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Oscillation of a second order half-linear difference equation and the discrete Hardy inequality

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Abstract. In local terms on finite and infinite intervals we obtain necessary and sufficient conditions for the conjugacy and disconjugacy of the following second order half-linear difference equation

$$\Delta(\rho_i|\Delta y_i|^{p-2}\Delta y_i) + v_i|y_{i+1}|^{p-2}y_{i+1} = 0, \qquad i = 0,1,2,\dots,$$

where $1 < p < \infty$, $\Delta y_i = y_{i+1} - y_i$, $\{\rho_i\}$ and $\{v_i\}$ are sequences of positive and non-negative real numbers, respectively. Moreover, we study oscillation and non-oscillation properties of this equation.

Keywords: half-linear equation, conjugate, disconjugate, oscillation, non-oscillation, discrete Hardy inequality.

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1 Introduction

During the last several decades the oscillation properties of the half-linear difference equation have been intensively investigated. There is a lot of works devoted to this problem (see [2–4, 7–12, 16–22] and references given there).

We consider the following second order half-linear difference equation

$$\Delta(\rho_i|\Delta y_i|^{p-2}\Delta y_i) + v_i|y_{i+1}|^{p-2}y_{i+1} = 0, \qquad i = 0,1,2,\dots, \tag{1.1}$$

where $1 < p < \infty$, $\Delta y_i = y_{i+1} - y_i$, $\{\rho_i\}$ and $\{v_i\}$ are sequences of positive and non-negative real numbers, respectively.

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural and integer numbers, respectively.

Let us remind some notions and statements related to (1.1). Let $m \geq 0$ be an integer number.

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– If there exists a non-trivial solution $y = \{y_i\}$ of the equation (1.1) such that $y_m \neq 0$ and $y_m y_{m+1} < 0$, then we say that the solution y has a generalized zero on the interval $(m, m+1]$.

– A non-trivial solution y of the equation (1.1) is called oscillatory if it has infinite number of generalized zeros, otherwise it is called non-oscillatory.

– The equation (1.1) is called oscillatory if all its non-trivial solutions are oscillatory, otherwise it is called non-oscillatory.

– Due to Sturm's separation theorem [18, Theorem 3], the equation (1.1) is oscillatory if one of its non-trivial solutions is oscillatory.

– The equation (1.1) is called disconjugate on the discrete interval $[m, n]$, $0 \leq m < n$, (further just "interval") if its any non-trivial solution has no more than one generalized zero on the interval $(m, n+1]$ and its non-trivial solution $\tilde{y}$ with the initial condition $\tilde{y}_m = 0$ has not a generalized zero on the interval $(m, n+1]$, otherwise it is called conjugate on the interval $[m, n]$.

– The equation (1.1) is called disconjugate on the interval $[m, \infty)$ if for any $n > m$ it is disconjugate on the interval $[m, n]$.

The main properties of solutions of the equation (1.1) are described by so-called "roundabout theorem" [18, Theorem 1] that gives two important methods [8] of the investigation of oscillation properties of the equation (1.1). Here we use one of these methods called "variational method". This method is based on the lemma given below that follows from the equivalence of the statements (i) and (ii) of Theorem 1 from [18].

Lemma A. *Let $0 \leq m < n < \infty$, m and n be integers. The equation (1.1) is disconjugate on the interval $[m, n]$ if and only if*

$$\sum_{i=m}^{n} (\rho_i |\Delta y_i|^p - v_i |y_{i+1}|^p) \geq 0 \tag{1.2}$$

for all non-trivial $y = \{y_k\}_{k=m}^{n+1}$, $y_m = 0$ and $y_{n+1} = 0$.

Let $y = \{y_i\}_{i=0}^{\infty}$ be a sequence of real numbers. Denote that $\operatorname{supp} y := \{i \geq 0 : y_i \neq 0\}$. Let $0 \leq m < n \leq \infty$. Denote by $\mathring{Y}(m, n)$ the set of all non-trivial sequences of real numbers $y = \{y_i\}_{i=0}^{\infty}$ such that $\operatorname{supp} y \subset [m+1, n]$, $n < \infty$. When $n = \infty$ we suppose that for any y there exists an integer $k = k(y) : \ m < k < \infty$ such that $\operatorname{supp} y \subset [m+1, k]$.

Lemma 1.1. *Let $0 \leq m < n \leq \infty$. The equation (1.1) is disconjugate on the interval $[m, n]$ ($[m, n] = [m, \infty)$ when $n = \infty$) if and only if*

$$\sum_{i=m}^{n} v_{i-1} |y_i|^p \leq \sum_{i=m}^{n} \rho_i |\Delta y_i|^p \tag{1.3}$$

for all $y \in \mathring{Y}(m, n)$, where $v_{-1} = 0$.

Proof. Let the equation (1.1) be disconjugate on the interval $[m, n]$, $n < \infty$. Then by Lemma A the condition (1.2) is valid for all $y \in \mathring{Y}(m, n)$. Since from $y_m = y_{n+1} = 0$ it follows that

$$\sum_{i=m}^{n} v_i |y_{i+1}|^p = \sum_{i=m+1}^{n+1} v_{i-1} |y_i|^p = \sum_{i=m}^{n} v_{i-1} |y_i|^p \tag{1.4}$$

and the sum in (1.2) is finite, then (1.2) is equivalent to the inequality (1.3). Let $n = \infty$. Let an arbitrary integer number n_1 be such that $m < n_1$. Then the equation (1.1) is disconjugate on the interval $[m, n_1]$ and by Lemma A the inequality (1.3) is valid for all $y \in \mathring{Y}(m, n_1) \subset \mathring{Y}(m, \infty)$.

Whence it appears that due to arbitrariness of the number n_1 the inequality (1.3) is correct for all $y \in \mathring{Y}(m,\infty)$.

Inversely, let (1.3) be valid for all $y \in \mathring{Y}(m,n)$. For $n < \infty$ due to (1.4) we have that (1.2) holds. Then by Lemma A the equation (1.1) is disconjugate on the interval $[m,n]$. For $n = \infty$ we have that (1.3) is correct for all $y \in \mathring{Y}(m,n_1) \subset \mathring{Y}(m,\infty)$ and arbitrary $n_1 > m$. Then by Lemma A the equation (1.1) is disconjugate on the interval $[m,n_1]$. Due to arbitrariness of n_1 we have that the equation (1.1) is disconjugate on the interval $[m,\infty)$. The proof of Lemma 1.1 is complete. □

The dual statement to Lemma 1.1 is the following lemma.

Lemma 1.2. *Let $0 \le m < n \le \infty$. The equation (1.1) is conjugate on the interval $[m,n]$ if and only if there exists $\tilde{y} \in \mathring{Y}(m,n)$ such that*

$$\sum_{i=m}^{n} v_{i-1}|\tilde{y}_i|^p > \sum_{i=m}^{n} \rho_i|\Delta\tilde{y}_i|^p. \tag{1.5}$$

The inequality (1.3) is the discrete Hardy inequality

$$\sum_{i=m}^{n} v_{i-1}|y_i|^p \le C\sum_{i=m}^{n} \rho_i|\Delta y_i|^p, \quad y \in \mathring{Y}(m,n), \tag{1.6}$$

where $0 < C \le 1$ and C is the least constant in (1.6).

A continuous analogue of the inequality (1.6) is investigated in many works (see e.g. [1], [14] and [15]). The resume of these works is given in [13]. Here we study the inequality (1.6) by methods different from the methods used for the continuous case in the mentioned works.

The paper is organized as follows. In Section 2 on the basis of the Hardy inequality (1.6) we find necessary and sufficient conditions for the conjugacy and disconjugacy of the equation (1.1) on the interval $[m,n]$. Moreover, in the same Section 2 on the basis of the first results we give necessary and sufficient conditions for the oscillation and non-oscillation of the equation (1.1). In Section 3 we present proofs of the results on the validity of the Hardy inequality (1.6).

Hereinafter "sequence" means a sequence of real numbers. The sums $\sum_{i=k}^{m}$ for $m < k$ and $\sum_{i\in\Omega}$ for empty Ω are equal to zero. Moreover, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. The numbers m, n, t, s, x and z with and without indexes are integers.

2 Main results

Let $0 \le m < n \le \infty$. Let us introduce the notations

$$B_p(m,n) = \sup_{m\le t\le s\le n} \frac{\sum_{i=t}^{s-1} v_i}{\left(\sum_{i=m}^{t}\rho_i^{1-p'}\right)^{1-p} + \left(\sum_{i=s}^{n}\rho_i^{1-p'}\right)^{1-p}},$$

$$\alpha_p = \inf_{\lambda>1}\frac{\lambda^p(\lambda^p-1)}{(\lambda-1)^p}, \qquad \gamma_0 = \gamma_0(p) = \frac{2p}{p+1}\left(\frac{2p}{p+1}\right)^p\left(\frac{2p}{p-1}\right)^{p-1},$$

$$\gamma_1 = \gamma_1(p) = p\left(\frac{2p}{p+1}\right)^p\left(\frac{2p}{p-1}\right)^{p-1}, \qquad \gamma_2 = \gamma_2(p) = pp^p\left(\frac{p}{p-1}\right)^{p-1}.$$

Theorem 2.1. *Let $0 \le m < n \le \infty$ and $\sum_{i=1}^{\infty} \rho_i^{1-p'} < \infty$ for $n = \infty$. The inequality (1.6) holds if and only if $B_p(m,n) < \infty$. Moreover,*

$$B_p(m,n) \le C \le 2\alpha_p B_p(m,n), \tag{2.1}$$

i.e., if the inequality (1.6) holds with C, then $B_p(m,n) \le C$; if $B_p(m,n) < \infty$, then the inequality (1.6) holds with the estimate $C \le 2\alpha_p B_p(m,n)$, where

$$\alpha_2 = \frac{3\sqrt{5}+7}{\sqrt{5}-1}, \qquad \gamma_0 < \alpha_p < \min\{\gamma_1, \gamma_2, 4^p\} \quad \text{for } p \ne 2, \tag{2.2}$$

and C is the least constant in (1.6).

Remark 2.2. We do not use the condition $\sum_{i=1}^{\infty} \rho_i^{1-p'} < \infty$ for the proof of Theorem 2.1. However, when $n = \infty$ and $\sum_{i=1}^{\infty} \rho_i^{1-p'} = \infty$, there exists an other method that estimates the least constant C in (1.6) better than (2.1). We turn back to this problem at the end of this section.

The proof of Theorem 2.1 is in the last Section 3. Now we study oscillation properties of the equation (1.1) that follow from Theorem 2.1, Lemmas 1.1 and 1.2. The relation (2.1) obviously gives the following corollary.

Corollary 2.3. *Let $0 \le m < n \le \infty$. If (1.3) holds, then $B_p(m,n) \le 1$; and if $2\alpha_p B_p(m,n) \le 1$, then (1.3) holds.*

Applying Corollary 2.3, Lemmas 1.1 and 1.2 to the problem of the conjugacy and disconjugacy of the equation (1.2) on the interval $[m,n]$, we get the following theorem (let us remind that $[m,n] = [m,\infty)$ when $n = \infty$).

Theorem 2.4. *Let $0 \le m < n \le \infty$ and $\sum_{i=1}^{\infty} \rho_i^{1-p'} < \infty$ for $n = \infty$. Then*

- (i) *for the disconjugacy of the equation (1.1) on the interval $[m,n]$ the condition $B_p(m,n) \le 1$ is necessary and the condition $2\alpha_p B_p(m,n) \le 1$ is sufficient;*
- (ii) *for the conjugacy of the equation (1.1) on the interval $[m,n]$ the condition $2\alpha_p B_p(m,n) > 1$ is necessary and the condition $B_p(m,n) > 1$ is sufficient.*

Proof. If the equation (1.1) is disconjugate on the interval $[m,n]$, then by Lemma 1.1 the inequality (1.3) holds. Hence, by Corollary 2.3 we have $B_p(m,n) \le 1$.

Inversely, if $2\alpha_p B_p(m,n) \ge 1$, then by Corollary 2.3 the inequality (1.3) holds. Hence, by Lemma 1.1 the equation (1.1) is disconjugate on the interval $[m,n]$. The proof of the statement (i) is complete.

Let the equation (1.1) be conjugate on the interval $[m,n]$. Then by Lemma 1.2 there exists $\tilde{y} \in \tilde{Y}(m,n)$ such that (1.5) holds. This means that the inequality (1.6) is not valid for all $y \in \mathring{Y}(m,n)$ when $C \le 1$, i.e., the least constant C in the inequality (1.6) must be larger than one. Then from (2.1) it follows that $2\alpha_p B_p(m,n) > 1$.

Inversely, let $B_p(m,n) > 1$. Then from (2.1) we have that the least constant C in the inequality (1.6) is larger than one, i.e., the inequality (1.3) is not valid for all $y \in \mathring{Y}(m,n)$. Therefore, there exists $\tilde{y} \in \mathring{Y}(m,n)$ such that the inequality (1.5) holds. Consequently, by Lemma 1.2 the equation (1.1) is conjugate on the interval $[m,n]$. The proof of the statement (ii) is complete. Thus, the proof of Theorem 2.4 is complete. □

Corollary 2.5. *Let the conditions of Theorem 2.4 hold. Then*

(i) *if there exist integers* $t,\ s:\ m \le t < s \le n$ *such that*

$$\sum_{i=t}^{s-1} v_i > \left(\sum_{i=m}^{t} \rho_i^{1-p'}\right)^{1-p} + \left(\sum_{i=s}^{n} \rho_i^{1-p'}\right)^{1-p}, \tag{2.3}$$

then the equation (1.1) *is conjugate on the interval* $[m,n]$;

(ii) *if the equation* (1.1) *is conjugate or disconjugate on the interval* $[m,n]$, *then there exist integers* $t,\ s:\ m \le t < s \le n$ *such that*

$$\sum_{i=t}^{s-1} v_i > \frac{1}{2\alpha_p}\left[\left(\sum_{i=m}^{t} \rho_i^{1-p'}\right)^{1-p} + \left(\sum_{i=s}^{n} \rho_i^{1-p'}\right)^{1-p}\right]$$

or

$$\sum_{i=t}^{s-1} v_i \le \left(\sum_{i=m}^{t} \rho_i^{1-p'}\right)^{1-p} + \left(\sum_{i=s}^{n} \rho_i^{1-p'}\right)^{1-p},$$

respectively.

In particular, from (2.3) we have the following simple condition of the conjugacy of the equation (1.1) on the interval $[m,n]$

$$\sum_{i=m}^{n-1} v_i > \rho_m + \rho_n. \tag{2.4}$$

The condition (2.4) coincides with the condition of Theorem 5 from [16].

Now we consider oscillation and non-oscillation properties of the equation (1.1).

Theorem 2.6. *Let* $\sum_{i=1}^{\infty} \rho_i^{1-p'} < \infty$.

(i) *For the equation* (1.1) *to be non-oscillatory the condition* $B_p(m,\infty) \le 1$ *for some* $m \ge 0$ *is necessary and the condition* $2\alpha_p B_p(n,\infty) \le 1$ *for some* $n \ge 0$ *is sufficient;*

(ii) *For the equation* (1.1) *to be oscillatory the condition* $2\alpha_p \limsup_{m\to\infty} B_p(m,\infty) \ge 1$ *is necessary and the condition* $\limsup_{m\to\infty} B_p(m,\infty) > 1$ *is sufficient.*

Proof. The statement (i) directly follows from the statement (i) of Theorem 2.4. Let us prove the statement (ii).

Let the equation (1.1) be oscillatory. Then there exists an integer $k:\ 0 \le k < \infty$ such that for all $m > k$ the equation (1.1) is conjugate on the interval $[m,\infty)$. Therefore, by Theorem 2.4 we have that $2\alpha_p B_p(m,\infty) > 1$ for all $m > k$. Whence it follows that $2\alpha_p \limsup_{m\to\infty} B_p(m,\infty) \ge 1$.

Inversely, let $\limsup_{m\to\infty} B_p(m,\infty) > 1$. Then there exists an increasing sequence of numbers $\{m_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $m_k \to \infty$ for $k \to \infty$ and $B_p(m_k,\infty) > 1$ for all $k \ge 1$. Then by Theorem 2.4 the equation (1.1) is conjugate on the interval $[m_k,\infty)$ for all $k \ge 1$, i.e., for all $k \ge 1$ there exists a non-trivial solution of the equation (1.1) that has at least two generalized zeros on the interval $[m_k,\infty)$. Hence, there exists a sequence $\{\tilde{m}_k\} \subset \{m_k\}$ such that on all intervals $[\tilde{m}_k, \tilde{m}_{k+1} - 1]$ some non-trivial solution of the equation (1.1) has two zeros. Then by Sturm's separation theorem [18, Theorem 2] there exists a non-trivial solution of the equation (1.1) that has at least one generalized zero on each interval $[m_k, m_{k+1} - 1]$, $k \ge 1$. Thus, this solution of the equation (1.1) is oscillatory. The proof of Theorem 2.6 is complete. □

From Theorem 2.6 we have the following corollary.

Corollary 2.7. *Let* $\sum_{i=1}^{\infty} \rho_i^{1-p'} < \infty$.

(*i*) *If there exist sequences of integers* m_k, t_k *and* s_k, $k \geq 1$, *such that* $0 < m_k \leq t_k < s_k$, $m_k \to \infty$ *for* $k \to \infty$ *and*

$$\sum_{i=t_k}^{s_k-1} v_i > \left(\sum_{i=m_k}^{t_k} \rho_i^{1-p'} \right)^{1-p} + \left(\sum_{i=s_k}^{\infty} \rho_i^{1-p'} \right)^{1-p} \tag{2.5}$$

for sufficiently large k, *then the equation* (1.1) *is oscillatory.*

(*ii*) *If the equation* (1.1) *is oscillatory, then there exist sequences of integers* m_k, t_k *and* s_k, $k \geq 1$, *such that* $0 < m_k \leq t_k < s_k$, $m_k \to \infty$ *for* $k \to \infty$ *and*

$$\sum_{i=t_k}^{s_k-1} v_i > \frac{1}{2\alpha_p} \left(\left(\sum_{i=m_k}^{t_k} \rho_i^{1-p'} \right)^{1-p} + \left(\sum_{i=s_k}^{\infty} \rho_i^{1-p'} \right)^{1-p} \right).$$

In particular, from (2.6) under the conditions of Corollary 2.7 for the equation (1.1) to be oscillatory we have the following condition

$$\sum_{i=t_k}^{s_k-1} v_i > \rho_{t_k} + \rho_{s_k}, \qquad t_k < s_k, \quad t_k \to \infty. \tag{2.6}$$

The condition (2.6) coincides with the condition of Corollary 2 from [16]. For example, from (2.6) for $s_k - 1 = t_k$ we have

$$v_{t_k} > \rho_{t_k} + \rho_{t_k+1}, \qquad t_k \to \infty. \tag{2.7}$$

Whence it follows that if $v_i = 0$, $i \neq t_k$, $v_{t_k} \neq 0$ and (2.7) holds, then under the conditions of Corollary 2.7 the equation (1.1) is oscillatory.

In the case

$$\sum_{i=1}^{\infty} \rho_i^{1-p'} = \infty \tag{2.8}$$

oscillation properties of the equation (1.1) are studied in the work [3] on the basis of the following lemma.

Lemma 2.8. *Let* $n = \infty$ *and* $\sum_{i=1}^{\infty} \rho_i^{1-p'} = \infty$. *Then the inequality* (1.6) *is equivalent to the discrete Hardy inequality*

$$\sum_{k=m}^{\infty} v_k \left| \sum_{i=m}^{k} a_i \right|^p \leq C \sum_{k=m}^{\infty} \rho_k |a_k|^p \tag{2.9}$$

for all sequences $\{a_k\}_{k=m}^{\infty}$ *of real numbers. Moreover, the least constants in* (1.6) *and* (2.9) *coincide.*

For complete presentation we prove Lemma 2.8 in the next Section 3 by a method different from those in [3].

The inequality (2.9) is well-studied. The main results on the inequality (2.9) are obtained in the works [5] and [6]. In [13] the summary of these results and estimates of the least constant C in (2.9) are presented.

Let us use the following notations:

$$A_1(n) = \sum_{k=n}^{\infty} v_k \left(\sum_{i=m}^{n} \rho_i^{1-p'} \right)^{1-p}, \qquad A_2(n) = \left(\sum_{i=m}^{n} \rho_i^{1-p'} \right)^{-1} \sum_{k=m}^{n} v_k \left(\sum_{i=m}^{k} \rho_i^{1-p'} \right)^{p},$$

$$A_3(n) = \left(\sum_{k=n}^{\infty} v_k \right)^{1-p} \left(\sum_{i=m}^{n} \rho_i^{1-p'} \left(\sum_{k=i}^{\infty} v_k \right)^{p'} \right)^{p-1}.$$

From [13, Theorem 7] we have the following theorem.

Theorem À. *The inequality (2.9) holds if and only if $A_i \equiv A_i(m) = \sup_{n \geq m} A_i(n) < \infty$ for $i = 1$, $i = 2$ or $i = 3$. Moreover, for the least constant C in (2.3) the following estimates*

$$A_1 \leq C \leq p \left(\frac{p}{p-1} \right)^{p-1} A_1, \tag{2.10}$$

$$\frac{1}{p} A_2 \leq C \leq (p')^p A_2 \tag{2.11}$$

and

$$\left(\frac{1}{p'} \right)^{p-1} A_3 \leq C \leq p^p A_3 \tag{2.12}$$

hold.

In the case (2.8) by Lemma 2.8 for the least constant C in the inequality (1.6) the estimates (2.10), (2.11) and (2.12) hold. Therefore, the following theorem is correct (see [3, Theorems 2 and 3]).

Theorem 2.9. *Let (2.8) hold. Then*

(i) the condition $\lim_{m \to \infty} A_i(m) \leq k_i$ for all $i = 1, 2, 3$ is necessary and the condition $\lim_{m \to \infty} A_i(m) \leq k_i$ for $i = 1$, $i = 2$ or $i = 3$ is sufficient for the equation (1.1) to be non-oscillatory;

(ii) the condition $\lim_{m \to \infty} A_i(m) > K_i$ for all $i = 1, 2, 3$ is necessary and the condition $\lim_{m \to \infty} A_i(m) > K_i$ for $i = 1$, $i = 2$ or $i = 3$ is sufficient for the equation (1.1) to be oscillatory, where $k_1 = 1$, $k_2 = p$, $k_3 = (p')^{p-1}$, $K_1 = \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$, $K_2 = \left(\frac{1}{p'}\right)^p$ and $K_3 = \left(\frac{1}{p}\right)^p$.

As an application of Theorem 2.4 let us consider the following example.

Example 2.10.

$$\Delta(k^{\alpha i} |\Delta y_i|^{p-2} \Delta y_i) + k^{\beta i} |y_{i+1}|^{p-2} y_{i+1} = 0, \tag{2.13}$$

where $k, \alpha, \beta \in \mathbb{R}$, $\alpha > p - 1$ and $k > 1$. From the condition $\alpha > p - 1$ it follows that

$$\sum_{i=m}^{\infty} k^{\alpha(1-p')} < \infty$$

for any $m \in \mathbb{N}$.

Denote that

$$a(t,s) = \frac{\sum_{i=t}^{s-1} k^{\beta i}}{\left(\sum_{i=m}^{t} k^{\alpha(1-p')}\right)^{1-p} + \left(\sum_{i=s}^{\infty} k^{\alpha(1-p')}\right)^{1-p}}.$$

Then

$$B_p(m, \infty) = \sup_{m<t<s} a(t,s).$$

Proposition 2.11. *Let $\alpha > p-1$ and $1 < p < \infty$. Then the equation (2.13) is*

(i) *non-oscillatory for $\beta < \alpha$;*

(ii) *oscillatory for $\beta > \alpha$.*

Proof. (i) Let $\beta < \alpha$. In the case $\beta \geq 0$ we have

$$\sum_{i=t}^{s-1} k^{\beta i} = k^{\beta t}\frac{k^{(s-t)\beta}-1}{k^{\beta}-1} \leq \frac{k^{s\beta}}{k^{p}-1},$$

$$\left(\sum_{i=s}^{\infty} k^{\alpha(1-p')}\right)^{1-p} = k^{\alpha s}(1-k^{\alpha(1-p')})^{p-1}$$

and

$$a(t,s) < \frac{k^{(\beta-\alpha)s}}{(k^{\beta}-1)(1-k^{\alpha(1-p')})^{p-1}} = b(s).$$

Since $\beta-\alpha<0$ and $k>1$, then $b(s)\downarrow 0$ for $s\to\infty$. Therefore, there exists $m\in\mathbb{N}$ such that

$$2\alpha_p B_p(m,\infty) < 2\alpha_p b(m) < 1. \tag{2.14}$$

In the case $\beta < 0$ we have

$$\sum_{i=t}^{s-1} k^{\beta i} < k^{\beta t}(s-t)$$

and

$$a(t,s) < k^{\beta t}\frac{s-t}{k^{\alpha s}(1-k^{\alpha(1-p')})^{p-1}}.$$

For $s>t$ the function $\frac{s-t}{k^{\alpha s}}$ has a maximum at the point $s = t+\frac{1}{\alpha\ln k}$. Therefore,

$$\sup_{t<s} a(t,s) < \frac{k^{(\beta-\alpha)t}k^{-\frac{1}{\ln k}}}{\alpha\ln k(1-k^{\alpha(1-p')})^{p-1}}.$$

In view of $\beta-\alpha<0$, the last inequality gives that (2.14) holds for some $m\in\mathbb{N}$, i.e., by Theorem 2.4 the equation (2.13) is non-oscillatory.

(ii) Let $\beta>\alpha$. Then we have the following estimates

$$\sum_{i=t}^{s-1} k^{\beta i} > k^{\beta t}(s-t),$$

$$\left(\sum_{i=m}^{t} k^{\alpha(1-p')i}\right)^{1-p} + \left(\sum_{i=s}^{\infty} k^{\alpha(1-p')i}\right)^{1-p} < k^{\alpha t}+k^{\alpha s} < 2k^{\alpha s}$$

for $s>t$. Therefore,

$$a(t,s) > \frac{k^{\beta t}(s-t)}{2k^{\alpha s}}.$$

As above, the last inequality gives that

$$\sup_{t<s} a(t,s) > \frac{1}{2}\sup_{t<s}\frac{k^{\beta t}(s-t)}{k^{\alpha s}} = \frac{1}{2}\frac{k^{(\beta-\alpha)t}k^{-\frac{1}{\ln k}}}{\alpha\ln k}.$$

Consequently, in view of $\beta-\alpha>0$, we have that $B_p(m,\infty)>1$ for all $m\in\mathbb{N}$. Hence, on the basis of Theorem 2.4 the equation (2.13) is oscillatory. The proof of Proposition 2.11 is complete. □

3 Proof of Theorem 2.1

Proof. Necessity. Let the inequality (1.6) hold with the least constant $C > 0$. Let α, t, s and β be integers satisfying the condition $m < \alpha \le t \le s \le \beta < n$.

We construct a test sequence $y = \{y_k\}$ in the following way

$$y_k = \begin{cases} \sum_{i=\alpha-1}^{k-1} \rho_i^{1-p'} \left(\sum_{i=\alpha-1}^{t-1} \rho_i^{1-p'}\right)^{-1}, & \alpha \le k \le t, \\ 1, & t \le k \le s, \\ \sum_{i=k}^{\beta} \rho_i^{1-p'} \left(\sum_{i=s}^{\beta} \rho_i^{1-p'}\right)^{-1}, & s \le k \le \beta, \\ 0, & m \le k < \alpha \text{ or } \beta < k \le n. \end{cases}$$

It is obvious that $y \in \mathring{Y}(m,n)$. Let us calculate Δy_k.

$$\Delta y_k = \begin{cases} \rho_k^{1-p'} \left(\sum_{i=\alpha-1}^{t-1} \rho_i^{1-p'}\right)^{-1}, & \alpha - 1 \le k \le t-1, \\ 0, & t \le k \le s, \\ -\rho_k^{1-p'} \left(\sum_{i=s}^{\beta} \rho_i^{1-p'}\right)^{-1}, & s \le k \le \beta, \\ 0, & m \le k < \alpha - 1 \text{ or } \beta < k \le n. \end{cases}$$

Then

$$\sum_{k=m}^{n} \rho_k |\Delta y_k|^p = \left(\sum_{i=\alpha-1}^{t-1} \rho_i^{1-p'}\right)^{1-p} + \left(\sum_{i=s}^{\beta} \rho_i^{1-p'}\right)^{1-p} \tag{3.1}$$

and

$$\sum_{i=m}^{n} v_{i-1} |y_i|^p \ge \sum_{i=t}^{s} v_{i-1} = \sum_{i=t-1}^{s-1} v_i. \tag{3.2}$$

From (3.1), (3.2) and (1.6) we have

$$\sum_{i=t-1}^{s-1} v_i \le C \left[\left(\sum_{i=\alpha-1}^{t-1} \rho_i^{1-p'}\right)^{1-p} + \left(\sum_{i=s}^{\beta} \rho_i^{1-p'}\right)^{1-p} \right].$$

Due to independence of the left-hand side of the last estimate from $\alpha: \ m < \alpha \le t$ and $\beta: \ s \le \beta < n$ and independence of the constant C from $t, s: \ m < t \le s < n$, we have

$$\sum_{i=t-1}^{s-1} v_i \le C \left[\left(\sum_{i=m}^{t-1} \rho_i^{1-p'}\right)^{1-p} + \left(\sum_{i=s}^{n} \rho_i^{1-p'}\right)^{1-p} \right]$$

or

$$B_p(m,n) \le C. \tag{3.3}$$

Sufficiency. Let $B_p(m,n) < \infty$. Let $y = \{y_i\} \in \mathring{Y}(m,n)$. Without loss of generality, we denote that $y_i \ge 0$, $i = 0,1,2,\dots$ Let $\lambda > 1$. For any $k \in \mathbb{Z}$ we define the set $T_k \equiv T_k(\lambda) = \{i \ge m: \ y_i > \lambda^k\}$. Due to boundedness of the set $\{y_i\}$ there exists a number $\tau = \tau(y,\lambda) \in \mathbb{Z}$ such that $T_\tau \ne \oslash$ and $T_{\tau+1} = \oslash$. Let $\Delta^- T_k := T_k - T_{k+1}$. Then

$$[m,n] = \bigcup_{k=-\infty}^{\tau} T_k = \bigcup_{k=-\infty}^{\tau} \Delta^- T_k. \tag{3.4}$$

The definition of T_k and the relation $T_\tau \neq \oslash$ give that $T_k \neq \oslash$ for all $k \leq \tau$. Let $k < \tau$. We present the set T_k in the form $T_k = \bigcup_j [t_k^j, s_k^j]$, $[t_k^j, s_k^j] \bigcap [t_k^i, s_k^i] = \oslash$ for $i \neq j$. We denote that $M_k^j = T_{k+1} \bigcap [t_k^j, s_k^j]$, $\Omega_k = \{j : \quad M_k^j \neq \oslash\}$. Moreover, for $j \in \Omega_k$ we define $x_k^j = \min M_k^j$ and $z_k^j = \max M_k^j$. Then $t_k^j \leq x_k^j$, $z_k^j \leq s_k^j$ and

$$T_{k+1} \subset \bigcup_{j \in \Omega_k} [x_k^j, z_k^j], \qquad \Delta^- T_k \supset \bigcup_{j \in \Omega_k} \left([t_k^j, x_k^j - 1] \bigcup [z_k^j + 1, s_k^j] \right). \tag{3.5}$$

Let $t_k^j < x_k^j$. Then $y_{t_k^j - 1} \leq \lambda^k$, $y_{x_k^j} > \lambda^{k+1}$ and

$$\lambda^k(\lambda - 1) = \lambda^{k+1} - \lambda^k \leq y_{x_k^j} - y_{t_k^j - 1} = \sum_{i=t_k^j}^{x_k^j - 1} \Delta y_i \leq \left(\sum_{i=t_k^j}^{x_k^j - 1} \rho_i^{1-p'} \right)^{\frac{1}{p'}} \left(\sum_{i=t_k^j}^{x_k^j - 1} \rho_i |\Delta y_i|^p \right)^{\frac{1}{p}}.$$

Whence it follows that

$$\lambda^{pk} \left(\sum_{i=t_k^j}^{x_k^j - 1} \rho_i^{1-p'} \right)^{1-p} \leq \frac{1}{(\lambda - 1)^p} \sum_{i=t_k^j}^{x_k^j - 1} \rho_i |\Delta y_i|^p. \tag{3.6}$$

Similarly, if $z_k^j < s_k^j$, then

$$\lambda^{pk} \left(\sum_{i=z_k^j}^{s_k^j} \rho_i^{1-p'} \right)^{1-p} \leq \frac{1}{(\lambda - 1)^p} \sum_{i=z_k^j}^{s_k^j} \rho_i |\Delta y_i|^p. \tag{3.7}$$

Let $z_k^j = s_k^j$. Then $y_{s_k^j} = y_{z_k^j} > \lambda^{k+1}$, $y_{z_k^j + 1} \leq \lambda^k$ and

$$\lambda^k(\lambda - 1) \leq y_{z_k^j} - y_{z_k^j + 1} = -\Delta y_{z_k^j} = \sum_{i=z_k^j}^{s_k^j} (-\Delta y_i)$$

or

$$\lambda^{pk} \left(\sum_{i=z_k^j}^{s_k^j} \rho_i^{1-p'} \right)^{1-p} \leq \frac{1}{(\lambda - 1)^p} \sum_{i=z_k^j}^{s_k^j} \rho_i |\Delta y_i|^p. \tag{3.8}$$

Similarly, if $t_k^j = x_k^j$, then

$$\lambda^{pk} \left(\sum_{i=t_k^j - 1}^{x_k^j - 1} \rho_i^{1-p'} \right)^{1-p} \leq \frac{1}{(\lambda - 1)^p} \sum_{i=t_k^j - 1}^{x_k^j - 1} \rho_i |\Delta y_i|^p. \tag{3.9}$$

Combining the inequalities (3.6), (3.7), (3.8) and (3.9), we write them in the following way

$$\lambda^{pk} \left(\sum_{i=\tilde{t}_k^j}^{x_k^j - 1} \rho_i^{1-p'} \right)^{1-p} \leq \frac{1}{(\lambda - 1)^p} \sum_{i=\tilde{t}_k^j}^{x_k^j - 1} \rho_i |\Delta y_i|^p, \tag{3.10}$$

$$\lambda^{pk}\left(\sum_{i=\tilde{t}_k^j}^{s_k^j}\rho_i^{1-p'}\right)^{1-p} \le \frac{1}{(\lambda-1)^p}\sum_{i=z_k^j}^{s_k^j}\rho_i|\Delta y_i|^p, \tag{3.11}$$

where $\tilde{t}_k^j = t_k^j$ for $t_k^j < x_k^j$ and $\tilde{t}_k^j = t_k^j - 1$ for $t_k^j = x_k^j$. From $B_p(m,n) < \infty$ we have

$$\sum_{i=x_k^j-1}^{z_k^j-1} v_i \le B_p(m,n)\left[\left(\sum_{i=\tilde{t}_k^j}^{x_k^j-1}\rho_i^{1-p'}\right)^{1-p} + \left(\sum_{i=z_k^j}^{s_k^j}\rho_i^{1-p'}\right)^{1-p}\right]. \tag{3.12}$$

We will use the following notations:

$$\omega_k^+ = \left\{j\in\Omega_k : t_k^j < x_k^j,\ z_k^j < s_k^j\right\}, \qquad \omega_{k,1} = \left\{j\in\Omega_k : t_k^j = x_k^j,\ z_k^j < s_k^j\right\},$$
$$\omega_{k,2} = \left\{j\in\Omega_k : t_k^j < x_k^j,\ z_k^j = s_k^j\right\}, \qquad \omega_k^- = \left\{j\in\Omega_k : t_k^j = x_k^j,\ z_k^j = s_k^j\right\},$$
$$\Delta_{k,1}^+ = \omega_k^+\bigcup\omega_{k,2},\quad \Delta_{k,2}^+ = \omega_k^+\bigcup\omega_{k,1},\qquad \Delta_{k,1}^- = \omega_k^-\bigcup\omega_{k,1},\quad \Delta_{k,2}^- = \omega_k^-\bigcup\omega_{k,2}.$$

It is obvious that $\Omega_k = \omega_k^+\cup\omega_{k,1}\cup\omega_{k,2}\cup\omega_k^-$. The relation for $\Delta^- T_k$ from (3.5) gives that

$$\Delta^- T_k \supset \left(\bigcup_{j\in\Delta_{k,1}^+}[t_k^j, x_k^j - 1]\right)\bigcup\left(\bigcup_{j\in\Delta_{k,2}^+}[z_k^j+1, s_k^j]\right). \tag{3.13}$$

Now we ready to estimate the left-hand side of the inequality (1.6). If $\Delta^- T_k \ne \oslash$, then $\lambda^k < y_i \le \lambda^{k+1}$ for $i\in\Delta^- T_k$ and

$$\sum_{i\in\Delta^- T_k} v_{i-1}|y_i|^p \le \lambda^{p(k+1)}\sum_{i\in\Delta^- T_k} v_{i-1}. \tag{3.14}$$

If $\Delta^- T_k = \oslash$, then by assumption the inequality (3.14) is also correct.

Using (3.4), (3.5), (3.14) and the equality $\lambda^{pk} = (1-\lambda^{-p})\sum_{t=-\infty}^k \lambda^{pt}$, we get

$$\begin{aligned} F &\equiv \sum_{k=m}^n v_{k-1}|y_k|^p = \sum_{k=-\infty}^{\tau-1}\sum_{i\in\Delta^- T_{k+1}} v_{i-1}|y_i|^p \le \sum_{k=-\infty}^{\tau-1}\lambda^{q(k+2)}\sum_{i\in\Delta^- T_{k+1}} v_{i-1} \\ &= \lambda^{2p}\sum_{k=-\infty}^{\tau-1}\lambda^{pk}\sum_{i\in\Delta^- T_{k+1}} v_{i-1} = \lambda^{2p}(1-\lambda^{-p})\sum_{k=-\infty}^{\tau-1}\sum_{i\in\Delta^- T_{k+1}} v_{i-1}\sum_{t=-\infty}^{k}\lambda^{pt} \\ &\le \lambda^p(\lambda^p-1)\sum_{k=-\infty}^{\tau-1}\lambda^{pk}\sum_{j\in\Omega_k}\sum_{i=x_k^j}^{z_k^j} v_{i-1} = \lambda^p(\lambda^p-1)\sum_{k=-\infty}^{\tau-1}\lambda^{pk}\sum_{j\in\Omega_k}\sum_{i=x_k^j-1}^{z_k^j-1} v_i. \end{aligned} \tag{3.15}$$

Substituting (3.12) in (3.15) and using (3.10) and (3.11), we obtain

$$\begin{aligned} F &\le \lambda^p(\lambda^p-1)B_p(m,n)\sum_{k=-\infty}^{\tau-1}\sum_{j\in\Omega_k}\left[\lambda^{pk}\left(\sum_{i=\tilde{t}_k^j}^{x_k^j-1}\rho_i^{1-p'}\right)^{1-p} + \lambda^{pk}\left(\sum_{i=z_k^j}^{s_k^j}\rho_i^{1-p'}\right)^{1-p}\right] \\ &\le \frac{\lambda^p(\lambda^p-1)}{(\lambda-1)^p}B_p(m,n)\sum_{k=-\infty}^{\tau-1}\sum_{j\in\Omega_k}\left[\sum_{i=\tilde{t}_k^j}^{x_k^j-1}\rho_i|\Delta y_i|^p + \sum_{i=z_k^j}^{s_k^j}\rho_i|\Delta y_i|^p\right]. \end{aligned} \tag{3.16}$$

Since

$$\begin{aligned}
&\sum_{j\in\Omega_k}\left(\sum_{i=\tilde{t}_k^j}^{x_k^j-1}\rho_i|\Delta y_i|^p+\sum_{i=z_k^j}^{s_k^j}\rho_i|\Delta y_i|^p\right)\\
&\quad=\sum_{j\in\omega_k^+}\left(\sum_{i=t_k^j}^{x_k^j-1}\rho_i|\Delta y_i|^p+\sum_{i=z_k^j}^{s_k^j}\rho_i|\Delta y_i|^p\right)+\sum_{j\in\omega_{k,1}}\left(\sum_{i=t_k^j-1}^{x_k^j-1}\rho_i|\Delta y_i|^p+\sum_{i=z_k^j}^{s_k^j}\rho_i|\Delta y_i|^p\right)\\
&\qquad+\sum_{j\in\omega_{k,2}}\left(\sum_{i=t_k^j}^{x_k^j-1}\rho_i|\Delta y_i|^p+\sum_{i=z_k^j}^{s_k^j}\rho_i|\Delta y_i|^p\right)+\sum_{j\in\omega_k^-}\left(\sum_{i=t_k^j-1}^{x_k^j-1}\rho_i|\Delta y_i|^p+\sum_{i=z_k^j}^{s_k^j}\rho_i|\Delta y_i|^p\right)\\
&\quad=\left(\sum_{j\in\Delta_{k,1}^+}\sum_{i=t_k^j}^{x_k^j-1}\rho_i|\Delta y_i|^p+\sum_{j\in\Delta_{k,2}^+}\sum_{i=z_k^j+1}^{s_k^j}\rho_i|\Delta y_i|^p\right)\\
&\qquad+\left(\sum_{j\in\Delta_{k,1}^-}\rho_{x_k^j-1}|\Delta y_{x_k^j-1}|^p+\sum_{j\in\Delta_{k,2}^-\bigcup\Delta_{k,2}^+}\rho_{z_k^j}|\Delta y_{z_k^j}|^p\right)\\
&\quad=F_{k,1}+F_{k,2},
\end{aligned}$$

from (3.16) we have

$$F\le\frac{\lambda^p(\lambda^p-1)}{(\lambda-1)^p}B_p(m,n)\sum_{k=-\infty}^{\tau}(F_{k,1}+F_{k,2}). \tag{3.17}$$

Due to (3.13) we have $F_{k,1}\le\sum_{i\in\Delta^-T_k}\rho_i|\Delta y_i|^p$ and then

$$\sum_{k=-\infty}^{\tau}F_{k,1}\le\sum_{k=-\infty}^{\tau}\sum_{i\in\Delta^-T_k}\rho_i|\Delta y_i|^p=\sum_{i=m}^{n}\rho_i|\Delta y_i|^p. \tag{3.18}$$

Since $t_k^j-1=x_k^j-1\le\lambda^k$ for $j\in\Delta_{k,1}^-$ and $z_k^j>\lambda^{k+1}$ for $j\in\Delta_{k,2}^-\bigcup\Delta_{k,2}^+$, there exist integers $k_1=k_1(k,j)<k$ and $k_2=k_2(k,j)>k$ such that $x_k^j-1\in\Delta^-T_{k_1}$ and $z_k^j\in\Delta^-T_{k_2}$. Let us note that $\Delta^-T_\tau=T_\tau$. Therefore,

$$\sum_{k=-\infty}^{\tau}F_{k,2}\le\sum_{k=-\infty}^{\tau}\sum_{i\in\Delta^-T_k}\rho_i|\Delta y_i|^p=\sum_{i=m}^{n}\rho_i|\Delta y_i|^p. \tag{3.19}$$

Thus, from (3.17), (3.18) and (3.19) we get

$$\sum_{i=m}^{n}v_{i-1}|y_i|^p\le2\frac{\lambda^p(\lambda^p-1)}{(\lambda-1)^p}B_p(m,n)\sum_{i=m}^{n}\rho_i|\Delta y_i|^p.$$

The left-hand side of this inequality does not depend on $\lambda>1$. Hence, taking infimum with respect to $\lambda>1$ in the right-hand side, we have

$$\sum_{i=m}^{n}v_i|y_i|^p\le2\alpha_pB_p(m,n)\sum_{i=m}^{n}\rho_i|\Delta y_i|^p,$$

i.e., the inequality (1.6) holds with the estimate

$$C\le2\alpha_pB_p(m,n)$$

for the least constant C. The last estimate together with (3.3) gives (2.1). The estimate (2.2) follows from Lemma 3.1 proved below. The proof of Theorem 2.1 is complete. □

Lemma 3.1. *Let $f(\lambda) = \frac{\lambda^p(\lambda^p-1)}{(\lambda-1)^p}$, $\lambda > 1$. Then there exists a point λ_p such that*

$$\frac{2p}{p+1} < \lambda_p < \min\{p, 2\}, \tag{3.20}$$

$$\inf_{\lambda>1} f(\lambda) = f(\lambda_p) = \frac{\lambda_p^{2p}}{(\lambda_p-1)^{p-1}}, \qquad f(\lambda_2) = \frac{\sqrt{5}+7}{\sqrt{5}-1} \tag{3.21}$$

and for $p \neq 2$ the estimate

$$\gamma_0 < f(\lambda_p) < \min\{\gamma_1, \gamma_2, 4^p\} \tag{3.22}$$

holds.

Proof. The function f is continuous for $\lambda > 1$ and $\lim_{\lambda\to 1^+} f(\lambda) = \infty$, $\lim_{\lambda\to\infty} f(\lambda) = \infty$. Therefore, f has a minimum. The derivative of f we present in the form

$$f'(\lambda) = \frac{p\lambda^{p-1}(\lambda^p-1)^{p+1}}{\lambda-1}\varphi(\lambda) = \frac{p\lambda^{p-1}}{(\lambda-1)^{p+2}}d(\lambda), \tag{3.23}$$

where $\varphi(\lambda) = \frac{\lambda^p}{\lambda^p-1} - \frac{1}{\lambda-1}$ and $d(\lambda) = \lambda^{p+1} - 2\lambda^p + 1$. For $p = 2$ we have that $d(\lambda) = (\lambda-1)(\lambda^2-\lambda-1)$. It means that $d(\lambda_2) = f'(\lambda_2) = 0$ when $\lambda_2 = \frac{1+\sqrt{5}}{2}$. Therefore, $\inf_{\lambda>1} f(\lambda) = f(\lambda_2) = \frac{\sqrt{5}+7}{\sqrt{5}-1}$ for $p = 2$.

Now we turn to the case $p \neq 2$. Let $\lambda = 1+\epsilon$, $\epsilon > 0$. Using Lagrange's mean value theorem, we have

$$\varphi(\lambda) = \frac{(1+\epsilon)^p}{(1+\epsilon)^p-1} - \frac{1}{\epsilon} \geq \frac{(1+\epsilon)^p}{p\epsilon(1+\epsilon)^{p-1}} - \frac{1}{\epsilon} = \frac{1}{p\epsilon}(\epsilon-(p-1)).$$

Whence it follows that $\varphi(\lambda) > 0$. Thereby $f'(\lambda) > 0$ for $\lambda > p$. The function $d(\lambda)$ at the point $\lambda = \frac{2p}{p+1}$ has a minimum, decreases for $1 < \lambda \leq \frac{2p}{p+1}$ and increases for $\lambda > \frac{2p}{p+1}$. Since $d(2) = 1 > 0$, then $d(\lambda) > 0$ and $f'(\lambda) > 0$ for $\lambda \geq 2$. Thus, $f'(\lambda) > 0$ for $\lambda > \min\{p, 2\}$. Since $d(1) = 0$, then $d(\frac{2p}{p+1}) < 0$. Therefore, $f'(\lambda) < 0$ for $\lambda \leq \frac{2p}{p+1}$. Due to continuity of the function $f'(\lambda)$ for $\lambda > 1$, there exists a point λ_p satisfying the condition (3.20). Since $f'(\lambda_p) = \varphi(\lambda_p) = 0$ and the point λ_p is a point of intersection of two decreasing functions $\frac{\lambda^p}{\lambda^p-1}$ and $\frac{1}{(\lambda-1)^p}$, then the function $f(\lambda)$ decreases for $1 < \lambda < \lambda_p$, increases for $\lambda > \lambda_p$ and has a minimum at the point λ_p, i.e., $\inf_{\lambda>1} f(\lambda) = f(\lambda_p)$. Since $\frac{\lambda_p^p}{\lambda_p^p-1} = \frac{1}{\lambda_p-1}$ or $\lambda_p^p - 1 = \lambda_p^p(\lambda_p-1)$, then, substituting the last equality in the expression $f(\lambda_p)$, we get $f(\lambda_p) = \frac{\lambda_p^{2p}}{(\lambda_p-1)^{p-1}}$, i.e., (3.21) holds. The function $g(t) = \frac{t^{2p}}{(t-1)^{p-1}}$ has the least value at the point $\frac{2p}{p+1}$. Since $\lambda_p > \frac{2p}{p+1}$, then $f(\lambda_p) > g(\frac{2p}{p+1})$. Therefore,

$$f(\lambda_p) = g(\lambda_p) > g\left(\frac{2p}{p+1}\right) = \frac{2p}{p+1}\left(\frac{2p}{p+1}\right)^p\left(\frac{2p}{p-1}\right)^{p-1} = \gamma_0. \tag{3.24}$$

On the other hand

$$f(\lambda_p) < \min\left\{f\left(\frac{2p}{p+1}\right), \ f(p), \ f(2)\right\}. \tag{3.25}$$

It is easy to see that $f(2) < 4^p$. Since $\varphi(\lambda_p) = 0$ and $\lambda_p < p$, then $\frac{p^p}{p^p-1} > \frac{1}{p-1}$ or $p^p(p-1) > p^p - 1$. Therefore,

$$f(p) = \frac{p^p(p^p-1)}{(p-1)^p} < \frac{p^{2p}}{(p-1)^{p-1}} = pp^p\left(\frac{p}{p-1}\right)^{p-1} = \gamma_2. \tag{3.26}$$

Moreover,

$$f\left(\frac{2p}{p+1}\right) = \left(\frac{2p}{p-1}\right)^p \left[\left(1+\frac{p-1}{p+1}\right)^p - 1\right] \leq \left(\frac{2p}{p-1}\right)^p p\frac{p-1}{p+1}\left(\frac{2p}{p+1}\right)^{p-1}$$
$$= p\left(\frac{2p}{p+1}\right)^p \left(\frac{2p}{p-1}\right)^{p-1} = \gamma_1. \tag{3.27}$$

From (3.24), (3.25), (3.26) and (3.27), taking into account $f(2) < 4^p$, we have (3.22). The proof of Lemma 3.1 is complete. □

Proof of Lemma 2.8. Suppose that $\dot{Y}(m,\infty)$ is the set of non-trivial sequences $y = \{y_i\}_{i=m}^{\infty}$, which has a finite number of first members equal to zero.

Let $\omega_p^1(m,\infty)$ be the set of all number sequences $y = \{y_i\}_{i=m}^{\infty}$ with the finite norm

$$\|y\|_{\omega_p^1} = |y_m| + \left(\sum_{i=m}^{\infty} \rho_i |\Delta y_i|^p\right)^{\frac{1}{p}}. \tag{3.28}$$

Let $\dot{\omega}_p^1(m,\infty)$ and $\mathring{\omega}_p^1(m,\infty)$ be the closures of the sets $\dot{Y}(m,\infty) \bigcap \omega_p^1(m,\infty)$ and $\mathring{Y}(m,\infty)$, respectively, with respect to the norm (3.28).

Let $y = \{y_i\}_{i=1}^{\infty}$ be an arbitrary element $\dot{\omega}_p^1(m,\infty)$. Then by the definition of $\dot{\omega}_p^1(m,\infty)$ there exists a sequence $\{y_n\} \subset \dot{Y}(m,\infty) \bigcap \omega_p^1(m,\infty)$ such that $\|y - y_n\|_{\omega_p^1} \to 0$ for $n \to 0$. Then from the definition of the norm (3.28) it follows that $|y_m - y_{n,m}| \to 0$ for $n \to \infty$. However $y_{n,m} = 0$ for all n, therefore, $y_m = 0$. Now we prove the equality $\dot{\omega}_p^1(m,\infty) = \mathring{\omega}_p^1(m,\infty)$. The inclusion $\dot{\omega}_p^1(m,\infty) \supset \mathring{\omega}_p^1(m,\infty)$ is obvious. Let us prove the inclusion $\mathring{\omega}_p^1(m,\infty) \supset \dot{\omega}_p^1(m,\infty)$. Let $z = \{z_i\}$ be an arbitrary element from $\dot{\omega}_p^1(m,\infty)$. In view of the condition $\sum_{i=m}^{\infty} \rho_i^{1-p'} = \infty$ for any $n > m$ there exists an integer number n^* such that $n^* > n$ and

$$|z_n| \left(\sum_{i=n}^{n^*} \rho_i^{1-p'}\right)^{-\frac{1}{p'}} \leq \left(\sum_{i=n}^{\infty} \rho_i |\Delta z_i|^p\right)^{\frac{1}{p}}. \tag{3.29}$$

Let

$$y_{n,i} = \begin{cases} z_i, & m \leq i \leq n-1, \\ z_n \left(\sum_{i=n}^{n^*} \rho_i^{1-p'}\right)^{-1} \sum_{j=i}^{n^*} \rho_j^{1-p'}, & n \leq i \leq n^*, \\ 0, & i > n^*. \end{cases}$$

It is obvious that $y_n = \{y_{n,i}\} \in \mathring{Y}(m,\infty)$ for all $n > m$.

Using the triangle inequality and (3.29), we have

$$\|z - y_n\|_{\omega_p^1} \leq \left(\sum_{i=n}^{n^*} \rho_i |\Delta z_i - \Delta y_{n,i}|^p\right)^{\frac{1}{p}} + \left(\sum_{i=n^*}^{\infty} \rho_i |\Delta z_i|^p\right)^{\frac{1}{p}}$$
$$\leq \left(\sum_{i=n}^{n^*} \rho_i |\Delta z_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=n}^{n^*} \rho_i |\Delta y_{n,i}|^p\right)^{\frac{1}{p}} + \left(\sum_{i=n^*}^{\infty} \rho_i |\Delta z_i|^p\right)^{\frac{1}{p}}$$
$$\leq 2\left(\sum_{i=n}^{\infty} \rho_i |\Delta z_i|^p\right)^{\frac{1}{p}} + |z_n| \left(\sum_{i=n}^{n^*} \rho_i |^{1-p'}\right)^{-\frac{1}{p'}} \leq 3\left(\sum_{i=n}^{\infty} \rho_i |\Delta z_i|^p\right)^{\frac{1}{p}}.$$

Whence it follows that $\|z - y_n\|_{\omega_p^1} \to 0$ for $n \to \infty$. Therefore, $z \in \mathring{\omega}_p^1(m,\infty)$, and due to arbitrariness of $z \in \dot{\omega}_p^1(m,\infty)$ we have $\dot{\omega}_p^1(m,\infty) \subset \mathring{\omega}_p^1(m,\infty)$. Thus, $\dot{\omega}_p^1(m,\infty) = \mathring{\omega}_p^1(m,\infty)$. Since the space $\mathring{\omega}_p^1(m,\infty)$ is the closure of the set $\mathring{Y}(m,\infty)$ with respect to the norm (3.28), then the correctness of the inequality (1.6) on the set $\mathring{Y}(m,\infty)$ is equivalent to its correctness on the set $\dot{\omega}_p^1(m,\infty) = \mathring{\omega}_p^1(m,\infty)$.

For any $y \in \mathring{\omega}_p^1(m,\infty)$ and any $k \geq m$ we have $y_{k+1} = \sum_{i=m}^{k} \Delta y_i + y_m = \sum_{i=m}^{k} \Delta y_i$. Therefore, the left-hand side of the inequality (1.6) for $n = \infty$ has the form

$$\sum_{k=m}^{\infty} v_{k-1}|y_k|^p = \sum_{k=m+1}^{\infty} v_{k-1}|y_k|^p = \sum_{i=m}^{\infty} v_i|y_{i+1}|^p = \sum_{k=m}^{\infty} v_k|y_{k+1}|^p = \sum_{k=m}^{\infty} v_k \left|\sum_{i=m}^{k} \Delta y_i\right|^p .$$

Now, assuming $\Delta y_i = a_i$ in the inequality (1.6), we have that the inequality (2.9) holds. Inversely, assuming $y_m = 0$ and $y_{k+1} = \sum_{i=m}^{k} \Delta y_i$ in (2.9), we get the correctness of the inequality (1.6) on the set $\mathring{\omega}_p^1(m,\infty)$. Thus, due to $\dot{\omega}_p^1(m,\infty) = \mathring{\omega}_p^1(m,\infty)$ it is also correct on the set $\mathring{Y}(m,\infty)$. The proof of Lemma 2.8 is complete. □

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