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Controllability of nonlinear delay oscillating systems

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Abstract. In this paper, we study the controllability of a system governed by second order delay differential equations. We introduce a delay Gramian matrix involving the delayed matrix sine, which is used to establish sufficient and necessary conditions of controllability for the linear problem. In addition, we also construct a specific control function for controllability. For the nonlinear problem, we construct a control function and transfer the controllability problem to a fixed point problem for a suitable operator. We give a sufficient condition to guarantee the nonlinear delay system is controllable. Two examples are given to illustrate our theoretical results by calculating a specific control function and inverse of a delay Gramian matrix.

Keywords: controllability, delay Gramian matrix, control function, delay oscillating systems.

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1 Introduction

It is well-known that delay differential equations arise naturally in economics, physics and control problems. It is not an easy task to construct a fundamental matrix for linear differential delay systems, even for a simple first order delay system $\dot{x}(t) = Ax(t) + Bx(t-\tau), t \geq 0$ with initial condition $x(t) = \varphi(t), t \in [-\tau, 0], \tau > 0$, where A, B are suitable constant matrices. Khusainov and Shuklin in [14] introduced the delayed matrix exponential $e_\tau^{Bt} : \mathbb{R} \to \mathbb{R}^n$ [14, Definition 0.3] and derived an explicit formula for solutions to such linear differential delay systems if we have $AB = BA$. Diblík and Khusainov [7] adopted the idea to construct the discrete matrix delayed exponential, and it was used to derive an explicit formula for solutions to a discrete delay system. There are a few recent results in the literature on existence, stability and control theory for delay differential, discrete and impulsive equations; see for example, [2–6, 8–11, 13, 15, 17–28, 30, 32]. We also remark that there exists possible connection between delay effect and memory property for fractional derivatives, which involved in fractional differential equations. For more recent development on stability and BVP for fractional differential equations, see for example, [1, 12, 29, 31].

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Khusainov et al. [13] studied the following Cauchy problem for a second order linear differential equation with pure delay:

$$\begin{cases} \ddot{x}(t) + \Omega^2 x(t-\tau) = f(t), & t \geq 0,\ \tau > 0, \\ x(t) = \varphi(t), \quad \dot{x}(t) = \dot{\varphi}(t), & -\tau \leq t \leq 0, \end{cases} \tag{1.1}$$

where $f : [0,\infty) \to \mathbb{R}^n$, Ω is a $n \times n$ nonsingular matrix, τ is the time delay and φ is an arbitrary twice continuously differentiable vector function. A solution of (1.1) has an explicit representation of the form [13, Theorem 2]:

$$\begin{aligned} x(t) = (\cos_\tau \Omega t)\varphi(-\tau) + \Omega^{-1}(\sin_\tau \Omega t)\dot{\varphi}(-\tau) + \Omega^{-1}\int_{-\tau}^{0} \sin_\tau \Omega(t-\tau-s)\ddot{\varphi}(s)ds \\ + \Omega^{-1}\int_0^t \sin_\tau \Omega(t-\tau-s) f(s)ds, \end{aligned} \tag{1.2}$$

where $\cos_\tau \Omega : \mathbb{R} \to \mathbb{R}^{n\times n}$ [13, Definition 1] and $\sin_\tau \Omega : \mathbb{R} \to \mathbb{R}^{n\times n}$ [13, Definition 2] denote the delayed matrix cosine of polynomial degree $2k$ on the intervals $(k-1)\tau \leq t < k\tau$ and the delayed matrix sine of polynomial degree $2k+1$ on the intervals $(k-1)\tau \leq t < k\tau$, respectively. More precisely,

$$\cos_\tau \Omega t = \begin{cases} \Theta, & -\infty < t < -\tau, \\ I, & -\tau \leq t < 0, \\ I - \Omega^2 \frac{t^2}{2!}, & 0 \leq t < \tau, \\ \vdots & \vdots \\ I - \Omega^2\frac{t^2}{2!} + \Omega^4 \frac{(t-\tau)^4}{4!} + \cdots + (-1)^k \Omega^{2k} \frac{[t-(k-1)\tau]^{2k}}{(2k)!}, & (k-1)\tau \leq t < k\tau,\ k \geq 0, \\ \vdots & \vdots \end{cases} \tag{1.3}$$

and

$$\sin_\tau \Omega t = \begin{cases} \Theta, & -\infty < t < -\tau, \\ \Omega(t+\tau), & -\tau \leq t < 0, \\ \Omega(t+\tau) - \Omega^3 \frac{t^3}{3!}, & 0 \leq t < \tau, \\ \vdots & \vdots \\ \Omega(t+\tau) - \Omega^3\frac{t^3}{3!} + \cdots + (-1)^k \Omega^{2k+1} \frac{[t-(k-1)\tau]^{2k+1}}{(2k+1)!}, & (k-1)\tau \leq t < k\tau,\ k \geq 0, \\ \vdots & \vdots \end{cases} \tag{1.4}$$

where Θ and I are the zero and identity matrices, respectively.

Diblík et al. [8] studied a control problem for a system governed by the following delay oscillating equations:

$$\begin{cases} \ddot{x}(t) + \Omega^2 x(t-\tau) = bu(t), & t \in [0, t_1],\ \tau > 0,\ t_1 > 0, \\ x(t) = \varphi(t), \quad \dot{x}(t) = \dot{\varphi}(t), & t \in [-\tau, 0], \end{cases} \tag{1.5}$$

where $b \in \mathbb{R}^n$ and $u : [0,\infty) \to \mathbb{R}$ and they give sufficient and necessary conditions of relative controllability [8, Theorem 3.8] for (1.5) from the point of view of the rank criteria

$$\operatorname{rank}\left(b, \Omega^2 b, \Omega^4 b, \ldots, \Omega^{2(n-1)} b\right) = n \tag{1.6}$$

provided by $t_1 > (n-1)\tau$. In addition, an explicit dependence of the control function related to $\sin_\tau \Omega$ and $\cos_\tau \Omega$ for (1.6) was given in [8, Theorem 3.9]

$$u^*(t) = b^{\mathrm{T}}(\Omega^{-1}\sin_\tau \Omega(t_1-\tau-t))^{\mathrm{T}}C_1^0 + b^{\mathrm{T}}(\cos_\tau \Omega(t_1-\tau-t))^{\mathrm{T}}C_2^0,$$

where $C_1^0 = (c_1^0, \dots, c_n^0)^{\mathrm{T}}$ and $C_2^0 = (c_{n+1}^0, \dots, c_{2n}^0)^{\mathrm{T}}$ are the solutions of the algebraic equation in [8, (3.45)].

In this paper, we use a different approach to that in [8] to study controllability of a system governed by the following Cauchy problem:

$$\begin{cases} \ddot{x}(t) + \Omega^2 x(t-\tau) = f(t, x(t)) + Bu(t), & \tau > 0,\ t \in [0, t_1], \\ x(t) = \varphi(t), \quad \dot{x}(t) = \dot{\varphi}(t), & -\tau \le t \le 0, \end{cases} \tag{1.7}$$

where $f : J \times \mathbb{R}^n \to \mathbb{R}^n$, B is a $n \times m$ matrix and an input $u : [0, t_1] \to \mathbb{R}^m$.

From (1.2), a solution of system (1.7) can be formulated as

$$\begin{aligned} x(t) = (\cos_\tau \Omega t)\varphi(-\tau) + \Omega^{-1}(\sin_\tau \Omega t)\dot{\varphi}(-\tau) + \Omega^{-1}\int_{-\tau}^{0} \sin_\tau \Omega(t-\tau-s)\ddot{\varphi}(s)ds \\ + \Omega^{-1}\int_0^t \sin_\tau \Omega(t-\tau-s) f(s, x(s))ds + \Omega^{-1}\int_0^t \sin_\tau \Omega(t-\tau-s)Bu(s)ds. \end{aligned} \tag{1.8}$$

We give sufficient and necessary conditions of controllability for the linear second-order delay differential system (1.7) with $f(\cdot, x) = 0$ from the point of view of the delay Gramian matrix. In addition, we construct a specific control function for the controllability problem of transferring an initial function to a prescribed point in the phase space. Then, we construct a specific control function involving a nonlinear term and apply a fixed point result to establish a sufficient condition of controllability for the nonlinear system (1.7) by using properties of the delayed matrix sine and the delayed matrix cosine.

2 Preliminary

Let $\mathbb{R}^n$ be the n-dimensional Euclid space with the vector norm $\|\cdot\|$. Set $J = [0, t_1]$, $t_1 > 0$. Denote by $C(J, \mathbb{R}^n)$ the Banach space of vector-valued continuous functions from $J \to \mathbb{R}^n$ endowed with the norm $\|x\|_{C(J)} = \max_{t \in J}\|x(t)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$. We also introduce the Banach space $C^2(J, \mathbb{R}^n) = \{x \in C(J, \mathbb{R}^n) : \ddot{x} \in C(J, \mathbb{R}^n)\}$ endowed with the norm $\|x\|_{C^2(J)} = \max_{t \in J}\{\|x(t)\|, \|\dot{x}(t)\|, \|\ddot{x}(t)\|\}$. Let X, Y be two Banach spaces and $L_b(X, Y)$ be the space of bounded linear operators from X to Y. Now, $L^p(J, Y)$ denotes the Banach space of functions $f : J \to Y$ which are Bochner integrable normed by $\|f\|_{L^p(J,Y)}$ for some $1 < p < \infty$. For $A : \mathbb{R}^n \to \mathbb{R}^n$, we consider its matrix norm $\|A\| = \max_{\|x\|=1}\|Ax\|$ generated by $\|\cdot\|$. In this paper we let $\|\varphi\|_C = \max_{s \in [-\tau, 0]}\|\varphi(s)\|$, $\|\dot{\varphi}\|_C = \max_{s \in [-\tau, 0]}\|\dot{\varphi}(s)\|$ and $\|\ddot{\varphi}\|_C = \max_{s \in [-\tau, 0]}\|\ddot{\varphi}(s)\|$.

Definition 2.1. System (1.7) is controllable if there exists a control function $u^* : [0, t_1] \to \mathbb{R}^m$ such that

$$\ddot{x}(t) + \Omega^2 x(t-\tau) = f(t, x(t)) + Bu^*(t)$$

has a solution $x = x^* : [-\tau, t_1] \to \mathbb{R}^n$ satisfying

$$\begin{aligned} x^*(t) &= \varphi(t), & \dot{x}^*(t) &= \dot{\varphi}(t), & -\tau \le t \le 0, \\ x^*(t_1) &= x_1, & \dot{x}^*(t_1) &= x_1', \end{aligned}$$

where $x_1, x_1' \in \mathbb{R}^n$ are any finite terminal conditions and t_1 is an arbitrary given terminal point.

For our investigation, we recall the following results.

Lemma 2.2 ([13, Lemmas 1 and 2]). *The following rules of differentiation are true for the matrix functions* (1.3) *and* (1.4)*:*

$$\frac{d}{dt}\cos_\tau \Omega t = -\Omega \sin_\tau \Omega(t-\tau), \qquad \frac{d}{dt}\sin_\tau \Omega t = \Omega \cos_\tau \Omega t, \qquad t \in \mathbb{R}.$$

Lemma 2.3 ([19, Lemmas 2.5 and 2.6]). *For any $t \in [(k-1)\tau, k\tau)$, $k = 0, 1, \ldots$, the following norm estimates hold:*

$$\|\cos_\tau \Omega t\| \le \cosh(\|\Omega\| t), \qquad \|\sin_\tau \Omega t\| \le \sinh[\|\Omega\|(t+\tau)].$$

Lemma 2.4 ([16, Krasnoselskii's fixed point theorem]). *Let $\mathcal{B}$ be a bounded closed and convex subset of a Banach space X and let F_1, F_2 be maps from $\mathcal{B}$ into X such that $F_1x + F_2y \in \mathcal{B}$ for every pair $x,\ y \in \mathcal{B}$. If F_1 is a contraction and $F_2 : \mathcal{B} \to X$ is continuous and compact, then the equation $F_1x + F_2x = x$ has a solution on $\mathcal{B}$.*

3 Controllability of linear delay system

In this section, we study controllability of a system governed by a second order linear delay differential equation:

$$\begin{cases} \ddot{x}(t) + \Omega^2 x(t-\tau) = Bu(t), & t \in [0, t_1],\ \tau > 0, \\ x(t) = \varphi(t), \quad \dot{x}(t) = \dot{\varphi}(t), & t \in [-\tau, 0]. \end{cases} \tag{3.1}$$

We introduce a delay Gramian matrix (an extension of the classical Gramian matrix for linear differential systems) as follows:

$$W_\tau[0, t_1] = \Omega^{-1} \int_0^{t_1} \sin_\tau \Omega(t_1 - \tau - s) BB^{\mathrm{T}} \sin_\tau \Omega^{\mathrm{T}}(t_1 - \tau - s) ds. \tag{3.2}$$

We give a new sufficient and necessary condition to guarantee (3.1) is controllable.

Theorem 3.1. *System* (3.1) *is controllable if and only if $W_\tau[0, t_1]$ defined in* (3.2) *is non-singular.*

Proof. First we establish sufficiency. Since $W_\tau[0, t_1]$ is non-singular, its inverse $W_\tau^{-1}[0, t_1]$ is well-defined. Thus, for any finite terminal conditions $x_1, x_1' \in \mathbb{R}^n$, one can construct the corresponding control input $u(t)$ as

$$u(t) = B^{\mathrm{T}} \sin_\tau \Omega^{\mathrm{T}}(t_1 - \tau - t) W_\tau^{-1}[0, t_1]\beta, \tag{3.3}$$

where

$$\beta = x_1 - (\cos_\tau \Omega t_1)\varphi(-\tau) - \Omega^{-1}(\sin_\tau \Omega t_1)\dot{\varphi}(-\tau) - \Omega^{-1} \int_{-\tau}^{0} \sin_\tau \Omega(t_1 - \tau - s)\ddot{\varphi}(s) ds. \tag{3.4}$$

From (1.8), the solution $x(t_1)$ of system (3.1) can be formulated as:

$$\begin{aligned} x(t_1) = {} & (\cos_\tau \Omega t_1)\varphi(-\tau) + \Omega^{-1}(\sin_\tau \Omega t_1)\dot{\varphi}(-\tau) + \Omega^{-1} \int_{-\tau}^{0} \sin_\tau \Omega(t_1 - \tau - s)\ddot{\varphi}(s) ds \\ & + \Omega^{-1} \int_0^{t_1} \sin_\tau \Omega(t_1 - \tau - s) Bu(s) ds. \end{aligned} \tag{3.5}$$

Put (3.3) into (3.5), and we obtain

$$x(t_1) = (\cos_\tau \Omega t_1)\varphi(-\tau) + \Omega^{-1}(\sin_\tau \Omega t_1)\dot{\varphi}(-\tau) + \Omega^{-1}\int_{-\tau}^{0} \sin_\tau \Omega(t_1 - \tau - s)\ddot{\varphi}(s)ds$$
$$+ \Omega^{-1}\int_{0}^{t_1} \sin_\tau \Omega(t_1 - \tau - s)BB^{\mathrm{T}} \sin_\tau \Omega^{\mathrm{T}}(t_1 - \tau - s)dsW_\tau^{-1}[0, t_1]\beta. \tag{3.6}$$

Now (3.2), (3.4) and (3.6) give

$$x(t_1) = (\cos_\tau \Omega t_1)\varphi(-\tau) + \Omega^{-1}(\sin_\tau \Omega t_1)\dot{\varphi}(-\tau) + \Omega^{-1}\int_{-\tau}^{0} \sin_\tau \Omega(t_1 - \tau - s)\ddot{\varphi}(s)ds + \beta$$
$$= x_1,$$

and now use Lemma 2.2 to obtain

$$\dot{x}(t_1) = \frac{d}{dt}\left\{(\cos_\tau \Omega t_1)\varphi(-\tau) + \Omega^{-1}(\sin_\tau \Omega t_1)\dot{\varphi}(-\tau) + \Omega^{-1}\int_{-\tau}^{0} \sin_\tau \Omega(t_1 - \tau - s)\ddot{\varphi}(s)ds + \beta\right\}$$
$$= x_1'.$$

Next, we check the initial conditions $x(t) = \varphi(t)$, $\dot{x}(t) = \dot{\varphi}(t)$ holds when $-\tau \le t \le 0$. From (1.3) and (1.4), the following relations hold:

$$\cos_\tau \Omega t = I, \qquad \sin_\tau \Omega t = \Omega(t + \tau), \qquad -\tau \le t \le 0,$$

$$\sin_\tau \Omega(t - \tau - s) = \begin{cases} \Theta, & t < s \le 0, \\ \Omega(t - s), & -\tau \le s \le t. \end{cases}$$

Linking (1.8) and the above relations, the solution of (3.1) can be expressed by

$$x(t) = \varphi(-\tau) + (t + \tau)\dot{\varphi}(-\tau) + \Omega^{-1}\int_{-\tau}^{t} \sin_\tau \Omega(t - \tau - s)\ddot{\varphi}(s)ds. \tag{3.7}$$

Integrating by parts and using Lemma 2.2 yields

$$\int_{-\tau}^{t} \sin_\tau \Omega(t - \tau - s)\ddot{\varphi}(s)ds = \int_{-\tau}^{t} \sin_\tau \Omega(t - \tau - s)d\dot{\varphi}(s)$$
$$= \sin_\tau \Omega(t - \tau - s)\dot{\varphi}(s)|_{-\tau}^{t} - \int_{-\tau}^{t} \dot{\varphi}(s)d \sin_\tau \Omega(t - \tau - s)$$
$$= -(t + \tau)\Omega\dot{\varphi}(-\tau) + \Omega\varphi(t) - \Omega\varphi(-\tau). \tag{3.8}$$

Put (3.8) into (3.7), and we get

$$x(t) = \varphi(-\tau) + (t + \tau)\dot{\varphi}(-\tau) + \Omega^{-1}\left[-\Omega(t + \tau)\dot{\varphi}(-\tau) + \Omega\varphi(t) - \Omega\varphi(-\tau)\right]$$
$$= \varphi(t).$$

Now $\dot{x}(t) = \dot{\varphi}(t)$ holds. Thus, (3.1) is controllable according to Definition 2.1.

Next we establish necessity. Assume the delay Gramian matrix $W_\tau[0, t_1]$ is singular, and then $W_\tau[0, t_1][\Omega^{-1}]^{\mathrm{T}}$ is singular too. Thus, there exists at least one nonzero state $\bar{x} \in \mathbb{R}^n$ such that

$$\bar{x}^{\mathrm{T}} W_\tau[0, t_1][\Omega^{-1}]^{\mathrm{T}}\bar{x} = 0.$$

It follows from (3.2) that

$$\begin{aligned}0 &= \bar{x}^{\mathrm{T}} W_\tau[0,t_1][\Omega^{-1}]^{\mathrm{T}}\bar{x}\\ &= \int_0^{t_1} \bar{x}^{\mathrm{T}}\Omega^{-1}\sin_\tau\Omega(t_1-\tau-s)BB^{\mathrm{T}}\sin_\tau\Omega^{\mathrm{T}}(t_1-\tau-s)[\Omega^{-1}]^{\mathrm{T}}\bar{x}ds\\ &= \int_0^{t_1}\left[\bar{x}^{\mathrm{T}}\Omega^{-1}\sin_\tau\Omega(t_1-\tau-s)B\right]\left[\bar{x}^{\mathrm{T}}\Omega^{-1}\sin_\tau\Omega(t_1-\tau-s)B\right]^{\mathrm{T}}ds\\ &= \int_0^{t_1}\left\|\bar{x}^{\mathrm{T}}\Omega^{-1}\sin_\tau\Omega(t_1-\tau-s)B\right\|^2 ds.\end{aligned}$$

This implies that

$$\bar{x}^{\mathrm{T}}\Omega^{-1}\sin_\tau\Omega(t_1-\tau-s)B = (\underbrace{0,\ldots,0}_{m}), \qquad \forall\, s\in J. \tag{3.9}$$

Since (3.1) is controllable, it can be driven from any continuously differentiable initial vector functions $\varphi,\dot{\varphi}:[-\tau,0]\to\mathbb{R}^n$ to an arbitrary state $x(t_1)\in\mathbb{R}^n$. Hence there exists a control $u_0(t)$ that drives the initial state to zero. This means that

$$\begin{aligned}x(t_1) =&\ \cos_\tau\Omega t_1\varphi(-\tau)+\Omega^{-1}\sin_\tau\Omega t_1\dot{\varphi}(-\tau)+\Omega^{-1}\int_{-\tau}^0\sin_\tau\Omega(t_1-\tau-s)\ddot{\varphi}(s)ds\\ &+\Omega^{-1}\int_0^{t_1}\sin_\tau\Omega(t_1-\tau-s)Bu_0(s)ds\\ =&\ \mathbf{0},\end{aligned} \tag{3.10}$$

where $\mathbf{0}$ denotes the n dimensional zero vector.

Moreover, there exists a control $\tilde{u}(t)$ that drives the initial state to the state $\bar{x}$, so

$$\begin{aligned}x(t_1) =&\ \cos_\tau\Omega t_1\varphi(-\tau)+\Omega^{-1}\sin_\tau\Omega t_1\dot{\varphi}(-\tau)+\Omega^{-1}\int_{-\tau}^0\sin_\tau\Omega(t_1-\tau-s)\ddot{\varphi}(s)ds\\ &+\Omega^{-1}\int_0^{t_1}\sin_\tau\Omega(t_1-\tau-s)B\tilde{u}(s)ds\\ =&\ \bar{x}.\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) gives

$$\bar{x} = \Omega^{-1}\int_0^{t_1}\sin_\tau\Omega(t_1-\tau-s)B[\tilde{u}(s)-u_0(s)]ds.$$

Multiplying both the sides of the equality by $\bar{x}^{\mathrm{T}}$, we get

$$\bar{x}^{\mathrm{T}}\bar{x} = \int_0^{t_1}\bar{x}^{\mathrm{T}}\Omega^{-1}\sin_\tau\Omega(t_1-\tau-s)B[\tilde{u}(s)-u_0(s)]ds.$$

Note that (3.9), we obtain $\bar{x}^{\mathrm{T}}\bar{x}=0$. That is, $\bar{x}=\mathbf{0}$, which conflicts with $\bar{x}$ being nonzero. Thus, the delay Gramian matrix $W_\tau[0,t_1]$ is non-singular. □

4 Controllability of nonlinear problem

In this section, we apply a fixed point method to establish a sufficient condition of controllability for (1.7).

We assume the following.

(H_1) $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous (here $J = [0, t_1]$), and there exist $L_f \in L^q(J, \mathbb{R}^+)$ and $q > 1$ such that

$$\|f(t, x_1) - f(t, x_2)\| \leq L_f(t)\|x_1 - x_2\|,$$

let $M_f = \sup_{t \in J} \|f(t, 0)\|$.

(H_2) Consider the operator $W : L^2(J, \mathbb{R}^m) \to \mathbb{R}^n$ given by

$$W = \Omega^{-1} \int_0^{t_1} \sin_\tau \Omega(t_1 - \tau - s) B u(s) ds.$$

Suppose that W^{-1} exists, and there exists a constant $M_1 > 0$ such that

$$\|W^{-1}\|_{L_b(\mathbb{R}^n, L^2(J, \mathbb{R}^m)/\ker W)} \leq M_1.$$

Next, consider a control function u_x of the form:

$$\begin{aligned} u_x(t) = W^{-1}\Bigg[& x_1 - (\cos_\tau \Omega t_1)\varphi(-\tau) - \Omega^{-1}(\sin_\tau \Omega t_1)\dot{\varphi}(-\tau) \\ & - \Omega^{-1} \int_{-\tau}^{0} \sin_\tau \Omega(t_1 - \tau - s)\ddot{\varphi}(s) ds \\ & - \Omega^{-1} \int_0^{t_1} \sin_\tau \Omega(t_1 - \tau - s) f(s, x(s)) ds \Bigg](t), \qquad t \in J. \end{aligned} \tag{4.1}$$

We define an operator $T : C([-\tau, t_1], \mathbb{R}^n) \to C([-\tau, t_1], \mathbb{R}^n)$ as follows:

$$\begin{aligned} (Tx)(t) = (\cos_\tau \Omega t)\varphi(-\tau) + \Omega^{-1}(\sin_\tau \Omega t)\dot{\varphi}(-\tau) + \Omega^{-1} \int_{-\tau}^{0} \sin_\tau \Omega(t - \tau - s)\ddot{\varphi}(s) ds \\ + \Omega^{-1} \int_0^t \sin_\tau \Omega(t - \tau - s) f(s, x(s)) ds + \Omega^{-1} \int_0^t \sin_\tau \Omega(t - \tau - s) B u_x(s) ds. \end{aligned} \tag{4.2}$$

For each positive number ϵ, let

$$O_\epsilon = \Big\{ x \in C([-\tau, t_1], \mathbb{R}^n) : \|x\|_{C[-\tau, t_1]} = \sup_{t \in [-\tau, t_1]} \|x(t)\| \leq \epsilon \Big\}.$$

Now O_ϵ is a bounded, closed and convex set of $C([-\tau, t_1], \mathbb{R}^n)$.

Now we use Krasnoselskii's fixed point theorem to prove our result. We first prove that the operator T has a fixed point x, which is a solution of (1.7). Then we check $(Tx)(t) = \varphi(t)$, $\frac{d}{dt}(Tx)(t) = \dot{\varphi}(t)$ when $-\tau \leq t \leq 0$ and $(Tx)(t_1) = x_1$, $\frac{d}{dt}(Tx)(t_1) = x_1'$ via the control u_x defined in (4.1), and this means system (1.7) is controllable.

Theorem 4.1. *Suppose (H_1) and (H_2) are satisfied. Then* (1.7) *is controllable if*

$$M_2 \left[1 + \frac{\cosh(\|\Omega\| t_1) - 1}{\|\Omega\|} \|\Omega^{-1}\| \|B\| M_1 \right] < 1, \tag{4.3}$$

where $M_2 = \|\Omega^{-1}\| \left[\frac{1}{2^p \|\Omega\|^p} (e^{\|\Omega\| p t_1} - 1) \right]^{\frac{1}{p}} \|L_f\|_{L^q(J, \mathbb{R}^+)}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$.

Proof. We divide our proof into three steps to verify the conditions required in Lemma 2.4.

Step 1. We show $T(O_\epsilon) \subseteq O_\epsilon$ for some positive number ϵ.

Consider any positive number ϵ and let $x^\epsilon \in O_\epsilon$.

Let $t \in [0, t_1]$. From (H_1) and Hölder inequality, we obtain

$$\begin{aligned}\int_0^t \sinh\Big[\|\Omega\|(t-s)\Big] L_f(s)ds &\leq \left(\int_0^t \Big(\sinh[\|\Omega\|(t-s)]\Big)^p ds\right)^{\frac{1}{p}} \left(\int_0^t L_f^q(s)ds\right)^{\frac{1}{q}} \\ &\leq \left(\int_0^t \frac{e^{\|\Omega\|p(t-s)}}{2^p} ds\right)^{\frac{1}{p}} \|L_f\|_{L^q(J,\mathbb{R}^+)} \\ &= \left[\frac{1}{2^p\|\Omega\|p}(e^{\|\Omega\|pt}-1)\right]^{\frac{1}{p}} \|L_f\|_{L^q(J,\mathbb{R}^+)}, \end{aligned} \tag{4.4}$$

where we use the fact that $\sinh t = \frac{e^t - e^{-t}}{2} \leq \frac{e^t}{2}$, for $\forall\ t \in \mathbb{R}$. Next,

$$\begin{aligned}\int_0^t \sinh\Big[\|\Omega\|(t-s)\Big] \|f(s,0)\|ds &\leq M_f \int_0^t \sinh\Big[\|\Omega\|(t-s)\Big]ds \\ &\leq \frac{M_f}{\|\Omega\|}\Big[\cosh(\|\Omega\|t) - 1\Big]. \end{aligned} \tag{4.5}$$

From (4.1), (H_1), (H_2), (4.4), (4.5) and Lemma 2.3, we obtain (here $\|\varphi\|_C = \max_{s\in[-\tau,0]}\|\varphi(s)\|$, $\|\dot{\varphi}\|_C = \max_{s\in[-\tau,0]}\|\dot{\varphi}(s)\|$ and $\|\ddot{\varphi}\|_C = \max_{s\in[-\tau,0]}\|\ddot{\varphi}(s)\|$),

$$\begin{aligned}\|u_x(t)\| &\leq \|W^{-1}\|_{L(\mathbb{R}^n, L^2(J,\mathbb{R}^m)/\ker W)}\Big(\|x_1\| + \|\cos_\tau \Omega t\|\|\varphi(-\tau)\| + \|\Omega^{-1}\|\|\sin_\tau \Omega t\|\|\dot{\varphi}(-\tau)\| \\ &\quad + \|\Omega^{-1}\|\int_{-\tau}^0 \|\sin_\tau \Omega(t-\tau-s)\|\|\ddot{\varphi}(s)\|ds \\ &\quad + \|\Omega^{-1}\|\int_0^t \|\sin_\tau \Omega(t-\tau-s)\|\|f(s,x(s))\|ds\Big) \\ &\leq M_1\|x_1\| + M_1\cosh(\|\Omega\|t)\|\varphi\|_C + M_1\|\Omega^{-1}\|\sinh\Big[\|\Omega\|(t+\tau)\Big]\|\dot{\varphi}\|_C \\ &\quad + M_1\|\Omega^{-1}\|\|\ddot{\varphi}\|_C\int_{-\tau}^0 \sinh\Big[\|\Omega\|(t-s)\Big]ds \\ &\quad + M_1\|\Omega^{-1}\|\int_0^t \sinh\Big[\|\Omega\|(t-s)\Big]L_f(s)\|x(s)\|ds \\ &\quad + M_1\|\Omega^{-1}\|\int_0^t \sinh\Big[\|\Omega\|(t-s)\Big]\|f(s,0)\|ds \\ &\leq M_1\|x_1\| + M_1\cosh(\|\Omega\|t)\|\varphi\|_C + M_1\|\Omega^{-1}\|\sinh\Big[\|\Omega\|(t+\tau)\Big]\|\dot{\varphi}\|_C \\ &\quad + \frac{M_1\|\Omega^{-1}\|\|\ddot{\varphi}\|_C}{\|\Omega\|}\Big(\cosh[\|\Omega\|(t+\tau)] - \cosh(\|\Omega\|t)\Big) \\ &\quad + M_1\|\Omega^{-1}\|\left[\frac{1}{2^p\|\Omega\|p}(e^{\|\Omega\|pt}-1)\right]^{\frac{1}{p}}\|L_f\|_{L^q(J,\mathbb{R}^+)}\|x\|_{C[0,t_1]} \\ &\quad + M_1\|\Omega^{-1}\|\frac{M_f}{\|\Omega\|}\Big[\cosh(\|\Omega\|t) - 1\Big] \\ &\leq M_1\|x_1\| + M_1\theta(t) + M_1M_2\epsilon \\ &\leq M_1\|x_1\| + M_1\theta(t_1) + M_1M_2\epsilon, \end{aligned} \tag{4.6}$$

where

$$\theta(t) = \cosh(\|\Omega\|t)\|\varphi\|_C + \|\Omega^{-1}\|\sinh\Big[\|\Omega\|(t+\tau)\Big]\|\dot{\varphi}\|_C + \|\Omega^{-1}\|\frac{M_f}{\|\Omega\|}\Big[\cosh(\|\Omega\|t)-1\Big]$$
$$+ \frac{\|\Omega^{-1}\|\|\ddot{\varphi}\|_C}{\|\Omega\|}\Big(\cosh[\|\Omega\|(t+\tau)] - \cosh(\|\Omega\|t)\Big),$$

(note we used the fact that $\frac{dt}{d}\theta(t) > 0,\ \forall\ t \in J$).

Now

$$\begin{aligned}
&\|(Tx^{\epsilon})(t)\| \\
&\quad\le \|\cos_\tau \Omega t\|\|\varphi(-\tau)\| + \|\Omega^{-1}\|\|\sin_\tau \Omega t\|\|\dot{\varphi}(-\tau)\| \\
&\qquad + \|\Omega^{-1}\|\int_{-\tau}^{0}\|\sin_\tau \Omega(t-\tau-s)\|\|\ddot{\varphi}(s)\|ds \\
&\qquad + \|\Omega^{-1}\|\int_{0}^{t}\|\sin_\tau \Omega(t-\tau-s)\|\|f(s,x(s))\|ds \\
&\qquad + \|\Omega^{-1}\|\int_{0}^{t}\|\sin_\tau \Omega(t-\tau-s)\|\|B\|\|u_x(s)\|ds \\
&\quad\le \cosh(\|\Omega\|t)\|\varphi\|_C + \|\Omega^{-1}\|\sinh\Big[\|\Omega\|(t+\tau)\Big]\|\dot{\varphi}\|_C \\
&\qquad + \frac{\|\Omega^{-1}\|\|\ddot{\varphi}\|_C}{\|\Omega\|}\Big(\cosh[\|\Omega\|(t+\tau)] - \cosh(\|\Omega\|t)\Big) \\
&\qquad + \|\Omega^{-1}\|\int_{0}^{t}\sinh\Big[\|\Omega\|(t-s)\Big]L_f(s)\|x(s)\|ds + \|\Omega^{-1}\|\int_{0}^{t}\sinh\Big[\|\Omega\|(t-s)\Big]\|f(s,0)\|ds \\
&\qquad + \|\Omega^{-1}\|\int_{0}^{t}\sinh\Big[\|\Omega\|(t-s)\Big]\|B\|\Big(M_1\|x_1\| + M_1\theta(t_1) + M_1M_2\epsilon\Big)ds \\
&\quad\le \theta(t_1) + M_2\epsilon + \frac{\cosh(\|\Omega\|t)-1}{\|\Omega\|}\|\Omega^{-1}\|\|B\|M_1\|x_1\| \\
&\qquad + \frac{\cosh(\|\Omega\|t)-1}{\|\Omega\|}\|\Omega^{-1}\|\|B\|M_1\theta(t_1) + \frac{\cosh(\|\Omega\|t)-1}{\|\Omega\|}\|\Omega^{-1}\|\|B\|M_1M_2\epsilon \\
&\quad\le \theta(t_1)\Big[1 + \frac{\cosh(\|\Omega\|t_1)-1}{\|\Omega\|}\|\Omega^{-1}\|\|B\|M_1\Big] + \frac{\cosh(\|\Omega\|t_1)-1}{\|\Omega\|}\|\Omega^{-1}\|\|B\|M_1\|x_1\| \\
&\qquad + M_2\Big[1 + \frac{\cosh(\|\Omega\|t_1)-1}{\|\Omega\|}\|\Omega^{-1}\|\|B\|M_1\Big]\epsilon.
\end{aligned}$$

Thus for some ϵ sufficiently large, and with this ϵ (which we take for the rest of the proof), from (4.3) we have $T(x^{\epsilon}) \in O_\epsilon$, so as a result $T(O_\epsilon) \subseteq O_\epsilon$.

Now we write the operator T defined in (4.2) as $T_1 + T_2$ where:

$$(T_1x)(t) = (\cos_\tau \Omega t)\varphi(-\tau) + \Omega^{-1}(\sin_\tau \Omega t)\dot{\varphi}(-\tau) + \Omega^{-1}\int_{-\tau}^{0}\sin_\tau \Omega(t-\tau-s)\ddot{\varphi}(s)ds$$
$$+ \Omega^{-1}\int_{0}^{t}\sin_\tau \Omega(t-\tau-s)Bu_x(s)ds, \tag{4.7}$$

$$(T_2x)(t) = \Omega^{-1}\int_{0}^{t}\sin_\tau \Omega(t-\tau-s)f(s,x(s))ds. \tag{4.8}$$

Step 2. We show $T_1 : O_\epsilon \to C([-\tau, t_1], \mathbb{R}^n)$ is a contraction.

Let $t \in [0, t_1]$. From (4.1), (4.4), (H_1) and (H_2), for $\forall\, x, y \in O_\epsilon$, we have

$$\begin{aligned}\|u_x(t) - u_y(t)\| &\le M_1 \|\Omega^{-1}\| \int_0^t \|\sin_\tau \Omega(t-\tau-s)\| L_f(s) \|x(s) - y(s)\| ds \\ &\le M_1 \|\Omega^{-1}\| \|x - y\|_{C[-\tau,t_1]} \int_0^t \sinh\Big[\|\Omega\|(t-s)\Big] L_f(s) ds \\ &\le M_1 M_2 \|x - y\|_{C[-\tau,t_1]}.\end{aligned}$$

Then from (4.7), we have

$$\begin{aligned}\|(T_1 x)(t) - (T_1 y)(t)\| &\le \|\Omega^{-1}\| \int_0^t \|\sin_\tau \Omega(t-\tau-s)\| \|B\| \|u_x(s) - u_y(s)\| ds \\ &\le \|\Omega^{-1}\| \|B\| M_1 M_2 \|x - y\|_{C[-\tau,t_1]} \int_0^t \sinh\Big[\|\Omega\|(t-s)\Big] ds \\ &\le \lambda \|x - y\|_{C[-\tau,t_1]},\end{aligned}$$

where $\lambda := \frac{\cosh(\|\Omega\| t_1) - 1}{\|\Omega\|} \|\Omega^{-1}\| \|B\| M_1 M_2$. From (4.3), note $\lambda < 1$, which implies T_1 is a contraction.

Step 3. We show that $T_2 : O_\epsilon \to C([-\tau, t_1], \mathbb{R}^n)$ is a continuous compact operator.

Let $x_n \in O_\epsilon$ with $x_n \to x$ in O_ϵ. For convenience, let $F_n(\cdot) = f(\cdot, x_n(\cdot))$ and $F(\cdot) = f(\cdot, x(\cdot))$, and note

$$\sinh\Big[\|\Omega\|(\cdot - s)\Big] F_n(s) \to \sinh\Big[\|\Omega\|(\cdot - s)\Big] F(s), \quad \text{a.e. } s \in J = [0, t_1].$$

From (H_1), we get

$$\sinh\Big[\|\Omega\|(\cdot - s)\Big] \|F_n(s) - F(s)\| \le 2\epsilon \sinh\Big[\|\Omega\|(\cdot - s)\Big] L_f(s) \in L^1(J, \mathbb{R}^+).$$

Then using (4.8) and Lebesgue's dominated convergence theorem, we obtain

$$\|(T_2 x_n)(t) - (T_2 x)(t)\| \le \|\Omega^{-1}\| \int_0^t \sinh\Big[\|\Omega\|(t-s)\Big] \|F_n(s) - F(s)\| ds \to 0, \quad \text{as } n \to \infty.$$

Thus $T_2 : O_\epsilon \to C([-\tau, t_1], \mathbb{R}^n)$ is continuous.

Next we show $T_2(O_\epsilon) \subset C([\tau, t_1], \mathbb{R}^n)$ is equicontinuous. For $x \in O_\epsilon$ and $0 < t \le t + h \le t_1$, from (4.8), we have

$$\begin{aligned}(T_2 x)(t+h) - (T_2 x)(t) &= \Omega^{-1} \int_0^{t+h} \sin_\tau \Omega(t+h-\tau-s) F(s) ds \\ &\quad - \Omega^{-1} \int_0^t \sin_\tau \Omega(t-\tau-s) F(s) ds \\ &= K_1 + K_2,\end{aligned} \tag{4.9}$$

where

$$K_1 = \Omega^{-1} \int_t^{t+h} \sin_\tau \Omega(t+h-\tau-s) F(s) ds,$$

and

$$K_2 = \Omega^{-1} \int_0^t \Big[\sin_\tau \Omega(t+h-\tau-s) - \sin_\tau \Omega(t-\tau-s)\Big] F(s) ds.$$

Thus

$$\|(T_2x)(t+h)-(T_2x)(t)\| \le \|K_1\|+\|K_2\|. \tag{4.10}$$

Now, we check $\|K_i\| \to 0$ as $h \to 0$, $i = 1,2$. For K_1 (similar to (4.4)) we obtain

$$\int_t^{t+h} \sinh\Big[\|\Omega\|(t+h-s)\Big] L_f(s)ds \le \left[\frac{1}{2^{p_1}\|\Omega\|p_1}(e^{\|\Omega\|p_1 h}-1)\right]^{\frac{1}{p_1}} \|L_f\|_{L^{q_1}(J,\mathbb{R}^+)}, \tag{4.11}$$

where $\frac{1}{p_1}+\frac{1}{q_1}=1,\ p_1,q_1>1$. Then using (H_1), (4.11) and Lemma 2.3, we get

$$\begin{aligned}
\|K_1\| &\le \|\Omega^{-1}\| \int_t^{t+h} \sinh\Big[\|\Omega\|(t+h-s)\Big]\|F(s)\|ds \\
&\le \|\Omega^{-1}\| \int_t^{t+h} \sinh\Big[\|\Omega\|(t+h-s)\Big](\|f(s,x(s))-f(s,0)\|+\|f(s,0)\|)ds \\
&\le \|\Omega^{-1}\| \int_t^{t+h} \sinh\Big[\|\Omega\|(t+h-s)\Big]L_f(s)\|x(s)\|ds \\
&\quad + M_f\|\Omega^{-1}\| \int_t^{t+h} \sinh\Big[\|\Omega\|(t+h-s)\Big]ds \\
&\le \epsilon\|\Omega^{-1}\| \left[\frac{1}{2^{p_1}\|\Omega\|p_1}(e^{\|\Omega\|p_1 h}-1)\right]^{\frac{1}{p_1}} \|L_f\|_{L^{q_1}(J,\mathbb{R}^+)} \\
&\quad + M_f\|\Omega^{-1}\|\frac{\cosh(\|\Omega\|h)-1}{\|\Omega\|} \longrightarrow 0, \quad \text{as } h \to 0.
\end{aligned}$$

For K_2, from Hölder's inequality, we have

$$\begin{aligned}
&\int_0^t \|\sin_\tau\Omega(t+h-\tau-s)-\sin_\tau\Omega(t-\tau-s)\|L_f(s)ds \\
&\quad \le \left(\int_0^t \|\sin_\tau\Omega(t+h-\tau-s)-\sin_\tau\Omega(t-\tau-s)\|^{p_2}ds\right)^{\frac{1}{p_2}} \|L_f\|_{L^{q_2}(J,\mathbb{R}^+)},
\end{aligned}$$

where $\frac{1}{p_2}+\frac{1}{q_2}=1,\ p_2,q_2>1$. Then we get

$$\begin{aligned}
\|K_2\| &\le \|\Omega^{-1}\| \int_0^t \|\sin_\tau\Omega(t+h-\tau-s)-\sin_\tau\Omega(t-\tau-s)\|\|F(s)\|ds \\
&\le \epsilon\|\Omega^{-1}\| \int_0^t \|\sin_\tau\Omega(t+h-\tau-s)-\sin_\tau\Omega(t-\tau-s)\|L_f(s)ds \\
&\quad + M_f\|\Omega^{-1}\| \int_0^t \|\sin_\tau\Omega(t+h-\tau-s)-\sin_\tau\Omega(t-\tau-s)\|ds \\
&\le \epsilon\|\Omega^{-1}\|\left(\int_0^{t_1} \|\sin_\tau\Omega(t+h-\tau-s)-\sin_\tau\Omega(t-\tau-s)\|^{p_2}ds\right)^{\frac{1}{p_2}} \|L_f\|_{L^{q_2}(J,\mathbb{R}^+)} \\
&\quad + M_f\|\Omega^{-1}\| \int_0^{t_1} \|\sin_\tau\Omega(t+h-\tau-s)-\sin_\tau\Omega(t-\tau-s)\|ds.
\end{aligned}$$

From (1.4), we know that the delayed matrix function $\sin_\tau\Omega t$ is uniformly continuous for $\forall\ t \in J$, and thus, we get $\|\sin_\tau\Omega(t+h-\tau-s)-\sin_\tau\Omega(t-\tau-s)\| \to 0$ as $h \to 0$. Finally, we get $\|K_2\| \to 0$. Now $\|K_1\| \to 0$ and $\|K_2\| \to 0$ with (4.10) yield

$$\|(T_2x)(t+h)-(T_2x)(t)\| \to 0 \quad \text{as } h \to 0,$$

for all $x \in O_\epsilon$. The other cases are treated similarly. From the Arzelà–Ascoli theorem we have that $T_2 : O_\epsilon \to C([\tau, t_1], \mathbb{R}^n)$ is compact.

From Krasnoselskii's fixed point theorem, T has a fixed point x on O_ϵ. From the definition of operator T, x is also the solution of system (1.7). Note $x(t_1) = x_1$ via the control function $u_x(t)$. Also $\dot{x}(t_1) = x_1'$. Finally, we get the initial conditions $x(t) = \varphi(t)$, $\dot{x}(t) = \dot{\varphi}(t)$ when $-\tau \le t \le 0$ using the same procedure in the proof of (3.1) in Theorem 3.1. Thus, system (1.7) is controllable. □

5 Examples

In this section, two examples are presented to illustrate the results.

Example 5.1. Consider the controllability of the following linear delay differential controlled system:

$$\begin{cases} \ddot{x}(t) + \Omega^2 x(t-0.6) = Bu(t), & t \in [0, 1.2], \\ x(t) \equiv \varphi(t), \quad \dot{x}(t) \equiv \dot{\varphi}(t), & t \in [-0.6, 0], \end{cases} \tag{5.1}$$

where

$$\Omega = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \varphi(t) = \begin{pmatrix} 3t \\ 2t \end{pmatrix}, \qquad \dot{\varphi}(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Note that B is a $n \times m$ matrix and an input $u : [0, t_1] \to \mathbb{R}^m$, we can see $n = 2$, $m = 1$, $\tau = 0.6$, $t_1 = 1.2$. Constructing the corresponding delay Gramian matrix of system (5.1) via (3.2), we obtain

$$W_{0.6}[0, 1.2] = \Omega^{-1} \int_0^{1.2} \sin_{0.6} \Omega(0.6 - s) BB^{\mathrm{T}} \sin_{0.6} \Omega^{\mathrm{T}}(0.6 - s) ds =: E_1 + E_2,$$

where

$$E_1 = \Omega^{-1} \int_0^{0.6} \sin_{0.6} \Omega(0.6 - s) BB^{\mathrm{T}} \sin_{0.6} \Omega^{\mathrm{T}}(0.6 - s) ds, \qquad (0.6 - s) \in (0, 0.6),$$
$$E_2 = \Omega^{-1} \int_{0.6}^{1.2} \sin_{0.6} \Omega(0.6 - s) BB^{\mathrm{T}} \sin_{0.6} \Omega^{\mathrm{T}}(0.6 - s) ds, \qquad (0.6 - s) \in (-0.6, 0),$$

and

$$\cos_{0.6} \Omega t = \begin{cases} \Theta, \ t \in (-\infty, -0.6) \\ I, \ t \in [-0.6, 0), \\ I - \Omega^2 \frac{t^2}{2!}, \ t \in [0, 0.6), \\ I - \Omega^2 \frac{t^2}{2!} + \Omega^4 \frac{(t-0.6)^4}{4!}, \\ t \in [0.6, 1.2), \\ \quad \vdots \end{cases} \qquad \sin_{0.6} \Omega t = \begin{cases} \Theta, \ t \in (-\infty, -0.6) \\ \Omega(t + 0.6), \ t \in [-0.6, 0), \\ \Omega(t + 0.6) - \Omega^3 \frac{t^3}{3!}, \ t \in [0, 0.6), \\ \Omega(t + 0.6) - \Omega^3 \frac{t^3}{3!} + \Omega^5 \frac{(t-0.6)^5}{5!}, \\ t \in [0.6, 1.2), \\ \quad \vdots \end{cases} \tag{5.2}$$

Next, we can calculate that

$$E_1 = \begin{pmatrix} \frac{21681}{15625} & \frac{102717}{218750} \\ \frac{227259}{156250} & \frac{269307}{546875} \end{pmatrix}, \qquad E_2 = \begin{pmatrix} \frac{27}{125} & \frac{9}{125} \\ \frac{27}{125} & \frac{9}{125} \end{pmatrix}.$$

Then, we get

$$W_{0.6}[0,1.2] = E_1 + E_2 = \begin{pmatrix} \frac{25056}{15625} & \frac{118467}{218750} \\ \frac{261009}{156250} & \frac{308682}{546875} \end{pmatrix}, \qquad W_{0.6}^{-1}[0,1.2] = \begin{pmatrix} \frac{428725000}{364257} & \frac{-411343750}{364257} \\ \frac{-422931250}{121419} & \frac{406000000}{121419} \end{pmatrix}.$$

Thus, system (5.1) is controllable by Theorem 3.1. In addition, for any finite terminal conditions $x(t_1) = x_1 = (x_{11}, x_{12})^{\mathrm{T}}$, $\dot{x}(t_1) = x_1' = (x_{11}', x_{12}')^{\mathrm{T}}$, it follows (3.3) that one can construct the corresponding control input $u(t) \in \mathbb{R}$ as

$$u(t) = B^{\mathrm{T}} \sin_{0.6} \Omega^{\mathrm{T}}(0.6 - t) W_{0.6}^{-1}[0,1.2]\beta, \tag{5.3}$$

where

$$\beta = x_1 - (\cos_{0.6} \Omega 1.2)\varphi(-0.6) - \Omega^{-1}(\sin_{0.6} \Omega 1.2)\dot{\varphi}(-0.6) = \begin{pmatrix} x_{11} - \frac{36132604990374860091}{70368744177664000000} \\ x_{12} - \frac{9439319638987191039}{35184372088832000000} \end{pmatrix}.$$

From (1.8) and (5.3), the solution of system (5.1) has the form:

$$\begin{aligned} x(t) &= (\cos_{0.6} \Omega t)\varphi(-0.6) + \Omega^{-1}(\sin_{0.6} \Omega t)\dot{\varphi}(-0.6) \\ &\quad + \Omega^{-1} \int_0^t \sin_{0.6} \Omega(t - 0.6 - s) B B^{\mathrm{T}} \sin_{0.6} \Omega^{\mathrm{T}}(0.6 - s) ds\, W_{0.6}^{-1}[0,1.2]\beta. \end{aligned} \tag{5.4}$$

Now we consider the integral term $\int_0^t \sin_{0.6} \Omega(t - 0.6 - s) B B^{\mathrm{T}} \sin_{0.6} \Omega^{\mathrm{T}}(0.6 - s) ds$ in (5.4).

For $0 < t < 0.6$, we can obtain $-0.6 < t - 0.6 - s < t - 0.6 < 0$ and $0 < 0.6 - t < 0.6 - s < 0.6$, so the solution (5.4) can be expressed to the following form via (5.2):

$$\begin{aligned} x(t) &= \left[I - \Omega^2 \frac{t^2}{2} \right] \varphi(-0.6) + \Omega^{-1} \left[\Omega(t + 0.6) - \Omega^3 \frac{t^3}{6} \right] \dot{\varphi}(-0.6) \\ &\quad + \Omega^{-1} \int_0^t [\Omega(t - s)] B B^{\mathrm{T}} \left[\Omega^{\mathrm{T}}(1.2 - s) - (\Omega^{\mathrm{T}})^3 \frac{(0.6 - s)^3}{6} \right] ds\, W_{0.6}^{-1}[0,1.2]\beta. \end{aligned}$$

For $0.6 < t < 1.2$, we get $0 < t - 0.6 - s < t - 0.6 < 0.6$ when $0 < s < t - 0.6$ and $-0.6 < t - 0.6 - s < 0$ when $t - 0.6 < s < t$. We can also obtain $0 < 0.6 - s < 0.6$ when $0 < s < 0.6$ and $-0.6 < 0.6 - t < 0.6 - s < 0$ when $0.6 < s < t$. Finally, (5.4) can be expressed to the following formula via (5.2):

$$\begin{aligned} x(t) &= \left[I - \Omega^2 \frac{t^2}{2} + \Omega^4 \frac{(t - 0.6)^4}{24} \right] \varphi(-0.6) \\ &\quad + \Omega^{-1} \left[\Omega(t + 0.6) - \Omega^3 \frac{t^3}{6} + \Omega^5 \frac{(t - 0.6)^5}{120} \right] \dot{\varphi}(-0.6) \\ &\quad + \Omega^{-1} \int_0^{t-0.6} \left[\Omega(t - s) - \Omega^3 \frac{(t - 0.6 - s)^3}{6} \right] B B^{\mathrm{T}} \left[\Omega^{\mathrm{T}}(1.2 - s) - (\Omega^{\mathrm{T}})^3 \frac{(0.6 - s)^3}{6} \right] ds \\ &\quad \times W_{0.6}^{-1}[0,1.2]\beta \\ &\quad + \Omega^{-1} \int_{t-0.6}^{0.6} [\Omega(t - s)] B B^{\mathrm{T}} \left[\Omega^{\mathrm{T}}(1.2 - s) - (\Omega^{\mathrm{T}})^3 \frac{(0.6 - s)^3}{6} \right] ds\, W_{0.6}^{-1}[0,1.2]\beta \\ &\quad + \Omega^{-1} \int_{0.6}^{t} [\Omega(t - s)] B B^{\mathrm{T}} \left[\Omega^{\mathrm{T}}(1.2 - s) \right] ds\, W_{0.6}^{-1}[0,1.2]\beta. \end{aligned}$$

Figure 5.1 shows the state $x(t)$ of system (5.1) when we set the terminal state $x_1 = (x_{11}, x_{12})^{\mathrm{T}} = (0,0)^{\mathrm{T}}$ and Figure 5.2 shows the state $x(t)$ of system (5.1) when we set $x_1 = (x_{11}, x_{12})^{\mathrm{T}} = (20,10)^{\mathrm{T}}$. Clearly, we can see the terminal states of system (5.1) is consistent with the achieved states.

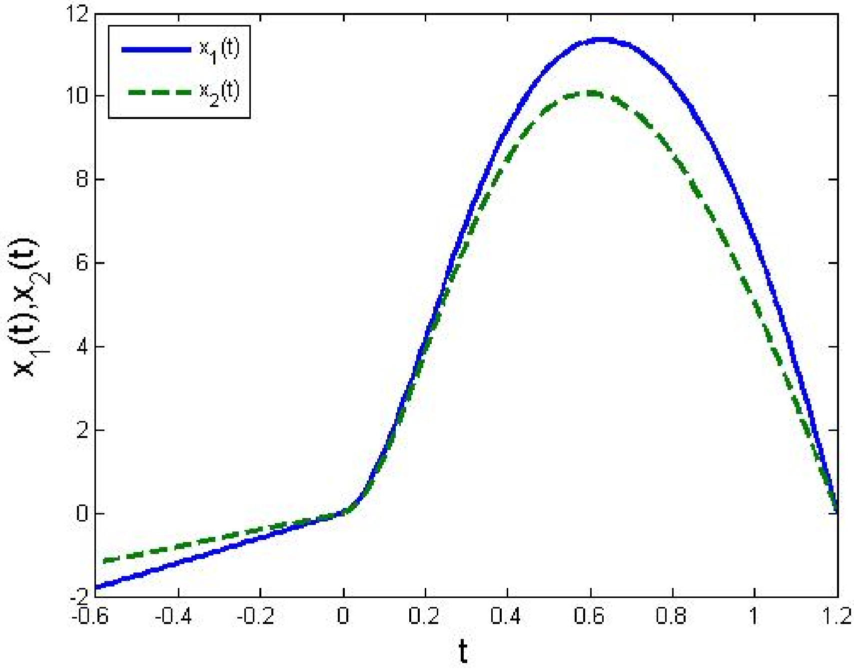

Figure 5.1: The state $x(t)$ of system (5.1) when we set $x_1 = (x_{11}, x_{12})^{\mathrm{T}} = (0, 0)^{\mathrm{T}}$.

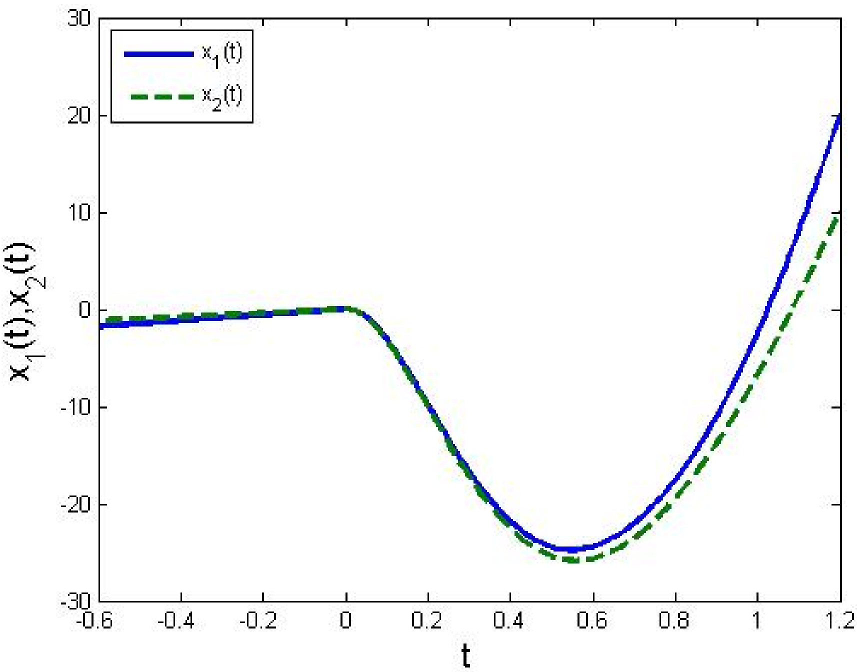

Figure 5.2: The state $x(t)$ of system (5.1) when we set $x_1 = (x_{11}, x_{12})^{\mathrm{T}} = (20, 10)^{\mathrm{T}}$.

Example 5.2. In this example, we consider the following nonlinear delay differential controlled system:

$$\begin{cases} \ddot{x}(t) + \Omega^2 x(t-0.4) = f(t, x(t)) + Bu(t), & t \in [0, 0.8], \\ x(t) \equiv \varphi(t), \quad \dot{x}(t) \equiv \dot{\varphi}(t), & t \in [-0.4, 0], \end{cases} \tag{5.5}$$

where we set

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = I_{2\times 2}, \varphi(t) = \begin{pmatrix} 5t+1 \\ 2t^2 \end{pmatrix}, \qquad \dot{\varphi}(t) = \begin{pmatrix} 5 \\ 4t \end{pmatrix},$$

$$f(t, x(t)) = \begin{pmatrix} 0.3(t-0.4)\sin[x_1(t)] \\ 0.3(t-0.4)\sin[x_2(t)] \end{pmatrix}.$$

Now, we set $u(t) = \widetilde{x}$, where $\widetilde{x} = \sum_{n=1}^{2} \langle \widetilde{x}, e_n \rangle e_n$, e_n is orthonormal basis of $\mathbb{R}^2$. From the definition of W in (H_2), we get

$$\begin{aligned} W &= \Omega^{-1} \int_0^{0.8} \sin_{0.4} \Omega(0.4-s) B ds\, \widetilde{x} \\ &= \Omega^{-1} \int_0^{0.4} \sin_{0.4} \Omega(0.4-s) ds\, \widetilde{x} + \Omega^{-1} \int_{0.4}^{0.8} \sin_{0.4} \Omega(0.4-s) ds\, \widetilde{x} \\ &= \begin{pmatrix} \frac{452}{1875} & 0 \\ 0 & \frac{452}{1875} \end{pmatrix} \widetilde{x} + \begin{pmatrix} \frac{2}{25} & 0 \\ 0 & \frac{2}{25} \end{pmatrix} \widetilde{x} \\ &= \begin{pmatrix} \frac{602}{1875} & 0 \\ 0 & \frac{602}{1875} \end{pmatrix} \widetilde{x}. \end{aligned}$$

Define the inverse $W^{-1} : \mathbb{R}^2 \to L^2(J_1, \mathbb{R}^2)$ by

$$(W^{-1}\widetilde{x})(t) := \begin{pmatrix} \frac{1875}{602} & 0 \\ 0 & \frac{1875}{602} \end{pmatrix} \widetilde{x},$$

where $J_1 = [0, 0.8]$.

Then, we get

$$\|(W^{-1}\widetilde{x})(t)\| \leq \left\| \begin{pmatrix} \frac{1875}{602} & 0 \\ 0 & \frac{1875}{602} \end{pmatrix} \right\| \|\widetilde{x}\| = 3.1146 \|\widetilde{x}\|,$$

and thus, we obtain $\|W^{-1}\| \leq 3.1146 =: M_1$. Hence, W satisfies the assumption (H_2).

Next, note that $|\sin a - \sin b| \leq |a - b|$, $\forall\, a, b \in \mathbb{R}$, we have

$$\begin{aligned} \|f(t,x) - f(t,y)\| &= |0.3(t-0.4)| \sqrt{(\sin[x_1(t)] - \sin[y_1(t)])^2 + (\sin[x_2(t)] - \sin[y_2(t)])^2} \\ &\leq |0.3(t-0.4)| \sqrt{[x_1(t) - y_1(t)]^2 + [x_2(t) - y_2(t)]^2} \\ &= |0.3(t-0.4)| \|x - y\|, \ \forall\, t \in J_1, x(t), y(t) \in \mathbb{R}^2. \end{aligned}$$

We can set $L_f = |0.3(t-0.4)| \in L^q(J_1, \mathbb{R}^+)$ in (H_1). When we choose $p = q = 2$, we get

$$\|L_f\|_{L^2(J_1, \mathbb{R}^+)} = \left(\int_0^{0.8} [0.3(s-0.4)]^2 ds \right)^{\frac{1}{2}} = 0.0620.$$

Then, we obtain

$$M_2 = \|\Omega^{-1}\| \left[\frac{1}{2^3 \|\Omega\|} (e^{1.6\|\Omega\|} - 1) \right]^{\frac{1}{2}} \|L_f\|_{L^2(J, \mathbb{R}^+)} = 0.0436.$$

Finally, we calculate that

$$M_2\left[1+\frac{\cosh(0.8\|\Omega\|)-1}{\|\Omega\|}\|\Omega^{-1}\|\|B\|M_1\right]=0.0894<1,$$

which implies that the condition (4.3) holds.

Now all the conditions required in Theorem 4.1 are satisfied, thus, system (5.5) is controllable.

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