please refer to (4.3)) that

$$\underline{u}_\varepsilon(x) \le u(x) + M, \qquad x \in \partial(D_+^\sigma).$$

We note that u and $\underline{u}_\varepsilon$ both satisfy (3.10) in $\tilde{D}_-^\sigma$ and D_+^σ, respectively. Moreover, by $(\mathbf{f_1})$ we obtain that $\overline{u}_\varepsilon + M$ and $u + M$ are both supersolutions in $\tilde{D}_-^\sigma$ and D_+^σ, respectively. It follows by Lemma 3.10 that

$$u(x) \le \overline{u}_\varepsilon(x) + M, \qquad x \in \tilde{D}_-^\sigma \tag{4.9}$$

and

$$\underline{u}_\varepsilon(x) \le u(x) + M, \qquad x \in D_+^\sigma. \tag{4.10}$$

Indeed, (4.9) combined with (4.8) implies that (4.4) holds, and (4.10) together with (4.3) implies that (4.5) holds. Hence, letting $\sigma \to 0$, we have for $x \in \Omega_{R_\varepsilon}$,

$$\frac{u(x)}{\psi\Big(\tau_1\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)} k(s) ds\big)^{(p-1)/p}\Big)} \le 1 + \frac{M}{\psi\Big(\tau_1\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)} k(s) ds\big)^{(p-1)/p}\Big)};$$
$$\frac{u(x)}{\psi\Big(\tau_2\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)} k(s) ds\big)^{(p-1)/p}\Big)} \ge 1 - \frac{M}{\psi\Big(\tau_2\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)} k(s) ds\big)^{(p-1)/p}\Big)}.$$

Consequently, by Lemma 3.2 (**ii**), we have

$$\begin{aligned} &\limsup_{|x| \to \infty} \frac{u(x)}{\psi\Big(\tau_1\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)} k(s) ds\big)^{(p-1)/p}\Big)} \le 1; \\ &\liminf_{|x| \to \infty} \frac{u(x)}{\psi\Big(\tau_2\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)} k(s) ds\big)^{(p-1)/p}\Big)} \ge 1. \end{aligned} \tag{4.11}$$

If $C_f \in (0, 1/p]$, then it follows by Lemma 3.2 (**iv**) that

$$\limsup_{|x|\to\infty} \frac{u(x)}{\psi\Big(\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)}$$
$$= \limsup_{|x|\to\infty} \frac{u(x)}{\psi\Big(\tau_1\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)} \lim_{|x|\to\infty} \frac{\psi\Big(\tau_1\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)}{\psi\Big(\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)}$$
$$\le \tau_1^{(pC_f-1)/pC_f};$$

$$\liminf_{|x|\to\infty} \frac{u(x)}{\psi\Big(\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)}$$
$$= \liminf_{|x|\to\infty} \frac{u(x)}{\psi\Big(\tau_2\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)} \lim_{|x|\to\infty} \frac{\psi\Big(\tau_2\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)}{\psi\Big(\big(\int_{|x|}^{\infty} s^{(1-\lambda)/(p-1)}k(s)ds\big)^{(p-1)/p}\Big)}$$
$$\ge \tau_2^{(pC_f-1)/pC_f}.$$

Letting $\varepsilon \to 0$, we obtain (1.4).

If $C_f = 0$, then (4.11) implies that (1.5) holds. □

5 Proof of Theorem 1.2

Proof. The existence of entire large solutions follows from Theorem 1.3 of [15]. Inspired by the ideas of Mohammed in [37], we prove the uniqueness. Suppose u_1 and u_2 are entire large solutions of problem (1.1). It follows by Theorems 1.1 that

$$\lim_{|x|\to\infty} \frac{u_1(x)}{u_2(x)} = 1.$$

So, for fixed $\varepsilon > 0$, there exists a large constant R_ε such that

$$(1-\varepsilon)u_2(x) \le u_1(x) \le (1+\varepsilon)u_2(x), \qquad x \in \Omega_{R_\varepsilon}. \tag{5.1}$$

Define

$$u^{\pm}(x) = (1 \pm \varepsilon)u_2(x), \qquad x \in \mathbb{R}^N.$$

By using $(\mathbf{f_4})$, we obtain

$$\Delta_p u^+ \le b(x)f(u^+) \quad \text{and} \quad \Delta_p u^- \le b(x)f(u^-) \quad \text{in } \mathbb{R}^N.$$

Let u_0 is the unique solution of

$$\Delta_p u_0 = b(x)u_0, \quad x \in \Omega_0, \quad u|_{\partial\Omega_0} = u_1,$$

where $\Omega_0 = \mathbb{R}^N \setminus \Omega_{R_\varepsilon}$. We conclude by Lemma (3.4) that

$$u^-(x) \le u_0(x) \le u^+(x), \qquad x \in \Omega_0. \tag{5.2}$$

Noting $u_0 = u_1$ on Ω_0, so it follows by combining (5.1) with (5.2) that

$$(1-\varepsilon)u_2(x) \le u_1(x) \le (1+\varepsilon)u_2(x), \qquad x \in \mathbb{R}^N = \Omega_0 \cup \Omega_{R_\varepsilon}.$$

Letting $\varepsilon \to 0$, we obtain $u_1 = u_2$ in $\mathbb{R}^N$. □

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References

[1] R. Alsaedi, H. Mâagli, N. Zeddini, Existence and global behavior of positive solution for semilinear problems with boundary blow-up, *J. Math. Anal. Appl.* **425**(2015), No. 2, 818–826. MR3303894; url

[2] R. Alsaedi, H. Mâagli, N. Zeddini, Exact behavior of the unique positive solution to some singular elliptic problem in exterior domains, *Nonlinear Anal.* **119**(2015), 186–198. MR3334183; url

[3] C. O. Alves, C. A. Santos, J. Zhou, Blow-up solutions for a p-Laplacian elliptic equation of logistic type with singular nonlinearity, appeared on arXiv:1503.05406, 2015.

[4] C. Anedda, G. Porru, Boundary behaviour for solutions of boundary blow-up problems in a borderline case, *J. Math. Anal. Appl.* **352**(2009), No. 1, 35–47. MR2499883; url

[5] C. Bandle, M. Marcus, 'Large' solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, *J. Anal. Math.* **58**(1992), 9–24. MR1226934; url

[6] C. Bandle, Asymptotic behaviour of large solutions of quasilinear elliptic problems, *Z. Angew. Math. Phys.* **54**(2003), No. 5, 731–738. MR2019176; url

[7] L. Bieberbach, $\Delta u = e^u$ und die automorphen Funktionen (in German), *Math. Ann.* **77**(1916), No. 2, 173–212. MR1511854; url

[8] R. Chemmam, A. Dhifli, H. Mâagli, Asymptotic behavior of ground state solutions for sublinear and singular nonlinear Dirichlet problems, *Electron. J. Differential Equations*, **2011**, No. 88, 1–12. MR2821533

[9] K.-S. Cheng, W.-M. Ni, On the structure of the conformal scalar curvature equation on R^n, *Indiana Univ. Math. J.* **41**(1992), No. 1, 261–278. MR1160913; url

[10] F. Şt. Cîrstea, V. D. Rădulescu, Blow-up boundary solutions of semilinear elliptic problems, *Nonlinear Anal.* **48**(2002), No. 4, 521–534. MR1871465; url

[11] F.-C. Cîrstea, V. Rădulescu, Uniqueness of the blow-up boundary solution of logistic equations with absorbtion, *C. R. Math. Acad. Sci. Paris* **335**(2002), No. 5, 447–452. MR1937111; url

[12] F.-C. Cîrstea, V. Rădulescu, Asymptotics for the blow-up boundary solution of the logistic equation with absorption, *C. R. Math. Acad. Sci. Paris* **336**(2003), No. 3, 231–236. MR1968264; url

[13] F. C. Cîrstea, V. Rădulescu, Nonlinear problems with boundary blow-up: a Karamata regular variation theory approach, *Asymptot. Anal.* **46**(2006), No. 3-4, 275–298. MR2215886

[14] F.-C. Cîrstea, V. Rădulescu, Boundary blow-up in nonlinear elliptic equations of Bieberbach–Rademacher type, *Trans. Amer. Math. Soc.* **359**(2007), No. 7, 3275–3286. MR2299455; url

[15] D.-P. Covei, Large and entire large solution for a quasilinear problem, *Nonlinear Anal.* **70**(2009), No. 4, 1738–1745. MR2483595; url

[16] G. Diaz, R. Letelier, Explosive solutions of quasilinear elliptic equations: existence and uniqueness, *Nonlinear Anal.* **20**(1993), No. 2, 97–125. MR1200384; url

[17] L. Dupaigne, M. Ghergu, O. Goubet, G. Warnault, Entire large solutions for semilinear elliptic equations, *J. Differential Equations* **253**(2012), No. 7, 2224–2251. MR2946970; url

[18] J. García-Melián, Boundary behavior for large solutions to elliptic equations with singular weights, *Nonlinear Anal.* **67**(2007), No. 3, 818–826. MR2319212; url

[19] J. García-Melián, Large solutions for equations involving the p-Laplacian and singular weights, *Z. Angew. Math. Phys.* **60**(2009), No. 4, 594–607. MR2520601; url

[20] J. García-Melián, Quasilinear equations with boundary blow-up and exponential reaction, *Adv. Nonlinear Stud.* **9**(2009), No. 1, 165–176. MR2473154; url

[21] F. Gladiali, G. Porru, Estimates for explosive solutions to p-Laplace equations, in: *Progress in partial differential equations, Vol. 1 (Pont-à-Mousson, 1997)*, Pitman Res. Notes Math. Ser., Vol. 383, Longman, Harlow, 1998, pp. 117–127. MR1628068

[22] Z. Guo, J. Shang, Remarks on uniqueness of boundary blow-up solutions, *Nonlinear Anal.* **66**(2007), No. 2, 484–497. MR2279541; url

[23] S. Huang, Q. Tian, Asymptotic behavior of large solutions to p-Laplacian of Bieberbach–Rademacher type, *Nonlinear Anal.* **71**(2009), No. 11, 5773–5780. MR2560241; url

[24] S. Huang, Asymptotic behavior of boundary blow-up solutions to elliptic equations, *Z. Angew. Math. Phys.* **67**(2016), No. 1, 1–20. MR3452266; url

[25] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math.* **10**(1957), 503–510. MR0091407; url

[26] A. V. Lair, A necessary and sufficient condition for existence of large solutions to semilinear elliptic equations, *J. Math. Anal. Appl.* **240**(1999), No. 1, 205–218. MR1728197; url

[27] A. V. Lair, Nonradial large solutions of sublinear elliptic equations, *Appl. Anal.* **82**(2003), No. 5, 431–437. MR1982888; url

[28] A. V. Lair, Large solutions of semilinear elliptic equations under the Keller–Osserman condition, *J. Math. Anal. Appl.* **328**(2007), No. 2, 1247–1254. MR2290049; url

[29] A. C. Lazer, P. J. McKenna, On a problem of Bieberbach and Rademacher, *Nonlinear Anal.* **21**(1993), No. 5, 327–335. MR1237124; url

[30] Y. Li, W.-M. Ni, On conformal scalar curvature equations in $\mathbb{R}^n$, *Duke Math. J.* **57**(1988), No. 3, 895–924. MR975127; url

[31] C. Loewner, L. Nirenberg, Partial differential equations invariant under conformal or projective transformations, in: *Contributions to analysis (a collection of papers dedicated to Lipman Bers)*, Academic Press, New York, 1974, pp. 245–272. MR0358078

[32] V. Maric, *Regular variation and differential equations*, Lecture Notes in Mathematics, Vol. 1726, Springer-Verlag, Berlin, 2000. MR1753584; url

[33] J. Matero, Quasilinear elliptic equations with boundary blow-up, *J. Anal. Math.* **69**(1996), 229–247. MR1428101; url

[34] L. Mi, Y. Qi, B. Liu, Blow-up rate of the unique solution for a class of one-dimensional p-Laplacian equations, *Nonlinear Anal. Real World Appl.* **13**(2012), No. 6, 2734–2746. MR2927221; url

[35] L. Mi, B. Liu, Second order expansion for blowup solutions of semilinear elliptic problems, *Nonlinear Anal.* **75**(2012), No. 4, 2591–2613. MR2870942; url

[36] A. Mohammed, Existence and asymptotic behavior of blow-up solutions to weighted quasilinear equations, *J. Math. Anal. Appl.* **298**(2004), No. 2, 621–637. MR2086979; url

[37] A. Mohammed, Boundary asymptotic and uniqueness of solutions to the p-Laplacian with infinite boundary values, *J. Math. Anal. Appl.* **325**(2007), No. 1, 480–489. MR2273539; url

[38] W.-M. Ni, On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalizations, and applications in geometry, *Indiana Univ. Math. J.* **31**(1982), No. 4, 493–529. MR662915; url

[39] R. Osserman, On the inequality $\Delta u \geq f(u)$, *Pacific J. Math.* **7**(1957), 1641–1647. MR0098239; url

[40] H. Rademacher, E. Rothe, Einige besondere Probleme partieller Differentialgleichungen (in German), in: Ph. Frank, R. v. Mises (Eds.), *Die Differential- und Integralgleichungen der Mechanik und Physik. I.*, 2nd ed., Rosenberg, New York, 1943, pp. 838–845. MR0150433

[41] S. I. Resnick, Extreme values, regular variation, and point processes, Springer-Verlag, New York, Berlin, 1987. MR900810; url

[42] E. Seneta, *Regularly varying functions*, Lecture Notes in Mathematics, Vol. 508, Springer-Verlag, 1976. MR0453936

[43] L. Véron, Semilinear elliptic equations with uniform blow-up on the boundary, *J. Anal. Math.* **59**(1992), 231–250. MR1226963; url

[44] H. Wan, Asymptotic behavior of entire large solutions to semilinear elliptic equations, *J. Math. Anal. Appl.* **448**(2017), No. 1, 44–59. MR3579872; url

[45] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.* **12**(1960), 21–37. MR0125546

[46] Z. Zhang, Y. Ma, L. Mi, X. Li, Blow-up rates of large solutions for elliptic equations, *J. Differential Equations* **249**(2010), No. 1, 180–199. MR2644132; url

[47] Z. Zhang, L. Mi, Blow-up rates of large solutions for semilinear elliptic equations, *Commun. Pure Appl. Anal.* **10**(2011), No. 6, 1733–1745. MR2805336; url

[48] Z. Zhang, The second expansion of large solutions for semilinear elliptic equations, *Nonlinear Anal.* **74**(2011), No. 11, 3445–3457. MR2803072; url

[49] Z. Zhang, Boundary behavior of large solutions to p-Laplacian elliptic equations, *Nonlinear Anal. Real World Appl.* **33**(2017), 40–57. MR3543111; url