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Existence and rapid convergence results for nonlinear Caputo nabla fractional difference equations

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Abstract. This paper is concerned with finding properties of solutions to initial value problems for nonlinear Caputo nabla fractional difference equations. We obtain existence and rapid convergence results for such equations by use of Schauder's fixed point theorem and the generalized quasi-linearization method, respectively. A numerical example is given to illustrate one of our rapid convergence results.

Keywords: Caputo nabla fractional difference equation, Schauder's fixed point theorem, generalized quasi-linearization, existence, rapid convergence.

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1 Introduction

It is well known that there is a large quantity of research on integer-order difference equations. Since the study was begun very early, much classical content has been established, and we refer specifically to the monographs [2,17]. However, the study of fractional difference equations is quite recent. The basic theory of linear and nonlinear fractional difference equations can be found in [13–16]. Note that the theory of nonlinear fractional difference equations is not complete and the convergence of approximate solutions is one of the most studied problems. This has an important affect on the development of the qualitative theory.

Generalized quasi-linearization is an efficient method for constructing approximate solutions of nonlinear problems. This method originated in dynamic programming theory and was initially applied by Bellman and Kalaba [8]. A systematic development of the method to ordinary differential equations was provided by Lakshmikantham and Vatsala [18], and there are some generalized results of the method to various types of differential equations and we refer to the monographs [19, 20], for functional differential equations [3, 11], for impulsive equations [4, 7], for partial differential equations [5, 9, 23], for differential equations on time scales [22], for fractional differential equations [10, 21, 26], for other types [12, 24, 25], and the references cited therein. For nonlinear fractional difference equations see the paper [6].

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In this paper, we attempt to extend the applications of generalized quasi-linearization with certain conditions on the forcing function, and study the rate of convergence of the approximate solutions for the nonlinear Caputo nabla fractional difference equation. In order to do this, we first prove the existence of solutions for such equations, and then using an appropriate iterative scheme, we obtain two monotone sequences which converge uniformly and rapidly to the solution of the problem. Finally, we provide a numerical example to illustrate the application of the obtained results.

2 Preliminary definitions

For the convenience of readers, we will list some relevant results here. We use the notation $\mathbb{N}_a := \{a, a+1, a+2, \dots\}$, where a is a real number. For the function $f : \mathbb{N}_a \to \mathbb{R}$, the backward difference or nabla operator is defined as $\nabla f(t) = f(t) - f(t-1)$ for $t \in \mathbb{N}_{a+1}$ and the higher order differences are defined recursively by $\nabla^n f(t) = \nabla(\nabla^{n-1} f(t))$ for $t \in \mathbb{N}_{a+n}$, $n \in \mathbb{N}$. In addition, we take ∇^0 as the identity operator. We define the definite nabla integral of $f : \mathbb{N}_a \to \mathbb{R}$ by

$$\int_a^b f(s)\nabla s = \begin{cases} \sum\limits_{s=a+1}^{b} f(s), & a < b, \\ 0, & a = b, \\ -\sum\limits_{s=b+1}^{a} f(s), & a > b, \end{cases} \tag{2.1}$$

where $b \in \mathbb{N}_a$.

Definition 2.1 (See [15, Definition 3.4])**.** The (generalized) rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)} \tag{2.2}$$

for those values of t and r for which the right-hand side of (2.2) is defined. Also, we use the convention that if t is a nonpositive integer, but $t+r$ is not a nonpositive integer, then $t^{\overline{r}} = 0$. We then define the ν-th order Taylor monomials based at a (see [15, Definition 3.56] by

$$H_\nu(t, a) = \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)},$$

for $\nu \neq -1, -2, \dots$, $t \in \mathbb{N}_a$.

For some important formulas for these Taylor fractional monomials see [15, Theorem 3.57 and Theorem 3.93].

Definition 2.2 (Nabla fractional sum [15, Definition 3.58])**.** Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and assume $\nu > 0$. Then

$$\nabla_a^{-\nu} f(t) = \int_a^t H_{\nu-1}(t, \rho(s)) f(s) \nabla s, \qquad t \in \mathbb{N}_a, \tag{2.3}$$

where $\rho(t) := t - 1$ and by convention $\nabla_a^{-\nu} f(a) = 0$.

Definition 2.3 (Nabla fractional difference [15, Definition 3.61]). Let $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ be given, and let $N := \lceil \nu \rceil$, where $\lceil \cdot \rceil$ is the ceiling function. Then we define the ν-order nabla fractional difference operator $\nabla_a^\nu f(t)$ by

$$\nabla_a^\nu f(t) = \nabla^N \nabla_a^{-(N-\nu)} f(t), \qquad t \in \mathbb{N}_{a+N}. \tag{2.4}$$

Definition 2.4 (Caputo nabla fractional difference [15, Definition 3.117]). Let $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ be given, and let $N := \lceil \nu \rceil$. Then we define the ν-order Caputo nabla fractional difference operator $\nabla_{a^*}^\nu f(t)$ by

$$\nabla_{a^*}^\nu f(t) = \nabla_a^{-(N-\nu)} \big[\nabla^N f(t)\big], \qquad t \in \mathbb{N}_{a+N}. \tag{2.5}$$

Now it follows from this definition that $\nabla_{a^*}^\nu c = 0$ for $\nu > 0$ with any constant c.

Lemma 2.5 (See [15, Definition 3.61 and Theorem 3.62]). *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$, $\nu \notin \mathbb{N}_1$, and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu < N$. Then*

$$\nabla_a^\nu f(t) = \int_a^t H_{-\nu-1}(t, \rho(s)) f(s) \nabla s, \qquad t \in \mathbb{N}_a, \tag{2.6}$$

where $\rho(t) := t - 1$ and by convention $\nabla_a^\nu f(a) = 0$.

Lemma 2.6 (See [1]). *Assume $f : \mathbb{N}_a \to \mathbb{R}$, for any $\nu > 0$, we have*

$$\nabla_{a^*}^\nu f(t) = \nabla_a^\nu f(t) - \sum_{k=0}^{N-1} H_{-\nu+k}(t, a) \nabla^k f(a). \tag{2.7}$$

In particular, when $0 < \nu < 1$, we have

$$\nabla_{a^*}^\nu f(t) = \nabla_a^\nu f(t) - H_{-\nu}(t, a) f(a). \tag{2.8}$$

3 Existence and comparison results

Consider the following initial value problem (IVP) for a nonlinear Caputo nabla fractional difference equation

$$\begin{cases} \nabla_{a^*}^\nu x(t) = f(t, x(t)), & t \in \mathbb{N}_{a+1}^b, \\ x(a) = x_0, \end{cases} \tag{3.1}$$

where $f : \mathbb{N}_{a+1}^b \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to x, $x : \mathbb{N}_a \to \mathbb{R}$, and $0 < \nu < 1$.

In this paper, we define the norm of x on $\mathbb{N}_a^b$ by $\|x\| = \max_{s \in \mathbb{N}_a^b} |x(s)|$. Throughout this paper, we use the notation $\frac{\partial^k f(t,x)}{\partial^k x} = f^{(k)}(t, x)$ $(k = 0, 1, 2 \ldots)$. We define the following set for convenience:

$$\Omega = \{(t, x) : \alpha_0(t) \le x(t) \le \beta_0(t),\ t \in \mathbb{N}_{a+1}^b\}.$$

where $\alpha_0(t)$ and $\beta_0(t)$ are defined on $\mathbb{N}_a^b$ with $\alpha_0(t) \le \beta_0(t)$ on $\mathbb{N}_a^b$.

Definition 3.1. The function $\alpha_0(t)$, $t \in \mathbb{N}_a^b$, is said to be a lower (an upper) solution of the IVP (3.1), if

$$\begin{cases} \nabla_{a^*}^\nu \alpha_0(t) \le (\ge) f(t, \alpha_0(t)), & t \in \mathbb{N}_{a+1}^b, \\ \alpha_0(a) \le (\ge) x_0. \end{cases} \tag{3.2}$$

Lemma 3.2. *The function $x(t)$ is a solution of the IVP (3.1) if and only if the function $x(t)$ has the following representation*

$$x(t) = x_0 + \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s)) f(s, x(s)). \tag{3.3}$$

Proof. Applying the operator $\nabla_a^{-\nu}$ on both sides of the first equality of the IVP (3.1), we have

$$\nabla_a^{-\nu}[\nabla_{a^*}^{\nu} x(t)] = \nabla_a^{-\nu} f(t, x(t)),$$

which can be written as

$$\nabla_a^{-\nu}\left[\nabla_a^{-(1-\nu)} \nabla x(t)\right] = \nabla_a^{-\nu} f(t, x(t)).$$

That is,

$$\nabla^{-1} \nabla x(t) = \nabla_a^{-\nu} f(t, x(t)).$$

Then, we have

$$x(t) = x_0 + \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s)) f(s, x(s)).$$

Conversely, assume that x has the representation (3.3). By means of (2.3), we obtain that (3.3) is equivalent to

$$x(t) = x_0 + \nabla_a^{-\nu} f(t, x(t)). \tag{3.4}$$

Applying $\nabla_{a^*}^{\nu}$ to both sides of (3.4), we get

$$\nabla_{a^*}^{\nu} x(t) = \nabla_{a^*}^{\nu} x_0 + \nabla_{a^*}^{\nu}\left[\nabla_a^{-\nu} f(t, x(t))\right].$$

Using (2.8), we obtain

$$\nabla_{a^*}^{\nu} x(t) = \nabla_{a^*}^{\nu} x_0 + \Big(\nabla_a^{\nu} \nabla_a^{-\nu} f(t, x(t)) - H_{-\nu}(t, a) \nabla_a^{-\nu} f(a, x(a))\Big).$$

Thus, we have

$$\nabla_{a^*}^{\nu} x(t) = f(t, x(t)).$$

The proof is complete. □

Now we present an existence result for the IVP (3.1), which we will use in our main results. Since the proof is a standard application of Schauder's fixed point theorem we will omit the proof of this lemma.

Lemma 3.3. *Assume that*

($H_{3.1}$) *the function $f : \overline{R} \to \mathbb{R}$ is continuous with repect to x, $|f(t, x)| \leq Q$ on $\overline{R}$, $D = H_\nu(b, a)$, and $D \leq \frac{M}{Q}$, where*

$$\overline{R} = \{(t, x) : t \in \mathbb{N}_{a+1}^{b}, \|x - x_0\| \leq M\}.$$

Then the IVP (3.1) has a solution.

Lemma 3.4. *Assume that*

($H_{3.2}$) *the function $f : \Omega \to \mathbb{R}$ is continuous in its second variable.*

($H_{3.3}$) *the functions α_0, $\beta_0 : \mathbb{N}_a^b \to \mathbb{R}$ are lower and upper solutions respectively of the the IVP (3.1) such that $\alpha_0(t) \leq \beta_0(t)$ on $\mathbb{N}_a^b$;*

Then there exists a solution $x(t)$ of the IVP (3.1) satisfying $\alpha_0(t) \leq x(t) \leq \beta_0(t)$ on $\mathbb{N}_a^b$.

Proof. Let $P : \mathbb{N}_{a+1}^b \times \mathbb{R} \to \mathbb{R}$ be defined by $P(t,x) = \max\{\alpha_0(t), \min\{x, \beta_0(t)\}\}$. Then $f(t, P(t,x))$ defines an extension of f to $\mathbb{N}_{a+1}^b \times \mathbb{R}$, which is bounded and continuous with respect its second variable on $\mathbb{N}_{a+1}^b$. Therefore, by Lemma 3.3, $\nabla_{a^*}^{\nu} x(t) = f(t, P(t,x))$, $x(a) = x_0$ has a solution on $\mathbb{N}_a^b$.

To complete the proof, we need to show that $\alpha_0(t) \le x(t) \le \beta_0(t)$ on $\mathbb{N}_a^b$. We now show that $x(t) \le \beta_0(t)$ on $\mathbb{N}_a^b$. Clearly, $x(a) \le \beta_0(a)$, we now only need to show $x(t) \le \beta_0(t)$ on $\mathbb{N}_{a+1}^b$. If it is not true, there exists a point $c \in \mathbb{N}_{a+1}^b$ such that $x(t) - \beta_0(t)$ has a positive maximum, that is,

$$x(c) - \beta_0(c) = \max\{x(t) - \beta_0(t) : t \in \mathbb{N}_{a+1}^b\} > 0,$$

and

$$x(t) - \beta_0(t) \le x(c) - \beta_0(c) \quad \text{on } \mathbb{N}_{a+1}^c.$$

First, we will show that $\nabla_{a^*}^{\nu} x(c) > \nabla_{a^*}^{\nu} \beta_0(c)$, that is,

$$\nabla_a^{\nu} x(c) - H_{-\nu}(c,a)x(a) > \nabla_a^{\nu} \beta_0(c) - H_{-\nu}(c,a)\beta_0(a).$$

Since $x(a) \le \beta_0(a)$ and

$$H_{-\nu}(c,a) = \frac{(c-a)^{\overline{-\nu}}}{\Gamma(1-\nu)} = \frac{\Gamma(c-a-\nu)}{\Gamma(c-a)\Gamma(1-\nu)} > 0, \qquad c \in \mathbb{N}_{a+1}^b,$$

we have

$$-H_{-\nu}(c,a)x(a) \ge -H_{-\nu}(c,a)\beta_0(a).$$

Next, we show $\nabla_a^{\nu} x(c) > \nabla_a^{\nu} \beta_0(c)$. In view of the fact that $\frac{(1)^{\overline{-\nu-1}}}{\Gamma(-\nu)} = \frac{\Gamma(-\nu)}{\Gamma(-\nu)\Gamma(1)} = 1$ and $H_{-\nu}(c,a) > 0$, it follows that

$$\begin{aligned}
\nabla_a^{\nu} x(c) - \nabla_a^{\nu} \beta_0(c) &= \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{c} (c-\rho(s))^{\overline{-\nu-1}} (x(s) - \beta_0(s)) \\
&= \left[\frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{c-1} (c-\rho(s))^{\overline{-\nu-1}} (x(s) - \beta_0(s)) \right] + (x(c) - \beta_0(c)) \\
&\ge \left[1 + \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{c-1} (c-\rho(s))^{\overline{-\nu-1}} \right] (x(c) - \beta_0(c)) \\
&= \left[\sum_{s=a+1}^{c} H_{-\nu-1}(c, \rho(s)) \right] (x(c) - \beta_0(c)) \\
&= H_{-\nu}(c,a)(x(c) - \beta_0(c)) > 0.
\end{aligned}$$

Then, we have $\nabla_a^{\nu} x(c) > \nabla_a^{\nu} \beta_0(c)$. Thus, we conclude that $\nabla_{a^*}^{\nu} x(c) > \nabla_{a^*}^{\nu} \beta_0(c)$.

On the other hand, due to $x(c) - \beta_0(c) > 0$, so $P(c, x(c)) = \beta_0(c)$. Hence

$$\nabla_{a^*}^{\nu} \beta_0(c) \ge f(c, P(c, x(c))) = \nabla_{a^*}^{\nu} x(c),$$

which is a contradiction. Thus, we obtain $x(t) \le \beta_0(t)$ on $\mathbb{N}_a^b$. Similarly, we can show that $\alpha_0(t) \le x(t)$ on $\mathbb{N}_a^b$. Therefore, it follows that $x(t)$ is actually a solution of IVP (3.1). The proof is complete. □

For the convenience of readers, we give a result for the linear Caputo nabla fractional difference equation.

Consider the Caputo nabla fractional difference inequality

$$\nabla_{a^*}^{\nu} x(t) - Cx(t) \leq 0, \ \ x(a) \leq 0, \qquad t \in \mathbb{N}_{a+1}^{b}. \tag{3.5}$$

Lemma 3.5. *Assume that*

$(H_{3.4})$ *the positive constant C satisfies $CH_\nu(b,a) < 1$.*

Then $x(t) \leq 0$ on $\mathbb{N}_a^b$.

Proof. Setting $x(t) = \nabla_a^{-\nu} y(t)$, according to the Definition 2.2, we have

$$\begin{aligned} x(t) = \nabla_a^{-\nu} y(t) &= \int_a^t H_{\nu-1}(t, \rho(s)) y(s) \nabla s \\ &= \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} y(s) \\ &= \sum_{s=a+1}^{t} \frac{\Gamma(t-s+\nu)}{\Gamma(\nu)\Gamma(t-s+1)} y(s). \end{aligned} \tag{3.6}$$

We get from (3.6) that $y(t) \leq 0$ implies $x(t) \leq 0$ on $\mathbb{N}_{a+1}^b$, so we only need to prove $y(t) \leq 0$ on $\mathbb{N}_{a+1}^b$. If this is false, there exists a $c \in \mathbb{N}_{a+1}^b$ such that $y(c) = \max\{y(t) : t \in \mathbb{N}_{a+1}^b\} > 0$, and $y(t) \leq y(c)$ on $\mathbb{N}_{a+1}^c$. It follows from (2.8) that (3.5) is equivalent to

$$\nabla_a^{\nu} x(t) - H_{-\nu}(t,a) x(a) - Cx(t) \leq 0. \tag{3.7}$$

Letting $x(t) = \nabla_a^{-\nu} y(t)$ in (3.7) yields

$$\nabla_a^{\nu} \nabla_a^{-\nu} y(t) - C \nabla_a^{-\nu} y(t) \leq 0,$$

which can be written as

$$y(t) \leq \frac{C}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\nu-1}} y(s).$$

Hence, we have

$$\begin{aligned} y(c) &\leq \frac{C}{\Gamma(\nu)} \sum_{s=a+1}^{c} (c-\rho(s))^{\overline{\nu-1}} y(s) \\ &\leq \frac{C}{\Gamma(\nu)} \Big(\sum_{s=a+1}^{c} (c-\rho(s))^{\overline{\nu-1}} \Big) y(c) \\ &= C y(c) H_\nu(c,a), \end{aligned}$$

that is,

$$(1 - CH_\nu(c,a)) y(c) \leq 0.$$

Since $y(c) > 0$, so we have $(1 - CH_\nu(c,a)) \leq 0$.

On the other hand, from the condition $(H_{3.4})$ and the increasing property of the function $H_\nu(t,a)$, we have $(1 - CH_\nu(c,a)) > 0$, which is a contradiction. Then, we have $y(t) \leq 0$ on $\mathbb{N}_{a+1}^b$. Hence, we conclude that $x(t) \leq 0$ on $\mathbb{N}_a^b$. The proof is complete.

□

4 Rapid convergence

In this section, we consider the IVP (3.1) with $f(t,x) = f_1(t,x) + f_2(t,x)$. We show that the convergence of the sequences of successive approximations is of order m where m is $2k+1$ or $2k$ ($k \geq 1$) by applying the method of generalized quasi-linearization for nonlinear Caputo nabla fractional difference equations.

Theorem 4.1. *Assume that the conditions* $(H_{3.3})$–$(H_{3.4})$ *hold, and*

$(A_{4.1})$ *the functions* f_1, $f_2 : \Omega \to \mathbb{R}$ *are such that* $f_1^{(i)}(t,x)$, $f_2^{(i)}(t,x)$ $(i = 0, 1, \ldots, 2k)$ *exist, are continuous in the second variable, and for* $C_1 > 0$, $C_2 > 0$, $C = C_1 + C_2$,

$$f_1^{(1)}(t,x) \leq C_1, \; f_2^{(1)}(t,x) \leq C_2 \quad \text{on } \Omega;$$

$(A_{4.2})$ *there exist* M_1, $M_2 > 0$ *such that for* $x_1 \geq x_2$, $y_1 \geq y_2$ *the functions* $f_1^{(2k)}(t,x)$, $f_2^{(2k)}(t,x)$ *satisfy the following conditions:*

$$0 \leq f_1^{(2k)}(t,x_1) - f_1^{(2k)}(t,x_2) \leq M_1(x_1 - x_2) \quad \text{on } \Omega,$$
$$0 \geq f_2^{(2k)}(t,y_1) - f_2^{(2k)}(t,y_2) \geq -M_2(y_1 - y_2) \quad \text{on } \Omega.$$

Then there exist two monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$, $n \geq 0$ *which converge uniformly and monotonically to a solution of the IVP* (3.1) *and the convergence is of order* $2k+1$.

Proof. From the condition $(A_{4.2})$, and the Taylor expansion with Lagrange remainder, we obtain

$$f_1(t,x_1) \geq \sum_{i=0}^{2k} \frac{f_1^{(i)}(t,x_2)}{i!}(x_1 - x_2)^i, \tag{4.1}$$

$$f_2(t,x_1) \geq \sum_{i=0}^{2k-1} \frac{f_2^{(i)}(t,x_2)}{i!}(x_1 - x_2)^i + \frac{f_2^{(2k)}(t,x_1)}{(2k)!}(x_1 - x_2)^{2k} \tag{4.2}$$

for (t,x_1), $(t,x_2) \in \Omega$, $x_2 \leq x_1$. Similarly, we have

$$f_1(t,x_1) \leq \sum_{i=0}^{2k} \frac{f_1^{(i)}(t,x_2)}{i!}(x_1 - x_2)^i, \tag{4.3}$$

$$f_2(t,x_1) \leq \sum_{i=0}^{2k-1} \frac{f_2^{(i)}(t,x_2)}{i!}(x_1 - x_2)^i + \frac{f_2^{(2k)}(t,x_1)}{(2k)!}(x_1 - x_2)^{2k} \tag{4.4}$$

for (t,x_1), $(t,x_2) \in \Omega$, $x_1 \leq x_2$.

Consider the following nonlinear Caputo nabla fractional difference equations:

$$\begin{cases} \nabla_{a^*}^{\nu} y(t) = \displaystyle\sum_{i=0}^{2k} \frac{f_1^{(i)}(t,\alpha)}{i!}(y - \alpha)^i + \sum_{i=0}^{2k-1} \frac{f_2^{(i)}(t,\alpha)}{i!}(y - \alpha)^i + \frac{f_2^{(2k)}(t,\beta)}{(2k)!}(y - \alpha)^{2k} \\ \qquad \equiv F(t,\alpha,\beta;y), \quad t \in \mathbb{N}_{a+1}^{b}, \\ y(a) = x_0, \end{cases} \tag{4.5}$$

and

$$\begin{cases} \nabla_{a^*}^{\nu} z(t) = \sum_{i=0}^{2k} \frac{f_1^{(i)}(t,\beta)}{i!}(z-\beta)^i + \sum_{i=0}^{2k-1} \frac{f_2^{(i)}(t,\beta)}{i!}(z-\beta)^i + \frac{f_2^{(2k)}(t,\alpha)}{(2k)!}(z-\beta)^{(2k)} \\ \qquad \equiv G(t,\alpha,\beta;z), \quad t \in \mathbb{N}_{a+1}^b, \\ z(a) = x_0. \end{cases} \tag{4.6}$$

We develop the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ using the above IVPs (4.5), (4.6), respectively. Letting $\alpha = \alpha_0$, $\beta = \beta_0$ in IVPs (4.5), (4.6). We first prove that $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of the IVP (4.5) respectively. In fact, from the condition $(H_{3.3})$, we have

$$\begin{aligned} &\nabla_{a^*}^{\nu} \alpha_0(t) \le f_1(t,\alpha_0) + f_2(t,\alpha_0) = F(t,\alpha_0,\beta_0;\alpha_0), \qquad t \in \mathbb{N}_{a+1}^b, \\ &\quad \alpha_0(a) \le x_0, \end{aligned}$$

and by using the inequalities (4.1), (4.2), it follows that

$$\begin{aligned} \nabla_{a^*}^{\nu} \beta_0(t) &\ge \sum_{i=0}^{2k} \frac{f_1^{(i)}(t,\alpha_0)}{i!}(\beta_0-\alpha_0)^i + \sum_{i=0}^{2k-1} \frac{f_2^{(i)}(t,\alpha_0)}{i!}(\beta_0-\alpha_0)^i + \frac{f_2^{(2k)}(t,\beta_0)}{(2k)!}(\beta_0-\alpha_0)^{(2k)} \\ &= F(t,\alpha_0,\beta_0;\beta_0), \quad t \in \mathbb{N}_{a+1}^b, \\ \beta_0(a) &\ge x_0, \end{aligned}$$

which imply that $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of the IVP (4.5), respectively. Furthermore, we can see that $F(t,\alpha_0,\beta_0;y)$ is continuous with respect to y. Thus, by Lemma 3.4, there exists a solution $\alpha_1(t)$ of the IVP (4.5) such that $\alpha_0(t) \le \alpha_1(t) \le \beta_0(t)$ on $\mathbb{N}_a^b$.

Similarly, applying the fact that $\alpha_1(t)$ is a solution of the IVP (4.5), the inequalities (4.1)–(4.4), and the conditions $(H_{3.3})$, $(A_{4.2})$, we obtain

$$\begin{aligned} \nabla_{a^*}^{\nu} \alpha_1(t) &\overset{(A_{4.2})}{\le} \sum_{i=0}^{2k} \frac{f_1^{(i)}(t,\alpha_0)}{i!}(\alpha_1-\alpha_0)^i + \sum_{i=0}^{2k-1} \frac{f_2^{(i)}(t,\alpha_0)}{i!}(\alpha_1-\alpha_0)^i + \frac{f_2^{(2k)}(t,\alpha_1)}{(2k)!}(\alpha_1-\alpha_0)^{2k} \\ &\overset{(4.1),(4.2)}{\le} f_1(t,\alpha_1) + f_2(t,\alpha_1) \\ &\overset{(4.3),(4.4)}{\le} \sum_{i=0}^{2k} \frac{f_1^{(i)}(t,\beta_0)}{i!}(\alpha_1-\beta_0)^i + \sum_{i=0}^{2k-1} \frac{f_2^{(i)}(t,\beta_0)}{i!}(\alpha_1-\beta_0)^i + \frac{f_2^{(2k)}(t,\alpha_1)}{(2k)!}(\alpha_1-\beta_0)^{2k} \\ &\overset{(A_{4.2})}{\le} \sum_{i=0}^{2k} \frac{f_1^{(i)}(t,\beta_0)}{i!}(\alpha_1-\beta_0)^i + \sum_{i=0}^{2k-1} \frac{f_2^{(i)}(t,\beta_0)}{i!}(\alpha_1-\beta_0)^i + \frac{f_2^{(2k)}(t,\alpha_0)}{(2k)!}(\alpha_1-\beta_0)^{2k} \\ &= G(t,\alpha_0,\beta_0;\alpha_1), \qquad t \in \mathbb{N}_{a+1}^b, \\ \alpha_1(a) &= x_0, \end{aligned}$$

and

$$\begin{aligned} &\nabla_{a^*}^{\nu} \beta_0(t) \ge f_1(t,\beta_0) + f_2(t,\beta_0) = G(t,\alpha_0,\beta_0;\beta_0), \qquad t \in \mathbb{N}_{a+1}^b, \\ &\quad \beta_0(a) \ge x_0, \end{aligned}$$

which show that $\alpha_1(t)$ and $\beta_0(t)$ are lower and upper solutions of the IVP (4.6), respectively. Furthermore, we can find that $G(t,\alpha_0,\beta_0;z)$ is continuous with respect to z. Hence, in view of Lemma 3.4, we see that there exists a solution $\beta_1(t)$ of the IVP (4.6) such that $\alpha_1(t) \le \beta_1(t) \le \beta_0(t)$ on $\mathbb{N}_a^b$.

Next, we must show that $\alpha_1(t)$ and $\beta_1(t)$ are lower and upper solutions respectively of the IVP (3.1). For this purpose, using the conclusion that $\alpha_1(t)$ is a solution of the IVP (4.5), the condition $(A_{4.2})$, and the inequalities (4.1), (4.2), we have

$$\begin{aligned}\nabla_{a^*}^{\nu}\alpha_1(t) &\le \sum_{i=0}^{2k}\frac{f_1^{(i)}(t,\alpha_0)}{i!}(\alpha_1-\alpha_0)^i + \sum_{i=0}^{2k-1}\frac{f_2^{(i)}(t,\alpha_0)}{i!}(\alpha_1-\alpha_0)^i + \frac{f_2^{(2k)}(t,\alpha_1)}{(2k)!}(\alpha_1-\alpha_0)^{2k}\\ &\le f_1(t,\alpha_1)+f_2(t,\alpha_1), \qquad t\in\mathbb{N}_{a+1}^b,\\ \alpha_1(a) &= x_0,\end{aligned}$$

which proves that $\alpha_1(t)$ is a lower solution of the IVP (3.1) on $\mathbb{N}_a^b$. Similar arguments show that

$$\begin{aligned}\nabla_{a^*}^{\nu}\beta_1(t) &\ge f_1(t,\beta_1)+f_2(t,\beta_1), \qquad t\in\mathbb{N}_{a+1}^b,\\ \beta_1(a) &= x_0,\end{aligned}$$

which shows that $\beta_1(t)$ is an upper solution of the IVP (3.1) on $\mathbb{N}_a^b$. Therefore, we obtain

$$\alpha_0(t)\le\alpha_1(t)\le\beta_1(t)\le\beta_0(t) \quad \text{on } \mathbb{N}_a^b.$$

By induction, we have

$$\alpha_0(t)\le\alpha_1(t)\le\cdots\le\alpha_n(t)\le\beta_n(t)\le\cdots\le\beta_1(t)\le\beta_0(t) \quad \text{on } \mathbb{N}_a^b.$$

In addition, using the fact that $\alpha_n(t)$, $\beta_n(t)$ are lower and upper solutions of the IVP (3.1) with $\alpha_n(t)\le\beta_n(t)$, and the conditions of Lemma 3.4 are satisfied, we can conclude that there exists a solution $x_n(t)$ of the IVP (3.1) such that $\alpha_n(t)\le x_n(t)\le\beta_n(t)$ on $\mathbb{N}_a^b$. From this we can obtain that

$$\alpha_0(t)\le\alpha_1(t)\le\cdots\le\alpha_n(t)\le x_n(t)\le\beta_n(t)\le\cdots\le\beta_1(t)\le\beta_0(t) \quad \text{on } \mathbb{N}_a^b.$$

For any fixed $t\in\mathbb{N}_a^b$, the monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are uniformly bounded by $\{\alpha_0(t)\}$ and $\{\beta_0(t)\}$. Hence, we have that both monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are convergent on $\mathbb{N}_a^b$, that is, there exist functions $\rho, r:\mathbb{N}_a^b\to\mathbb{R}$ such that

$$\lim_{n\to\infty}\alpha_n(t)=\rho(t)\le r(t)=\lim_{n\to\infty}\beta_n(t).$$

Taking $\alpha=\alpha_n$, $\beta=\beta_n$ in IVPs (4.5), (4.6), then we can show easily that $\rho(t)$, $r(t)$ are solutions of the IVP (3.1). Next, we show that $\rho(t)\ge r(t)$. Let $p(t):=r(t)-\rho(t)$ so that $r(a)-\rho(a)=0$. Using the condition $(A_{4.1})$, and the mean value theorem, we have

$$\begin{aligned}\nabla_{a^*}^{\nu}p(t) &= f_1(t,r)+f_2(t,r)-f_1(t,\rho)-f_2(t,\rho)\\ &= f_1^{(1)}(t,\xi_1)(r-\rho)+f_2^{(1)}(t,\xi_2)(r-\rho)\\ &\le C_1p+C_2p\\ &= Cp,\end{aligned}$$

where $\rho(t)\le\xi_1(t),\xi_2(t)\le r(t)$. By Lemma 3.5, we conclude that $r(t)\le\rho(t)$ on $\mathbb{N}_a^b$. This proves that $\rho(t)=r(t)$. By the squeeze theorem, we have $\lim_{n\to\infty}x_n(t)$ exists, and

$$\lim_{n\to\infty}\alpha_n(t)=\rho(t)=\lim_{n\to\infty}x_n(t)=r(t)=\lim_{n\to\infty}\beta_n(t).$$

Set $\lim_{n\to\infty} x_n(t) = x(t)$, then we have $\rho(t) = x(t) = r(t)$. Hence $\alpha_n(t)$ and $\beta_n(t)$ converge uniformly and monotonically to a solution of the IVP (3.1).

Finally, we shall show that the convergence of the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ to a solution $x(t)$ of the IVP (3.1) is of order $2k+1$. For this purpose, set

$$\begin{aligned} p_n(t) &= x(t) - \alpha_n(t) \geq 0, \\ q_n(t) &= \beta_n(t) - x(t) \geq 0 \end{aligned}$$

for $t \in \mathbb{N}_a^b$ with $p_n(a) = q_n(a) = 0$.

From the conditions $(A_{4.1})$, $(A_{4.2})$, the Taylor's expansion with Lagrange remainder, and the mean value theorem, we obtain

$$\begin{aligned}
&\nabla_{a^*}^{\nu} p_{n+1}(t) \\
&\quad = f_1(t,x) + f_2(t,x) \\
&\qquad - \left[\sum_{i=0}^{2k} \frac{f_1^{(i)}(t,\alpha_n)}{i!}(\alpha_{n+1}-\alpha_n)^i + \sum_{i=0}^{2k-1} \frac{f_2^{(i)}(t,\alpha_n)}{i!}(\alpha_{n+1}-\alpha_n)^i + \frac{f_2^{(2k)}(t,\beta_n)}{(2k)!}(\alpha_{n+1}-\alpha_n)^{2k} \right] \\
&\quad = f_1(t,x) + f_2(t,x) \\
&\qquad - \left[f_1(t,\alpha_{n+1}) - \frac{f_1^{(2k)}(t,\xi_3)}{(2k)!}(\alpha_{n+1}-\alpha_n)^{2k} + \frac{f_1^{(2k)}(t,\alpha_n)}{(2k)!}(\alpha_{n+1}-\alpha_n)^{2k} + f_2(t,\alpha_{n+1}) \right. \\
&\qquad\quad \left. - \frac{f_2^{(2k)}(t,\xi_4)}{(2k)!}(\alpha_{n+1}-\alpha_n)^{2k} + \frac{f_2^{(2k)}(t,\beta_n)}{(2k)!}(\alpha_{n+1}-\alpha_n)^{2k} \right] \\
&\quad \overset{(A_{4.2})}{\leq} f_1^{(1)}(t,\eta_1)(x-\alpha_{n+1}) + f_2^{(1)}(t,\eta_2)(x-\alpha_{n+1}) \\
&\qquad + \frac{M_1}{(2k)!}(\alpha_{n+1}-\alpha_n)^{2k}(\xi_3-\alpha_n) + \frac{M_2}{(2k)!}(\alpha_{n+1}-\alpha_n)^{2k}(\beta_n-\xi_4) \\
&\quad \overset{(A_{4.1})}{\leq} C_1 p_{n+1} + C_2 p_{n+1} + K_1 p_n^{2k+1} + K_2 p_n^{2k}(p_n+q_n) \\
&\quad = C p_{n+1} + (K_1+K_2) p_n^{2k+1} + K_2 p_n^{2k} q_n \\
&\quad \leq C p_{n+1} + K p_n^{2k}(p_n+q_n),
\end{aligned}$$

where $\alpha_n \leq \xi_3, \xi_4 \leq \alpha_{n+1}$, $\alpha_{n+1} \leq \eta_1, \eta_2 \leq x$, $\frac{M_1}{(2k)!} = K_1$, $\frac{M_2}{(2k)!} = K_2$, $K = K_1 + K_2$. According to Lemma 3.5, we have $p_{n+1}(t) \leq x(t)$ on $\mathbb{N}_a^b$, where $x(t)$ is the solution of

$$\begin{cases} \nabla_{a^*}^{\nu} x(t) = Cx + K p_n^{2k}(p_n+q_n), & t \in \mathbb{N}_{a+1}^b, \\ x(a) = 0. \end{cases} \tag{4.7}$$

Hence, using the expression for $x(t)$ in Lemma 3.2, the solution $x(t)$ of (4.7) is given by

$$\begin{aligned}
x(t) &= \sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s)) \left[Cx(s) + K p_n^{2k}(s)(p_n(s)+q_n(s)) \right] \\
&\leq \left[C \max_{s\in\mathbb{N}_{a+1}^t} x(s) + K \max_{s\in\mathbb{N}_{a+1}^t} p_n^{2k}(s)(p_n(s)+q_n(s)) \right] \sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s)) \\
&= \left[C \max_{s\in\mathbb{N}_{a+1}^t} x(s) + K \max_{s\in\mathbb{N}_{a+1}^t} p_n^{2k}(s)(p_n(s)+q_n(s)) \right] \int_a^t H_{\nu-1}(t,\rho(s)) \nabla s
\end{aligned}$$

$$= \left[C \max_{s\in\mathbb{N}_{a+1}^{t}} x(s) + K \max_{s\in\mathbb{N}_{a+1}^{t}} p_n^{2k}(s)(p_n(s)+q_n(s)) \right] H_\nu(t,a)$$
$$\leq H_\nu(b,a) \left[C \max_{s\in\mathbb{N}_{a+1}^{b}} x(s) + K \max_{s\in\mathbb{N}_{a+1}^{b}} p_n^{2k}(s)(p_n(s)+q_n(s)) \right].$$

Then, we have

$$\max_{s\in\mathbb{N}_{a+1}^{b}} x(s) \leq CH_\nu(b,a) \max_{s\in\mathbb{N}_{a+1}^{b}} x(s) + KH_\nu(b,a) \max_{s\in\mathbb{N}_{a+1}^{b}} p_n^{2k}(s)(p_n(s)+q_n(s)).$$

According to the condition $(H_{3.4})$, the above inequality can be written as

$$\max_{s\in\mathbb{N}_{a+1}^{b}} x(s) \leq (1-CH_\nu(b,a))^{-1} KH_\nu(b,a) \max_{s\in\mathbb{N}_{a+1}^{b}} p_n^{2k}(s)(p_n(s)+q_n(s)).$$

Thus, we have

$$\max_{s\in\mathbb{N}_{a+1}^{b}} p_{n+1}(s) \leq (1-CH_\nu(b,a))^{-1} KH_\nu(b,a) \max_{s\in\mathbb{N}_{a+1}^{b}} p_n^{2k}(s)(p_n(s)+q_n(s)). \tag{4.8}$$

Therefore, we conclude that the following inequality holds

$$\|p_{n+1}\| \leq L\|p_n\|^{2k}(\|p_n\|+\|q_n\|),$$

where $L = \frac{KH_\nu(b,a)}{1-CH_\nu(b,a)}$ is a positive constant, which is the desired result.

Similarly, utilizing the conditions $(A_{4.1})$, $(A_{4.2})$, the Taylor's expansion with Lagrange remainder, and the mean value theorem, we have

$$\begin{aligned}
&\nabla_{a^*}^{\nu} q_{n+1}(t) \\
&\quad= \left[\sum_{i=0}^{2k} \frac{f_1^{(i)}(t,\beta_n)}{i!}(\beta_{n+1}-\beta_n)^i + \sum_{i=0}^{2k-1} \frac{f_2^{(i)}(t,\beta_n)}{i!}(\beta_{n+1}-\beta_n)^i + \frac{f_2^{(2k)}(t,\alpha_n)}{(2k)!}(\beta_{n+1}-\beta_n)^{2k} \right] \\
&\qquad - f_1(t,x) - f_2(t,x) \\
&\quad= \left[f_1(t,\beta_{n+1}) - \frac{f_1^{(2k)}(t,\xi_5)}{(2k)!}(\beta_{n+1}-\beta_n)^{2k} + \frac{f_1^{(2k)}(t,\beta_n)}{(2k)!}(\beta_{n+1}-\beta_n)^{2k} \right. \\
&\qquad \left. + f_2(t,\beta_{n+1}) - \frac{f_2^{(2k)}(t,\xi_6)}{(2k)!}(\beta_{n+1}-\beta_n)^{2k} + \frac{f_2^{(2k)}(t,\alpha_n)}{(2k)!}(\beta_{n+1}-\beta_n)^{2k} \right] \\
&\qquad - f_1(t,x) - f_2(t,x) \\
&\quad\overset{(A_{4.2})}{\leq} f_1^{(1)}(t,\eta_3)(\beta_{n+1}-x) + f_2^{(1)}(t,\eta_4)(\beta_{n+1}-x) \\
&\qquad + \frac{M_1}{(2k)!}(\beta_{n+1}-\beta_n)^{2k}(\beta_n-\xi_5) + \frac{M_2}{(2k)!}(\beta_{n+1}-\beta_n)^{2k}(\xi_6-\alpha_n) \\
&\quad\overset{(A_{4.1})}{\leq} C_1 q_{n+1} + C_2 q_{n+1} + K_1 q_n^{2k+1} + K_2 q_n^{2k}(q_n+p_n) \\
&\quad= Cq_{n+1} + (K_1+K_2)q_n^{2k+1} + K_2 q_n^{2k} p_n \\
&\quad\leq Cq_{n+1} + Kq_n^{2k}(q_n+p_n),
\end{aligned}$$

where $\beta_{n+1} \leq \xi_5, \xi_6 \leq \beta_n$, $x \leq \eta_3, \eta_4 \leq \beta_{n+1}$. By Lemma 3.5, we obtain $q_{n+1}(t) \leq x(t)$ on $\mathbb{N}_a^b$, where $x(t)$ is the solution of

$$\begin{cases} \nabla_{a^*}^{\nu} x(t) = Cx + Kq_n^{2k}(q_n+p_n), \quad t \in \mathbb{N}_{a+1}^{b}, \\ x(a) = 0. \end{cases} \tag{4.9}$$

Similar to the inequality (4.8) for $p_{n+1}(t)$, we can get

$$\max_{s\in\mathbb{N}_{a+1}^{b}} q_{n+1}(s) \le (1-CH_{\nu}(b,a))^{-1}KH_{\nu}(b,a)\max_{s\in\mathbb{N}_{a+1}^{b}} q_n^{2k}(s)(q_n(s)+p_n(s)). \tag{4.10}$$

Consequently, we get

$$\|q_{n+1}\| \le L\|q_n\|^{2k}(\|q_n\|+\|p_n\|).$$

The proof is complete. $\square$

Theorem 4.2. *Assume that the conditions $(H_{3.3})$–$(H_{3.4})$ hold, and*

$(A_{4.3})$ *the functions $f_1, f_2 : \Omega \to \mathbb{R}$ are such that $f_1^{(i)}(t,x)$, $f_2^{(i)}(t,x)$ $(i = 0,1,\ldots,2k-1)$ exist, are continuous in its second variable, and for $C_1 > 0$, $C_2 > 0$, $C = C_1 + C_2$,*

$$f_1^{(1)}(t,x) \le C_1, \quad f_2^{(1)}(t,x) \le C_2 \quad \text{on } \Omega.$$

$(A_{4.4})$ *there exist $M_1, M_2 > 0$ such that for $x_1 \ge x_2$, $y_1 \ge y_2$ the functions $f_1^{(2k-1)}(t,x)$, $f_2^{(2k-1)}(t,x)$ satisfy the following conditions:*

$$\begin{aligned} 0 &\le f_1^{(2k-1)}(t,x_1) - f_1^{(2k-1)}(t,x_2) \le M_1(x_1-x_2) && \text{on } \Omega,\\ 0 &\ge f_2^{(2k-1)}(t,y_1) - f_2^{(2k-1)}(t,y_2) \ge -M_2(y_1-y_2) && \text{on } \Omega. \end{aligned}$$

Then there exist two sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$, $n \ge 0$ which converge uniformly and monotonically to a solution of the IVP (3.1) and the convergence is of order $2k$.

Proof. In order to construct monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$, $n \ge 0$ which converge uniformly and monotonically to the solution of the IVP (3.1), we need to consider the following nonlinear Caputo nabla fractional difference equations:

$$\begin{cases} \nabla_{a^*}^{\nu}\alpha_n(t) = \displaystyle\sum_{i=0}^{2k-1}\frac{f_1^{(i)}(t,\alpha_{n-1})}{i!}(\alpha_n-\alpha_{n-1})^i + \sum_{i=0}^{2k-2}\frac{f_2^{(i)}(t,\alpha_{n-1})}{i!}(\alpha_n-\alpha_{n-1})^i \\ \qquad\qquad + \dfrac{f_2^{(2k-1)}(t,\beta_{n-1})}{(2k-1)!}(\alpha_n-\alpha_{n-1})^{2k-1} \\ \qquad\qquad \equiv F(t,\alpha_{n-1},\beta_{n-1};\alpha_n), \quad t\in\mathbb{N}_{a+1}^{b}, \\ \alpha_n(a) = x_0, \end{cases} \tag{4.11}$$

and

$$\begin{cases} \nabla_{a^*}^{\nu}\beta_n(t) = \displaystyle\sum_{i=0}^{2k-2}\frac{f_1^{(i)}(t,\beta_{n-1})}{i!}(\beta_n-\beta_{n-1})^i + \frac{f_1^{(2k-1)}(t,\alpha_{n-1})}{(2k-1)!}(\beta_n-\beta_{n-1})^{2k-1} \\ \qquad\qquad + \displaystyle\sum_{i=0}^{2k-1}\frac{f_2^{(i)}(t,\beta_{n-1})}{i!}(\beta_n-\beta_{n-1})^i \\ \qquad\qquad \equiv G(t,\alpha_{n-1},\beta_{n-1};\beta_n), \quad t\in\mathbb{N}_{a+1}^{b}, \\ \beta_n(a) = x_0. \end{cases} \tag{4.12}$$

Similar to Theorem 4.1, we can show that the convergence is of order $2k$. $\square$

5 Numerical result

Now, we give a numerical example to illustrate the application of the results established in the previous section.

Example 5.1. Consider the following nonlinear Caputo nabla fractional difference equation

$$\begin{cases} \nabla_{0^*}^{\frac{1}{2}} x(t) = -x^5(t) + x^3(t) + x^2(t) - 5x(t) - 1, & t \in \mathbb{N}_1^5, \\ x(0) = 0. \end{cases} \tag{5.1}$$

Taking $\alpha_0(t) = -1$, $\beta_0(t) = 1$, it is easy to verify that $\alpha_0(t)$, $\beta_0(t)$ are lower and upper solutions of the IVP (5.1), respectively. According to the Definitions 2.2, 2.4, we have $\nabla_{0^*}^{\frac{1}{2}} x(t) = \nabla_0^{-\frac{1}{2}} \nabla x(t) = \sum_{s=1}^{t} \frac{\Gamma(t-s+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(t-s+1)} \nabla x(s)$. Let $f(t,x)$ denote the right-hand side of (5.1), and split it into two functions as $f(t,x) = f_1(t,x) + f_2(t,x)$, where $f_1(t,x) = x^3(t) + x^2(t) - 5x(t) - 1$, $f_2(t,x) = -x^5(t)$.

Table 5.1: Table of four α-iterates of (5.1).

t	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$	$\alpha_4(t)$	$x(t)$
1	-0.595929	-0.227396	-0.163045	-0.162943	-0.162943
2	-0.611197	-0.248267	-0.176153	-0.175965	-0.175965
3	-0.615457	-0.254921	-0.180447	-0.180254	-0.180254
4	-0.617607	-0.258339	-0.182687	-0.182480	-0.182480
5	-0.618955	-0.260491	-0.184105	-0.183888	-0.183888

Table 5.2: Table of four β-iterates of (5.1).

t	$\beta_4(t)$	$\beta_3(t)$	$\beta_2(t)$	$\beta_1(t)$
1	-0.162943	-0.161992	-0.011633	0.525454
2	-0.175965	-0.174333	-0.000297	0.539207
3	-0.180254	-0.178300	0.003561	0.543062
4	-0.182480	-0.180338	0.005570	0.545007
5	-0.183888	-0.181619	0.006843	0.546226

It is easy to show that

$$f_1^{(1)}(t,x) = 3x^2(t) + 2x(t) - 5 = 3\left(x(t) + \frac{1}{3}\right)^2 - \frac{16}{3} \le 0 \quad \text{on } \Omega,$$
$$f_2^{(1)}(t,x) = -5x^4(t) \le 0 \quad \text{on } \Omega,$$

so we can choose $C = \frac{1}{3}$ such that

$$CH_{\frac{1}{2}}(5,0) = \frac{1}{3} \cdot \frac{315}{128} = \frac{105}{128} < 1.$$

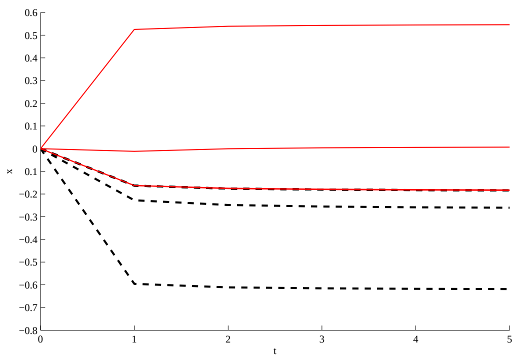

Figure 5.1: α-iterates (broken lines), β-iterates (unbroken lines).

Furthermore, we can check that for $x_1(t) \geq x_2(t)$, $y_1(t) \geq y_2(t)$,

$$0 \leq f_1^{(2)}(t, x_1) - f_1^{(2)}(t, x_2) = 6(x_1(t) - x_2(t)) \quad \text{on } \Omega,$$
$$0 \geq f_2^{(2)}(t, y_1) - f_2^{(2)}(t, y_2) = -20y_1^3(t) + 20y_2^3(t) \geq -60(y_1(t) - y_2(t)) \quad \text{on } \Omega.$$

Hence, we can apply the iterates of Theorem 4.1. After only four iterates of α and β, we can find the α, β-iterates as given in Tables 5.1, 5.2. Figure 5.1 shows the graphs of some α-iterates (broken lines) and some β-iterates (unbroken lines). Note that the actual graphs of the $\alpha_i(t)$ and the $\beta_i(t)$, $1 \leq i \leq 3$, $1 \leq t \leq 5$, consist just of the points corresponding to the values of these functions at $t = 1, 2, 3, 4, 5$. Note that for all practical purposes the graphs of $\alpha_3(t)$, $\beta_3(t)$ in Figure 5.1 and the solution $x(t)$ are the same to several decimal places.

6 Conclusion

In the above parts, we proved the existence of solutions for nonlinear Caputo nabla fractional difference equations. Based on this fact, we have developed two monotone sequences which converge rapidly to the solution of such equations. In addition, a numerical example is given to show one of the established results.

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