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Minimal positive solutions for systems of semilinear elliptic equations

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Abstract. The paper is devoted to a system of semilinear PDEs containing gradient terms. Applying the approach based on Sattinger's iteration procedure we use sub and supersolutions methods to prove the existence of positive solutions with minimal growth. These results can be applied for both sublinear and superlinear problems.

Keywords: semilinear elliptic problems, positive solutions, minimal solutions, sub and supersolutions methods.

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1 Introduction

We discuss the existence of minimal positive solutions for the following problem

$$\begin{cases} \Delta u(x) + f_1(x, u(x), v(x)) + g_1(\|x\|)x \cdot \nabla u(x) = 0, \\ \Delta v(x) + f_2(x, u(x), v(x)) + g_2(\|x\|)x \cdot \nabla v(x) = 0, \end{cases} \quad \text{for } x \in G_R, \tag{1.1}$$

subject to the limit conditions

$$\lim_{\|x\| \to \infty} u(x) = 0, \qquad \lim_{\|x\| \to \infty} v(x) = 0, \tag{1.2}$$

where $n > 2$, $R > 0$, $\|x\| := \sqrt{\sum_{i=1}^n x_i^2}$, $G_R = \{x \in \mathbb{R}^n,\ \|x\| > R\}$, $g_1, g_2 : [1, +\infty) \to \mathbb{R}$ are continuously differentiable and $g_1(l)$, $g_2(l)$ are positive for l sufficiently large.

There exists rich literature devoted to similar problems which arise in many applications e.g. in pseudoplastic fluids [6], reaction–diffusion processes or chemical heterogeneous catalysts [3], heat conduction in electrically conducting materials [7].

Here we have to mention also Constantin's results (see e.g. [8, 9]) describing the case of a single equation and further papers (e.g. [14–18]). Recently the research concerning the existence and properties of positive solutions of systems of nonlinear elliptic problems has been very active and enjoying of increasing interest (see e.g. [10–13, 21, 22, 25, 28] and references therein). The sub and supersolutions methods are applied in many of these papers (e.g. [11, 12, 21,22] or [25]). Other techniques are also met. In [13] the results are based on approximations. In [10] we can find the variational approach which allowed to show the existence of a few solutions of the following problem

$$\begin{aligned} -\Delta U(x) &= \nabla H(x, U(x)) \quad \text{in } \Omega \\ U(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $U = (u_1, u_2) : \Omega \to \mathbb{R}^2$, Ω is a bounded regular domain in $\mathbb{R}^n$, the right-hand side is a Carathéodory function and satisfies, among others, some growth conditions. The main result of the paper [10, Theorem 1.1] says that the above problem possesses at least nine nontrivial solutions $U = (u_1, u_2)$, satisfying the following sign conditions: both u_1 and u_2 are strictly positive or negative in the first four solutions; four others are such that one of the two components is of the one sign while the other is of changing sign, and finally both components change their sing in the ninth solution.

We also have to mention the paper [5], where we can find, among others, the results concerning the existence or nonexistence of radially symmetric solutions for the Emden–Fowler system involving p-Laplace operators and some real parameters. The approach is based on suitable transformations which play a crucial role in the reduction of the main problem to a quadratic system. Moreover, in the case when the main system is variational the behaviour of the ground states was described. Considering parameters satisfying additional conditions and applying a new type of energy function, the authors investigate the existence of ground states also in the case when the system is not variational.

Two further papers [20] and [19] are devoted to more general problems associated with elliptic inequalities. The first one describes the existence and nonexistence of nonnegative and nontrivial entire weak solutions for a single inequality. The approach is based on a generalized version of the Keller–Ossermann condition (see e.g. [4] and [23]). In the latter paper the system of elliptic inequalities of divergence type is investigated. The author obtains the results employing the method of test functions (see e.g. in [24]). These results can be applied in the case of p-Laplace operators as well as mean curvature operators.

The main motivation of this paper is a recent paper by Covei [12] in which also the Lane–Emden–Fowler system is investigated

$$\begin{aligned} -\Delta u(x) &= a_1(x) F_1(x, u(x), v(x)) \\ -\Delta v(x) &= a_2(x) F_2(x, u(x), v(x)) \\ u &= v = 0 \quad \text{in } \partial\Omega \end{aligned}$$

for bounded domains $\Omega \subset \mathbb{R}^n$ or $\Omega = \mathbb{R}^n$. The author assumes that for $i \in \{1, 2\}$, $F_i : \Omega \times (0, +\infty) \times (0, +\infty) \to (0, +\infty)$ is locally Hölder and satisfies the following estimates

$$F_i(x, t_1, t_2) \le g_i(t_i) \quad \text{for all } (x, t_1, t_2) \in \Omega \times (0, +\infty) \times (0, +\infty),$$

where $g_i : (0, +\infty) \to (0, +\infty)$ is continuous, the mapping $s \mapsto \frac{g_i(s)}{s}$ is decreasing on $(0, +\infty)$ and $\lim_{s\to+\infty} \frac{g_i(s)}{s} = 0$, and

$$F_i(x, t_1, t_2) \ge h_i(t_1, t_2) \quad \text{for all } (x, t_1, t_2) \in \Omega \times (0, T_1) \times (0, T_2),$$

where $h_i : (0, T_1) \times (0, T_2) \to (0, +\infty)$ is a continuous, nonincreasing function such that $\lim_{s\to+\infty} h_i(s, s) \in (0, +\infty]$, with $T_1, T_2 \in (0, 1)$.

Covei's results are also based on the sub and supersolutions method which was introduced by Keller and Amann. The existence result for solutions of elliptic problems under the assumption concerning the existence of subsolutions and supersolutions was first proved by H. Amann in [1]. One year later in [27], Sattinger proved similar theorems for regular elliptic boundary value problems using the L^p estimates of Agmon, Douglis and Nirenberg ([2]). In our paper we want to employ these classical ideas. Motivated by [12], we are interested in another type of nonlinearities. It is worth emphasizing that we do not require our system to be either potential or radial symmetric. Moreover we do not need either growth conditions on f_i or the equality $f_i(\cdot, 0, 0) \equiv 0$, $i = 1, 2$. Precisely, in the sequel we assume the following conditions

(A1) *for* $i = 1, 2$, $g_i : [1, +\infty) \to \mathbb{R}$ *belongs to* $C^1([1, +\infty))$ *and there exists* $l_0 \geq 1$ *such that* $g_i(l) \geq 0$ *for all* $l \geq l_0$;

(A2) *for* $i = 1, 2$, $f_i : G_1 \times \mathbb{R}^2 \to \mathbb{R}$, *where* $G_1 = \{x \in \mathbb{R}^n,\ \|x\| > 1\}$, *is locally Hölder continuous with exponent* $\alpha \in (0, 1)$, $x \mapsto f_i(x, 0, 0)$ *is positive and there exist* $d_1, d_2 > 0$ *and continuous functions* $\widetilde{f}_i$ *such that*

$$\sup_{(u,v)\in[0,d_1]\times[0,d_2]} \sup_{\|x\|=r} f_i(x, u, v) \leq \widetilde{f}_i(r)$$

for all $r \in [1, +\infty)$ *and*

$$\int_1^\infty r^{n-1}\widetilde{f}_i(r)dr \leq 4\,(n-2)\,d_i;$$

(A3) f_1 *is continuously differentiable in* u *and nondecreasing in* v *and* f_2 *is continuously differentiable in* v *and nondecreasing in* u *in* $G_R \times [0, d_1] \times [0, d_2]$;

(A4) *there exist* $A_1, A_2 > 0$ *and* $L_1, L_2 > 1$ *such that*

$$\begin{aligned} f_1(x, u, 0) &\geq A_1(n-2)g_1(x)\|x\|^{2-n} \quad \textit{for all } u \in [0, d_1] \textit{ and } \|x\| > L_1,\\ f_2(x, 0, v) &\geq A_2(n-2)g_2(x)\|x\|^{2-n} \quad \textit{for all } v \in [0, d_2] \textit{ and } \|x\| > L_2. \end{aligned}$$

Remark 1.1. Assumptions (A1)–(A4) are not too restrictive and many elementary functions satisfy them. It is easy to find many examples of f_1, f_2 satisfying (A2), (A3) and (A4) among functions of the form

$$f_i(x, u, v) = \overline{f}_i(u, v)\left(\|x\|^q + a(x)\right)^{-1},$$

where $q > n$, a is positive and sufficiently smooth and $\overline{f}_i$ is a polynomial, exponential or rational function or their combinations, e.g. $\overline{f}_i(u, v) = c(u^5 + u^4 + (u+v)^2 + 1)$ or $\overline{f}_i(u, v) = c\big(e^{u+v} + \frac{u^3+v^3}{(4-u)(5-v)}\big)$. We can also investigate problems of the Emden–Fowler type when $\overline{f}_i(u, v) = c(u^\alpha + v^\beta + M)$ with $\alpha, \beta, M > 0$. We will discuss an example of the problem with exponential and rational nonlinearities at the end of this paper.

It is worth emphasizing that we need the monotonicity and differentiability of f_i in u and/or v only on some right-hand neighborhood of the origin. Moreover, we can consider both sublinear and superlinear f_i which is associated with the fact that we have to control only the value of nonlinearities. Thus we can omit growth conditions concerning second and third variables. In the proof of the existence of a positive solution of (1.1) we do not need (A4). We use this condition only to show that the solution is minimal. Our main tool is Theorem 1.2 which says that the existence of a sub-subsolution $(\underline{u}, \underline{v})$ and a super-supersolution $(\overline{u}, \overline{v})$ of our problem such that $0 \le \underline{u} \le \overline{u} \le d_1$ and $0 \le \underline{v} \le \overline{v} \le d_2$ implies the existence of solution (u,v) of (1.1) which is squeezed between them. In the proof of Theorem 1.2 we apply classical ideas based on Sattinger's monotone iteration procedure. The iteration scheme is patterned after that in [22] where the existence results are proved in the case when the problem does not contain the gradient term. Thus we start with standard definitions of a solution, a super-supersolution and a sub-subsolution of (1.1)–(1.2) (see e.g. [22]).

By a solution of our problem we understand a pair $(u,v) \in C^{2+\alpha}_{\mathrm{loc}}(G_R) \times C^{2+\alpha}_{\mathrm{loc}}(G_R)$ satisfying (1.1) and vanishing at infinity, that is conditions (1.2). We say that a positive solution of our problem is minimal when the functions: $x \mapsto \|x\|^{n-2}u(x)$ and $x \mapsto \|x\|^{n-2}v(x)$ are bounded above and below by positive constants in some exterior domain (see, among others, [25]).

By a super-supersolution of (1.1)–(1.2) in G_R we understand a vector function $(\overline{u},\overline{v}) \in C^{2+\alpha}_{\mathrm{loc}}(G_R) \times C^{2+\alpha}_{\mathrm{loc}}(G_R)$ satisfying the following differential inequalities

$$\begin{cases} \Delta\overline{u}(x) + f_1(x,\overline{u}(x),\overline{v}(x)) + g_1(\|x\|)x\cdot\nabla\overline{u}(x) \le 0 \\ \Delta\overline{v}(x) + f_2(x,\overline{u}(x),\overline{v}(x)) + g_2(\|x\|)x\cdot\nabla\overline{v}(x) \le 0 \\ \lim_{\|x\|\to\infty}\overline{u}(x) = 0,\ \lim_{\|x\|\to\infty}\overline{v}(x) = 0. \end{cases}$$

Analogously, as for a sub-subsolution $(\underline{u},\underline{v})$ of (1.1)–(1.2) in G_R, the sign of the inequality should be reversed.

Since $f_i(x,0,0)$ is positive in G_1, $(\underline{u},\underline{v})$, where $\underline{u}(x) = \underline{v}(x) = 0$ in G_R, is the trivial sub-subsolution. We obtain the super-supersolution of our system as a radial solution of a certain auxiliary linear problem considered in the complement of the unit ball centered at the origin. Now we formulate the theorem which will be our main tool in the proof of the existence result.

Theorem 1.2. *Assume that conditions (A1)–(A3) hold and suppose that $(\overline{u},\overline{v})$ and $(\underline{u},\underline{v})$ are, respectively, a super-supersolution and a sub-subsolution of (1.1)–(1.2) such that*

$$0 \le \underline{u} \le \overline{u} \le d_1 \quad \text{and} \quad 0 \le \underline{v} \le \overline{v} \le d_2 \quad \text{on } \overline{G}_R.$$

Then there exists a vector function $(u^0,v^0) \in C^{2+\alpha}_{\mathrm{loc}}(G_R) \times C^{2+\alpha}_{\mathrm{loc}}(G_R)$ such that

$$\begin{cases} \Delta u^0(x) + f_1(x,u^0(x),v^0(x)) + g_1(\|x\|)x\cdot\nabla u^0(x) = 0 \\ \Delta u^0(x) + f_2(x,u^0(x),v^0(x)) + g_2(\|x\|)x\cdot\nabla v^0(x) = 0 \end{cases} \quad \text{on } G_R$$

$$\lim_{\|x\|\to\infty} u^0(x) = 0, \qquad \lim_{\|x\|\to\infty} v^0(x) = 0.$$

Moreover, $u^0(x) = \overline{u}(x)$ and $v^0(x) = \overline{v}(x)$ when $\|x\| = R$, and $\underline{u} \le u^0 \le \overline{u}$ and $\underline{v} \le v^0 \le \overline{v}$ on $\overline{G}_R$.

Remark 1.3. To prove the above result we will use the ideas described by Kawano in [22] for the case when the elliptic problem contains the gradient terms. We can follow his steps because of the fact that the differential operator is also linear in our problem. Although the proof of Theorem 1.2 is standard we present the sketch of the reasoning in the Appendix for the reader's convenience.

2 The existence of a super-supersolution

In this section we show the existence of a positive super-supersolution of (1.1) as radial solutions w_i, $i = 1, 2$, of the following auxiliary linear problems

$$\begin{cases} -\Delta w(x) = \widetilde{f}_i(\|x\|) & \text{for } x \in G_1, \\ w(x) = 0 \quad \text{for } \|x\| = 1, \ \lim_{\|x\|\to\infty} w(x) = 0. \end{cases} \tag{2.1}$$

We prove that w_i is radially decreasing in exterior domain G_R for R sufficiently large. We also describe more precisely the asymptotic behaviour of w_i. These properties will play the crucial role in the proof of the fact that (w_1, w_2) is a super-supersolution of our problem in a certain exterior domain and our solution of (1.1) has a minimal growth at infinity.

Lemma 2.1. *Assume (A1)–(A3). For each $i = 1, 2$, there exists a positive minimal radial solution w_i for (2.1) such that $w_i(x) \le d_i$ in G_1. Moreover, (w_1, w_2) is a super-supersolution of (1.1).*

Proof. Fix $i \in \{1, 2\}$. We start with the well-known fact that, via a suitable transformation, the investigation of the existence of a radial solution for the problem (2.1) leads, to the solvability of the Dirichlet problem with singularity at the end-point 1, namely

$$\begin{cases} -z''(t) = h_i(t)\,, & \text{in } (0,1) \\ z(0) = z(1) = 0 \end{cases} \tag{2.2}$$

with

$$h_i(t) = \frac{1}{(n-2)^2}\,(1-t)^{\frac{2n-2}{2-n}}\,\widetilde{f}_i((1-t)^{\frac{1}{2-n}})$$

for all $t \in (0,1)$. Precisely, when we have a radial solution $w(x) = \widetilde{z}(\|x\|)$ with $\widetilde{z} : [1, +\infty) \to \mathbb{R}$ of (2.1), we get the solution for (2.2) of the form $z(t) = \widetilde{z}\big((1-t)^{\frac{1}{2-n}}\big)$. Conversely, having a solution z of (2.2) we derive that $w(x) = z(1 - \|x\|^{2-n})$ satisfies (2.1). Assumptions made on f_i allow us to state that h_i is continuous, $h_i(\cdot) > 0$ and

$$\int_0^1 h_i(l)dl \le 4d_i. \tag{2.3}$$

Simple calculation leads to the conclusion that

$$z_i(t) = \int_0^1 \mathbf{G}(s,t)h_i(s)ds, \tag{2.4}$$

where

$$\mathbf{G}(s,t) := \begin{cases} s(1-t) & \text{for } 0 \le s \le t, \\ (1-s)t & \text{for } t < s \le 1, \end{cases} \quad s, t \in (0,1)\,,$$

is a solution of (2.2). Since $0 \le \mathbf{G}(s,t) \le \frac{1}{4}$ for all $(s,t) \in [0,1] \times [0,1]$ we get for all $t \in [0,1]$,

$$0 \le z_i(t) = \int_0^1 \mathbf{G}(s,t)h_i(s)ds \le d_i.$$

Taking into account the facts that the solution z_i of (2.2) is nontrivial and concave, we can state that z_i is positive in $(0,1)$. As in [17] and [26], we prove the existence of $\bar{t}_i \in (0,1)$ such that $z_i'(t) \le 0$ for all $t \in [\bar{t}, 1)$. Applying Rolle's theorem, we get that the set $S := \{t \in (0,1);$

$z_i'(t) = 0\}$ is nonempty. Since z_i' in nonincreasing in $(0,1)$, we have $z_i'(t) = 0$ in $[t_1, t_2]$ for all $t_1, t_2 \in S$ such that $t_1 \le t_2$. Thus S is an interval. Let $\bar{t}_i := \sup S$. It is easy to show that $\bar{t}_i \ne 1$. Indeed, if $\bar{t}_i = 1$ then, we derive the existence of a sequence $\{t_m\} \subset S$ such that $\lim_{m\to\infty} t_m = 1$. Without loss of generality we can assume that for all $m \in N$, $t_1 \le t_m$. Thus for all $m \in N$, $z_i'(t) = 0$ in $[t_1, t_m]$, and consequently $z_i'(t) = 0$ in $[t_1, 1)$, which gives, by the continuity of z_i in $[0,1]$, that $z_i(t) = 0$ in $[t_1, 1]$. But it is impossible with respect to the fact that z_i is positive. In consequence, for all $t \in [\bar{t}, 1)$, $z'(t) \le 0$.

Now we try to describe more precisely the behaviour of z_i in the left-hand neighborhood of 1. First we note that $z_i'(t) = -\int_0^1 sh_i(s)ds + \int_t^1 h_i(s)ds$ and further

$$(0, +\infty) \ni M := \int_0^1 s\bar{h}_i(s)ds = -\lim_{t\to 1^-} z_i'(t) = \lim_{t\to 1^-} \frac{z_i(t)}{(1-t)}.$$

To sum up, we proved the existence of the positive solution z_i of (2.2) which satisfies the following conditions

$$z_i'(t) \le 0 \quad \text{for all } t \in [\bar{t}_i, 1) \tag{2.5}$$

with certain $\bar{t}_i \in (0,1)$ and

$$z_i(t) = O(1-t) \quad \text{for } t \to 1^-. \tag{2.6}$$

Coming to the radial solution of (2.1), we use the positive solution z_i of (2.2) described above. Then $w_i(x) := z_i(1 - \|x\|^{2-n})$ in G_1 is a positive radial solution of (2.1) and taking into account the substitution $t := 1 - \|x\|^{2-n}$ and (2.6) we can derive that

$$\lim_{\|x\|\to\infty} \frac{w_i(x)}{\|x\|^{2-n}} = M_i \in (0, +\infty).$$

This implies that there exists $\widetilde{L}_i > 0$ such that for all $x \in \mathbb{R}^n$, $\|x\| > \widetilde{L}_i$,

$$\frac{M_i}{2}\|x\|^{2-n} < w_i(x) < \frac{3M_i}{2}\|x\|^{2-n},$$

which means that w_i is the minimal solution of (2.1).

Now, taking into account (2.5), we have for all $x \in \mathbb{R}^n$, $\|x\| \ge \overline{R} := \max_{i=1,2} \left(1 - \bar{t}_i\right)^{\frac{1}{2-n}}$,

$$x \cdot \nabla w_i(x) = z_i'(1 - \|x\|^{2-n})\,(n-2)\,\|x\|^{2-n} \le 0$$

where the last inequality follows from the fact that for $\|x\| \ge \overline{R}$ we have $t := 1 - \|x\|^{2-n} \in [\bar{t}_i, 1)$, $i = 1,2$. Finally, we get for $i = 1,2$ and $x \in \mathbb{R}^n$ such that $\|x\| \ge R := \max\{\overline{R}, l_0\}$

$$\Delta w_i(x) + f_i(x, w_1(x), w_2(x)) + g_i(\|x\|)x \cdot \nabla w_i(x) \le \Delta w_i(x) + \widetilde{f}_i(\|x\|) = 0. \qquad \square$$

Theorem 2.2. *Suppose that (A1)–(A3) hold. Then (1.1) possesses a positive solution (u_0, v_0) in G_R. If we assume additionally (A4) then the solution (u_0, v_0) is minimal.*

Proof. Lemma 2.1 gives the existence of a positive super-supersolution (w_1, w_2) such that each function $x \mapsto \|x\|^{n-2} w_i(x)$ is bonded above and below by positive constants in some exterior domain. On the other hand (1.1) possesses the trivial sub-subsolution in G_R. Applying Theorem 1.2, we derive that there exists a solution of (1.1) such that

$$0 \le u_0(x) \le w_1(x) \quad \text{and} \quad 0 \le v_0(x) \le w_2(x) \quad \text{on } G_R$$

and

$$u_0(x) = w_1(x) \quad \text{and} \quad v_0(x) = w_2(x) \quad \text{on } \partial G_R.$$

Our last task is to show that the solution is minimal (see also [25]). We start with the proof that u_0 and v_0 are bounded below by functions of the form $x \mapsto B\|x\|^{2-n}$ in a certain exterior domain. To this end we apply (A4) which gives the existence of $L_1 > 1$ and $A_1 > 0$ such that for all $u \in [0, d_1]$, $x \in \mathbb{R}^n$, $\|x\| > L_1$, $f_1(x,u,0) - A_1 (n-2) g_1(\|x\|)\|x\|^{2-n} > 0$. We can assume (without loss of generality) that $L_1 > \max\{R, l_0\}$. Now we consider $w(x) = B_1\|x\|^{2-n}$ with $B_1 > 0$ and $B_1 < \min\{A_1, L_1^{n-2} \min_{\|x\|=L_1} u_0(x)\}$. Let us note that assumption (A4) guarantees for all $x \in \mathbb{R}^n$ with the norm $\|x\| > L_1$ the following chain of assertions

$$\begin{aligned}
&- \Delta (u_0(x) - w(x)) - g_1(\|x\|)x \cdot \nabla(u_0(x) - w(x)) \\
&\quad = f_1(x, u_0(x), v_0(x)) - B_1 (n-2) g_1(\|x\|)\|x\|^{2-n} \\
&\quad \geq f_1(x, u_0(x), 0) - A_1 (n-2) g_1(\|x\|)\|x\|^{2-n} \geq 0.
\end{aligned}$$

It is also obvious that for all $x \in \mathbb{R}^n$, $\|x\| = L_1$, $u_0(x) - w(x) = u_0(x) - B_1 L_1^{2-n} > 0$ and $\lim_{\|x\|\to\infty} (u_0(x) - w(x)) = 0$. Finally, the maximum principle allows us to state that $u_0(x) - w(x) \geq 0$ for all $x \in \mathbb{R}^n$, $\|x\| > L_1$, which gives

$$\|x\|^{n-2} u_0(x) \geq B_1.$$

On the other hand, by the asymptotics of the super-supersolution, we can state that for all $x \in \mathbb{R}^n$, $\|x\| > \widetilde{L}_1$, we have

$$\|x\|^{n-2} u_0(x) \leq \frac{3M_1}{2}\ .$$

Finally, the function $x \mapsto \|x\|^{n-2} u_0(x)$ is bounded below and above by respectively, B_1 and $\frac{3M_1}{2}$ in the exterior domain $G_{\overline{L}_1}$, with $\overline{L}_1 := \max\{L_1, \widetilde{L}_1, R\}$. In the same way we can obtain the similar conclusion for v_0. To sum up, we have shown that the solution (u_0, v_0) of our problem is minimal. □

Example 2.3. Let us consider the following problem

$$\begin{cases}
\Delta u(x) + \frac{1}{8}\left(e^{u(x)+v(x)} + \frac{u^3(x)+v^3(x)}{(4-u(x))(5-v(x))}\right) \left(\|x\|^4 + 1\right)^{-1} + g_1(\|x\|)x \cdot \nabla u(x) = 0 \\
\Delta v(x) + \left(u^{\frac{3}{2}}(x) + v^{\frac{4}{3}}(x) + 1\right) \left(\|x\|^6 + 1\right)^{-1} + g_2(\|x\|)x \cdot \nabla v(x) = 0
\end{cases}$$

with asymptotic conditions

$$\lim_{\|x\|\to\infty} u(x) = 0 \quad \text{and} \quad \lim_{\|x\|\to\infty} v(x) = 0,$$

for $x \in G_R$, where $G_R = \left\{x \in \mathbb{R}^3,\ \|x\| > R\right\}$ and $g_1(x) = \frac{1}{\|x\|^{k-\frac{1}{2}}}$ and $g_2(x) = \frac{1}{\|x\|^{l-\frac{1}{3}}}$ with $k \geq 3$ and $l \geq 5$.

It is obvious that for g_i, $i = 1, 2$, (A1) holds. Moreover, in our case

$$f_1(x,u,v) = \frac{1}{8}\left(e^{u+v} + \frac{u^3+v^3}{(4-u)(5-v)}\right)\left(\|x\|^4 + 1\right)^{-1}$$

and

$$f_2(x,u,v) = \left(u^{\frac{3}{2}} + v^{\frac{4}{3}} + 1\right)\left(\|x\|^6 + 1\right)^{-1}$$

satisfy (A3) for $d_1 = d_2 = 2$. Simple calculations allow us to state that for $r \geq 1$,

$$\sup_{(u,v)\in[0,d_1]\times[0,d_2]}\ \sup_{\|x\|=r} f_1(x,u,v) \leq 8r^{-4}$$

and

$$\sup_{(u,v)\in[0,d_1]\times[0,d_2]}\ \sup_{\|x\|=r} f_2(x,u,v) \le 7r^{-6}.$$

Taking $\widetilde{f}_1(r) := 8r^{-4}$ and $\widetilde{f}_2(r) := 7r^{-4}$ we have

$$\int_1^{\infty} r^2 \widetilde{f}_i(r)dr \le 8.$$

Thus (A2) also holds. Finally, Theorem 1.2 leads to the existence of at least one positive solution (u_0, v_0) of (1.1) such that $u_0(x) \le 2$ and $v_0(x) \le 2$. In order to prove that (u_0, v_0) is minimal we have to show that (A4) holds. To this effect we note that for a certain $L_1 > 1$ sufficiently large and $A_1 \in \left(0, \frac{1}{8}\right)$, we get for all $x \in \mathbb{R}^3$ such that $\|x\| > L_1$,

$$f_1(x,u,0) - A_1 g_1(x)\|x\|^{-1} \ge \frac{\|x\|^{k+1} - \frac{1}{2}\|x\| - 8A_1\|x\|^4 - 8A_1}{8\left(\|x\|^4+1\right)\left(\|x\|^k - \frac{1}{2}\right)\|x\|} > 0.$$

Similarly we can show the existence of $L_2 > 1$ and $A_2 \in (0,1)$ such that for all $x \in \mathbb{R}^3$ such that $\|x\| > L_2$,

$$f_2(x,0,v) - A_2 g_2(x)\|x\|^{-1} \ge \frac{\|x\|^{l+1} - \frac{1}{3}\|x\| - A_2\|x\|^6 - A_2}{\left(\|x\|^6+1\right)\left(\|x\|^l - \frac{1}{3}\right)\|x\|} > 0.$$

Thus (A4) is also satisfied. This fact guarantees that (u_0, v_0) is the minimal solution of our problem.

3 Final remarks

Remark 3.1. As in [22] we can formulate and prove the results of Theorem 1.2 also in the case when (A3) is replaced by the following condition

(A3') *f_1 is continuously differentiable in u and nonincreasing in v and f_2 is continuously differentiable in v and nonincreasing in u in $G_R \times [0,d_1] \times [0,d_2]$.*

The difference is associated with the starting point of the monotone procedure. In this case we have to consider a super-subsolution $(\overline{u}, \underline{v})$ and next a sub-supersolution $(\underline{u}, \overline{v})$ of our problem to obtain two sequences which give us solutions on bounded domains.

In consequence we can prove the following result analogous to Theorem 2.2.

Theorem 3.2. *Suppose that (A1), (A2) and (A3') hold. Then (1.1) possesses a positive solution (u_0, v_0) in G_R. If we assume additionally*

(A4') *there exist $A_1, A_2 > 0$ and $L_1, L_2 > 1$ such that*

$$f_1(x,u,d_2) \ge A_1(n-2)g_1(x)\|x\|^{2-n} \quad \text{for all } u \in [0,d_1] \text{ and } \|x\| > L_1,$$
$$f_2(x,d_1,v) \ge A_2(n-2)g_2(x)\|x\|^{2-n} \quad \text{for all } v \in [0,d_2] \text{ and } \|x\| > L_2.$$

then the solution (u_0, v_0) is minimal.

4 Appendix

As in [22], we start with the existence of solutions on bounded domains.

Lemma 4.1. *Let* $\Omega_k := \{x \in \mathbb{R}^n,\ R < \|x\| < R+k\}$, *where* $k \in N := \{1,2,\dots\}$. *Assume that conditions (A1)–(A3) hold and suppose that* $(\overline{u},\overline{v})$ *and* $(\underline{u},\underline{v})$ *are, respectively, a super-supersolution and a sub-subsolution of* (1.1)–(1.2) *such that* $0 \le \underline{u} \le \overline{u} \le d_1$ *and* $0 \le \underline{v} \le \overline{v} \le d_2$ *on* $\overline{G}_R$. *Then there exist vector functions* $(\overline{u}_k,\overline{v}_k)$ *and* $(\underline{u}_k,\underline{v}_k) \in C^{2+\alpha}(\overline{\Omega}_k)\times C^{2+\alpha}(\overline{\Omega}_k)$ *satisfying* (1.1) *in* Ω_k. *Moreover* $\underline{u} \le \underline{u}_k \le \overline{u}_k \le \overline{u}$ *and* $\underline{v} \le \underline{v}_k \le \overline{v}_k \le \overline{v}$.

Proof. Let us consider the following auxiliary system

$$\begin{cases} \Delta u(x) + f_1(x,u(x),v(x)) + g_1(\|x\|)x\cdot\nabla u(x) = 0 \text{ in } \Omega_k \\ \Delta v(x) + f_2(x,u(x),v(x)) + g_2(\|x\|)x\cdot\nabla v(x) = 0 \text{ in } \Omega_k \\ u = \varphi_1 \text{ and } v = \varphi_2 \text{ on } \partial\Omega_k, \end{cases} \tag{4.1}$$

with $\varphi_1,\varphi_2 \in C^{2+\alpha}_{loc}(G_R)$ such that $\underline{u} \le \varphi_1 \le \overline{u}$ and $\underline{v} \le \varphi_2 \le \overline{v}$ on G_R. By assumption (A3) we state the existence of positive constants K_1, K_2 such that $\frac{\partial f_1(x,u,v)}{\partial u} + K_1 \ge 0$ and $\frac{\partial f_2(x,u,v)}{\partial v} + K_2 \ge 0$ for all $(x,u,v) \in \Omega_k \times [0,d_1]\times[0,d_2]$. The starting point of the iteration procedure based on the Sattinger's schema is associated with super-supersolution of our problem. Precisely, we take $\overline{u}_0 = \overline{u}$ and $\overline{v}_0 = \overline{v}$ and obtain the existence of a classical solution $(\overline{u}_1,\overline{v}_1)$ of the linear problem considered in Ω_k :

$$\begin{cases} (\Delta + g_1(\|x\|)x\cdot\nabla - K_1)u(x) = -(f_1(x,\overline{u}_0(x),\overline{v}_0(x)) + K_1\overline{u}_0(x)) \\ (\Delta + g_2(\|x\|)x\cdot\nabla - K_2)v(x) = -(f_2(x,\overline{u}_0(x),\overline{v}_0(x)) + K_2\overline{v}_0(x)) \\ u = \varphi_1 \text{ and } v = \varphi_2 \text{ on } \partial\Omega_k. \end{cases}$$

Since $(\overline{u},\overline{v})$ is a super-supersolution of (1.1)–(1.2), it is easy to get

$$(\Delta + g_1(\|x\|)x\cdot\nabla - K_1)(\overline{u}_1(x) - \overline{u}_0(x)) \ge 0.$$

Taking into account the fact that $\overline{u}_1(x) = \varphi_1(x)$ on $\partial\Omega_k$, we state, by the maximum principle, $\overline{u}_1(x) \le \overline{u}_0(x)$. Analogously we obtain $\overline{v}_1(x) \le \overline{v}_0(x)$ in Ω_k. Having a pair $(\overline{u}_{m-1},\overline{v}_{m-1}) \in C^{2+\alpha}(\overline{\Omega}_k)\times C^{2+\alpha}(\overline{\Omega}_k)$ we can iterate this process and obtain $(\overline{u}_m,\overline{v}_m)\, C^{2+\alpha}(\overline{\Omega}_k)\times C^{2+\alpha}(\overline{\Omega}_k)$ as a classical solution of the linear problem in Ω_k

$$\begin{cases} (\Delta + g_1(\|x\|)x\cdot\nabla - K_1)u(x) = -(f_1(x,\overline{u}_{m-1}(x),\overline{v}_{m-1}(x)) + K_1\overline{u}_{m-1}(x)) \\ (\Delta + g_2(\|x\|)x\cdot\nabla - K_2)v(x) = -(f_2(x,\overline{u}_{m-1}(x),\overline{v}_{m-1}(x)) + K_2\overline{v}_{m-1}(x)) \\ u = \varphi_1 \text{ and } v = \varphi_2 \text{ on } \partial\Omega_k. \end{cases} \tag{4.2}$$

Let us note that for all $m \in N$,

$$\overline{u}_m(x) \le \overline{u}_{m-1}(x) \quad \text{and} \quad \overline{v}_m(x) \le \overline{v}_{m-1}(x) \quad \text{in } \overline{\Omega}_k. \tag{4.3}$$

For $m = 1$, (4.3) was proved. Fix integer $m \ge 1$. If we assume that $\overline{u}_m(x) \le \overline{u}_{m-1}(x)$ and $\overline{v}_m(x) \le \overline{v}_{m-1}(x)$ in Ω_k , the properties of f_1 and f_2 give two assertions: for all $x \in \overline{\Omega}_k$,

$$\begin{aligned}&(\Delta + g_1(\|x\|)x\cdot\nabla - K_1)(\overline{u}_{m+1}(x) - \overline{u}_m(x)) \\ &\quad = -(f_1(x,\overline{u}_m(x),\overline{v}_m(x)) + K_1\overline{u}_m(x)) + f_1(x,\overline{u}_{m-1}(x),\overline{v}_{m-1}(x)) + K_1\overline{u}_{m-1}(x) \ge 0,\end{aligned}$$

$$\begin{aligned}&(\Delta + g_2(\|x\|)x\cdot\nabla - K_2)(\overline{v}_{m+1}(x) - \overline{v}_m(x))\\ &\quad= -(f_2(x,\overline{u}_m(x),\overline{v}_m(x)) + K_2\overline{v}_m(x)) + f_2(x,\overline{u}_{m-1}(x),\overline{v}_{m-1}(x)) + K_2\overline{v}_{m-1}(x) \geq 0.\end{aligned}$$

Applying the boundary conditions and again the maximum principle one infers $\overline{u}_{m+1}(x) \leq \overline{u}_m(x)$ and $\overline{v}_{m+1}(x) \leq \overline{v}_m(x)$ in $\overline{\Omega}_k$. Finally, the induction principle allows us to state that for all $m \in N$, (4.3) holds.

Taking the sub-subsolution of (1.1)–(1.2) in G_R as a starting point of our procedure we can use the same reasoning and construct another sequence $\{(\underline{u}_m, \underline{v}_m)\} \in C^{2+\alpha}(\overline{\Omega}_k) \times C^{2+\alpha}(\overline{\Omega}_k)$. Moreover, we can show that for all $m \in N$, $\underline{u}_m(x) \geq \underline{u}_{m-1}(x)$ and $\underline{v}_m(x) \geq \underline{v}_{m-1}(x)$ in $\overline{\Omega}_k$.

Applying again properties of f_1 and f_2, the maximum principle and the induction, one also proves the following chains of inequalities

$$\begin{aligned}&\underline{u}\,(x) \leq \underline{u}_1(x) \leq \cdots \leq \underline{u}_m(x) \leq \underline{u}_{m+1}(x) \leq \cdots\\ &\qquad \leq \overline{u}_{m+1}(x) \leq \overline{u}_m(x) \leq \cdots \leq \overline{u}_1(x) \leq \overline{u}\,(x)\,,\\ &\underline{v}\,(x) \leq \underline{v}_1(x) \leq \cdots \leq \underline{v}_m(x) \leq \underline{v}_{m+1}(x) \leq \cdots\\ &\qquad \leq \overline{v}_{m+1}(x) \leq \overline{v}_m(x) \leq \cdots \leq \overline{v}_1(x) \leq \overline{v}\,(x)\,.\end{aligned} \tag{4.4}$$

To sum up, we have constructed two monotonic and bounded sequences $\{(\overline{u}_m, \overline{v}_m)\}$ and $\{(\underline{u}_m, \underline{v}_m)\}$. They are both pointwisely convergent in Ω_k to some vector functions $(\overline{u}^k, \overline{v}^k)$ and $(\underline{u}^k, \underline{v}^k)$ respectively. Our task is now to show that $(\overline{u}^k, \overline{v}^k)$ and $(\underline{u}^k, \underline{v}^k)$ are solutions of (4.1).

The standard reasoning, based on the L^p-estimates of Agmon–Douglis–Nirenberg, allows us to prove the existence of $C > 0$ such that for all $m \in N$,

$$\|\overline{u}_m\|_{C^{2+\alpha}(\Omega_k)} \leq C \quad \text{and} \quad \|\overline{v}_m\|_{C^{2+\alpha}(\Omega_k)} \leq C$$

(see [22] or [25] for details). Therefore the compact embedding $C^{2+\alpha}(\Omega_k) \times C^{2+\alpha}(\Omega_k) \hookrightarrow C^2(\Omega_k) \times C^2(\Omega_k)$, implies that $\{(\overline{u}_m, \overline{v}_m)\}$ tends to $(\overline{u}^k, \overline{v}^k)$ in $C^2(\Omega_k) \times C^2(\Omega_k)$, and consequently, $(\overline{u}^k, \overline{v}^k)$ is a solution of (4.1). Analogously we can show that $(\underline{u}^k, \underline{v}^k)$ satisfies (4.1). By (4.4) we get $\underline{u} \leq \underline{u}^k \leq \overline{u}^k \leq \overline{u}$ and $\underline{v} \leq \underline{v}^k \leq \overline{v}^k \leq \overline{v}$ on $\overline{\Omega}_k$. □

Proof of Theorem 1.2. Applying Lemma 4.1 with $\varphi_1 = \overline{u}$ and $\varphi_2 = \overline{v}$, we state for each $k \in N$, the existence of a solution (u^k, v^k) of the problem

$$\begin{cases}\Delta u(x) + f_1(x,u(x),v(x)) + g_1(\|x\|)x\cdot\nabla u(x) = 0 \text{ in } \Omega_k\\ \Delta v(x) + f_2(x,u(x),v(x)) + g_2(\|x\|)x\cdot\nabla v(x) = 0 \text{ in } \Omega_k\\ u = \overline{u} \text{ and } v = \overline{v} \text{ on } \partial\Omega_k,\end{cases}$$

with $\Omega_k := \{x \in \mathbb{R}^n,\ R < \|x\| < R+k\}$, such that

$$\underline{u} \leq u^k \leq \overline{u} \quad \text{and} \quad \underline{v} \leq v^k \leq \overline{v} \quad \text{on } \Omega_k. \tag{4.5}$$

Now we investigate the properties of the sequence $\{(u^k, v^k)\}$. Employing the reasoning similar to that in the proof of Lemma 4.1 one proves that for any integer fixed $k_0 > 0$ and $k > k_0 + 1$, this sequence is bounded in $C^{2+\alpha}(\Omega_{k_0}) \times C^{2+\alpha}(\Omega_{k_0})$, namely there exists a positive constant C_1 such that for all $k > k_0 + 1$,

$$\|u^k\|_{C^{2+\alpha}(\Omega_{k_0})} \leq C_1 \quad \text{and} \quad \|v^k\|_{C^{2+\alpha}(\Omega_{k_0})} \leq C_1.$$

In the next step we use the compact embedding $C^{2+\alpha}(\Omega_1) \times C^{2+\alpha}(\Omega_1)$ into $C^2(\Omega_1) \times C^2(\Omega_1)$, which allows us to state the existence of a subsequence $\{(u^{k_{l_1}}, v^{k_{l_1}})\}$ of $\{(u^k, v^k)\}$ which tends to a certain $(\widehat{u}^1, \widehat{v}^1)$ in $C^2(\Omega_1) \times C^2(\Omega_1)$. It is obvious that $(\widehat{u}^1, \widehat{v}^1)$ satisfies (1.1) in Ω_1. Moreover, $\underline{u} \le \widehat{u}^1 \le \overline{u}$ and $\underline{v} \le \widehat{v}^1 \le \overline{v}$ in $\overline{\Omega}_1$ and $\widehat{u}^1 = \overline{u}$ and $\widehat{v}^1 = \overline{u}$ on $\{x \in \mathbb{R}^n,\ \|x\| = R\}$. Applying the same reasoning we obtain the subsequence $\{(u^{k_{l_2}}, v^{k_{l_2}})\}$ of $\{(u^{k_{l_1}}, v^{k_{l_1}})\}$ such that $\{(u^{k_{l_2}}, v^{k_{l_2}})\}$ converges in $C^2(\Omega_2) \times C^2(\Omega_2)$ to $(\widehat{u}^2, \widehat{v}^2)$ being a solution of (1.1) in Ω_2. We also have $\underline{u} \le \widehat{u}^2 \le \overline{u}$, $\underline{v} \le \widehat{v}^2 \le \overline{u}$ in $\overline{\Omega}_2$ and $\widehat{u}^2 = \overline{u}$ and $\widehat{v}^2 = \overline{v}$ on $\{x \in \mathbb{R}^n,\ \|x\| = R\}$. Iterate this process for each $m \in N$, we construct a sequence $\{(u^{k_{l_m}}, v^{k_{l_m}})\}$ which is convergent in $C^2(\Omega_m) \times C^2(\Omega_m)$ and such that $\{(u^{k_{l_m}}, v^{k_{l_m}})\}$ is a subsequence of $\{(u^{k_{l_{m-1}}}, v^{k_{l_{m-1}}})\}$. Let $\widehat{u}^m(x) := \lim_{l_m \to \infty} u^{k_{l_m}}(x)$ and $\widehat{v}^m(x) := \lim_{l_m \to \infty} v^{k_{l_m}}(x)$ in Ω_m. To sum up, one can state that $(\widehat{u}^m, \widehat{v}^m)$ is a solution of (1.1) in Ω_m, $\underline{u} \le \widehat{u}^m \le \overline{u}$ and $\underline{v} \le \widehat{v}^m \le \overline{v}$ in $\overline{\Omega}_m$ and $\widehat{u}^m = \overline{u}$ and $\widehat{v}^m = \overline{v}$ on $\{x \in \mathbb{R}^n,\ \|x\| = R\}$. By the construction we know that $\widehat{u}^m|_{\Omega_{m-1}} = \widehat{u}^{m-1}$ and $\widehat{v}^m|_{\Omega_{m-1}} = \widehat{v}^{m-1}$. Finally, we define $U, V : G_R \to R$ in the following way

$$U(x) := \widehat{u}^m \quad \text{and} \quad V(x) := \widehat{v}^m \quad \text{in } \Omega_m.$$

Then $(U, V) \in C^{2+\alpha}_{\text{loc}}(G_R) \times C^{2+\alpha}_{\text{loc}}(G_R)$ and satisfies (1.1) in G_R. Moreover we have $\underline{u} \le U \le \overline{u}$, $\underline{v} \le V \le \overline{v}$ in $\overline{\Omega}_m$, $U = \overline{u}$ and $V = \overline{v}$ on $\{x \in \mathbb{R}^n,\ \|x\| = R\}$. The asymptotics of $\overline{u}$ and $\overline{v}$ implies that $\lim_{\|x\| \to \infty} U(x) = 0$, $\lim_{\|x\| \to \infty} V(x) = 0$. □

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