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Constant sign solution for a simply supported beam equation

Alberto Cabada and **Lorena Saavedra**

Instituto de Matemáticas, Facultade de Matemáticas, Universidade de Santiago de Compostela
Santiago de Compostela, Galicia, Spain

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Abstract. The aim of this paper is to ensure the existence of constant sign solutions for the fourth order boundary value problem:

$$\begin{cases} u^{(4)}(t) - p\,u''(t) + c(t)\,u(t) = h(t)(\geq 0)\,, & t \in I \equiv [a,b]\,, \\ u(a) = u''(a) = u(b) = u''(b) = 0\,, \end{cases}$$

where $c,\ h \in C(I)$ and $p \geq 0$.

This problem models the behavior of a suspension bridge assuming that the vertical displacement is small enough.

By using variational methods, we weaken the previously known sufficient conditions on c to ensure that the obtained solution is of constant sign.

Keywords: differential equations, boundary value problems, constant sign solutions, variational approach.

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1 Introduction

The study of different fourth order differential equations coupled with the simply supported beam boundary conditions has been wider treated along the literature. For instance, in [9] and [10] there are obtained sufficient conditions that ensure that the problem:

$$\begin{cases} u^{(4)}(t) + c(t)\,u(t) = h(t)(\geq 0)\,, & t \in [0,1]\,, \\ u(a) = u(b) = u''(a) = u''(b) = 0\,, \end{cases} \tag{1.1}$$

has a unique solution, which has constant sign on the interval $[0,1]$. Both papers improve previous results obtained in [4] and [16].

In [7] the strongly inverse positive (negative) character of the operator

$$u^{(4)}(t) + p_1(t)\,u^{(3)}(t) + p_2(t)\,u''(t) + M\,u(t)\,,$$

✉ Corresponding author. Email: lorena.saavedra@usc.es

coupled with the same simply supported beam boundary conditions as in (1.1), where $p_1 \in C^3(I)$ and $p_2 \in C^2(I)$, is determined by the spectrum of the operator with suitable related boundary conditions.

As it has been shown in [11], the study of this kind of problems is very important, since they are used to model different kind of bridges.

In addition, in [12, Chapter 2] the author describes several models for suspension bridges. For instance, in [12, Section 2.6.3], he considers a hinged beam (which represents the roadway) subject to non-linear forces along two-sided springs (the hangers of the suspension bridge). In such a case, the one dimension mathematical model for the vertical displacement of the roadway is given by:

$$\begin{cases} E\,I\,u^{(4)}(t) - T\,u''(t) + g(u(t)) = q(t)\,, \quad t \in I\,, \\ u(a) = u(b) = u''(a) = u''(b) = 0\,, \end{cases} \tag{1.2}$$

where a and b are the extremes of the studied bridge, E and I are two positive constants given by the material of the beam (the Young's module and the moment of inertia), $T \geq 0$ is the constant strength tension, $q(t)$ is a downwards distributed load which acts on the beam and g is the restoring force. This model involves a non-linear part given by the function g. But in order to study it, it is very important to know first its the linear part. Indeed, in some cases, for instance if the vertical displacement is small enough, we can consider g as a linear function in the way $g(u(t)) = k\,u(t)$, for a constant $k \in \mathbb{R}$. A more general problem consists on considering this restoring force as a non-autonomous function $g(t, u(t)) = f(t)\,u(t)$, where f is a continuous function on I. In particular the fact that the displacement of the bridge occurs in the same direction as the external force is fundamental in order to ensure the stability of the considered structure.

In [4, 13] the existence of one or multiple positive solutions of some suitable non-linear problems are considered. The used tools are strongly involved with the constant sign of the related Green's function.

In this paper we study the existence of constant sign solutions of the following fourth-order problem:

$$T[p,c]\,u(t) \equiv u^{(4)}(t) - p\,u''(t) + c(t)\,u(t) = h(t)\,, \quad t \in I\,, \tag{1.3}$$

coupled with the boundary conditions:

$$u(a) = u(b) = u''(a) = u''(b) = 0\,. \tag{1.4}$$

Remark 1.1. Realize that if in (1.2) we consider the non-autonomous function $g(t, u(t)) = f(t)\,u(t)$ and divide by $E\,I$, then problem (1.3)–(1.4) is a particularization of (1.2) with $p = \frac{T}{E\,I} \geq 0$, $c(t) = \frac{f(t)}{E\,I}$ and $h(t) = \frac{q(t)}{E\,I} \geq 0$ (because q is a downwards load).

Let us denote the correspondent space of definition as follows:

$$X = \left\{ u \in C^4(I) \mid u(a) = u(b) = u''(a) = u''(b) = 0 \right\}. \tag{1.5}$$

Realize that problem (1.1), studied in [9, 10], is a particular case of (1.3)–(1.4) with $p = 0$.

We want to recall that despite the problem studied in [7] is more general than (1.3)–(1.4), here we weaken the sufficient conditions on c to ensure the existence of constant sign solutions. In that reference, it is imposed that the continuous function c remains between two values which we obtain by means of spectral theory. With the results which we will prove below, we allow c to pass throw these values in some sense.

In the next section, for convenience of the reader we introduce some previous results which we use along the paper.

Then in Section 3, before showing the main existence results, we formulate the variational approach of problem (1.3)–(1.4) and we obtain different previous results which will be used along the paper.

In Section 4, we will obtain sufficient conditions to ensure that problem (1.3)–(1.4) has a unique solution. In fact, Section 4 is devoted to prove sufficient conditions which guarantee that the operator $T[p,c]$ is either strongly inverse positive in X or strongly inverse negative in X.

In Section 5, we obtain different conditions for functions $h > 0$ and c that ensure that the unique solution of the problem (1.3)–(1.4) is either positive or negative.

Finally, in Section 6, we show an example where we apply our results.

2 Preliminaries

In this section, we introduce several tools and results which will be used along the paper.

We consider a general n^{th}-order linear operator:

$$L_n[M]\,u(t) \equiv u^{(n)}(t) + p_1(t)\,u^{(n-1)}(t) + \cdots + p_{n-1}(t)\,u'(t) + (p_n(t) + M)\,u(t)\,, \tag{2.1}$$

with $t \in I$ and $p_k \in C^{n-k}(I)$, $k = 1, \ldots, n$.

Definition 2.1. The n^{th}-order linear differential equation:

$$L_n[M]\,u(t) = 0, \qquad t \in I, \tag{2.2}$$

is said to be disconjugate on I if every non trivial solution has less than n zeros at I, multiple zeros being counted according to their multiplicity.

We introduce a definition to our particular problem (1.3) in the space X.

Definition 2.2. The operator $T[p,c]$ is said to be strongly inverse positive (strongly inverse negative) in X, if every function $u \in X$ such that $T[p,c]\,u \gneqq 0$ in I, satisfies $u > 0$ ($u < 0$) on (a,b) and, moreover $u'(a) > 0$ and $u'(b) < 0$ ($u'(a) < 0$ and $u'(b) > 0$).

Let us denote $g_{p,c}$ the related Green's function to operator $T[p,c]$ in X. Next results, collected in [7], show a relationship between the Green's function's sign and the previous definition.

Theorem 2.3. *Green's function related to operator $T[M]$ in X is positive a.e. on $(a,b) \times (a,b)$ and, moreover, $\frac{\partial}{\partial t} g_{p,c}(t,s)_{|t=a} > 0$ and $\frac{\partial}{\partial t} g_{p,c}(t,s)_{|t=b} < 0$ a.e. on (a,b), if, and only if, operator $T[M]$ is strongly inverse positive in X.*

Theorem 2.4. *Green's function related to operator $T[M]$ in X is negative a.e. on $(a,b) \times (a,b)$ and, moreover, $\frac{\partial}{\partial t} g_{p,c}(t,s)_{|t=a} < 0$ and $\frac{\partial}{\partial t} g_{p,c}(t,s)_{|t=b} > 0$ a.e. on (a,b), if, and only if, operator $T[M]$ is strongly inverse negative in X.*

- Let $\lambda_1^p > 0$ be the least positive eigenvalue of $T[p,0]$ in X.
- Let $\lambda_2^p < 0$ be the maximum between:

$\lambda_2^{p'} < 0$, the biggest negative eigenvalue of $T[p,0]$ in

$$X_1 = \left\{u \in C^4(I) \mid u(a) = u(b) = u'(b) = u''(b) = 0\right\},$$

$\lambda_2^{p''} < 0$, the biggest negative eigenvalue of $T[p,0]$ in

$$X_3 = \left\{u \in C^4(I) \mid u(a) = u'(a) = u''(a) = u(b) = 0\right\}.$$

- Let $\lambda_3^p > 0$ be the minimum between:

$\lambda_3^{p'} > 0$, the least positive eigenvalue of $T[p,0]$ in

$$U = \left\{u \in C^4(I) \mid u(a) = u'(a) = u(b) = u''(b) = 0\right\},$$

$\lambda_3^{p''} > 0$, the least positive eigenvalue of $T[p,0]$ in

$$V = \left\{u \in C^4(I) \mid u(a) = u''(a) = u(b) = u'(b) = 0\right\}.$$

Remark 2.5. In [5] we prove that the second order linear differential equation

$$u''(t) + m\,u(t) = 0,$$

is disconjugate on I if, and only if, $m \in \left(-\infty, \left(\frac{\pi}{b-a}\right)^2\right)$. In particular: if $p \geq 0$, then $u''(t) - p\,u(t) = 0$ is a disconjugate equation in every real interval I.

Hence, under this disconjugacy condition, in [7] it is proved the existence of $\lambda_1^p > 0$, $\lambda_2^{p'} < 0$, $\lambda_2^{p''} < 0$, $\lambda_3^{p'} > 0$ and $\lambda_3^{p''} > 0$. Thus, the previous eigenvalues are well-defined.

As a consequence of [7, Theorem 6.1] we can state the following result.

Corollary 2.6. *Consider the operator $T[p,c]\,u(t) \equiv u^{(4)}(t) - p\,u''(t) + c(t)\,u(t)$, where $p \in \mathbb{R}$ and $p \geq 0$. Then,*

- *If $-\lambda_1^p < c(t) \leq -\lambda_2^p$ for every $t \in I$, then $T[p,c]$ is strongly inverse positive in X.*
- *If $-\lambda_3^p \leq c(t) < -\lambda_1^p$ for every $t \in I$, then $T[p,c]$ is strongly inverse negative in X.*

Moreover, in [7], there are obtained the values of λ_1^p, λ_2^p and λ_3^p. In particular, we have: The eigenvalues of the operator $T[p,0]$ in X are given by

$$\lambda_1^p(k) = k^4\left(\frac{\pi}{b-a}\right)^4 + k^2\,p\left(\frac{\pi}{b-a}\right)^2, \tag{2.3}$$

where $k \in \{1,2,3,\dots\}$.

Obviously, the least positive eigenvalue is given by $\lambda_1^p \equiv \lambda_1^p(1) = \left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2$. Moreover, we denote as $\lambda_1^p(2) = 16\,\left(\frac{\pi}{b-a}\right)^4 + 4\,p\,\left(\frac{\pi}{b-a}\right)^2$ the second positive eigenvalue of $T[p,0]$ in X.

It is clear that if we denote λ as an eigenvalue of $T[p,0]$ and its associated eigenfunction as $u \in X_1$, then function $v(t) := u(1-t)$ is an eigenfunction associated to λ in X_3. As a consequence, the eigenvalues of $T[p,0]$ on the spaces X_1 and X_3 are the same. So, in the previous definitions $\lambda_2^p = \lambda_2^{p'} = \lambda_2^{p''}$.

One can verify that such eigenvalues are given as $-\lambda$, where λ is a positive solution of

$$\frac{\tan\left(\frac{b-a}{2}\sqrt{2\sqrt{\lambda}-p}\right)}{\sqrt{2\sqrt{\lambda}-p}}=\frac{\tanh\left(\frac{b-a}{2}\sqrt{2\sqrt{\lambda}+p}\right)}{\sqrt{2\sqrt{\lambda}+p}}, \tag{2.4}$$

in particular, λ_2^p is the opposite of the least positive solution of this equation.

Similarly, the eigenvalues of $T[p,0]$ in U and V are the same and we can conclude that $\lambda_3^p=\lambda_3^{p'}=\lambda_3^{p''}$.

In particular, the eigenvalues are given as the positive solutions of the following equality:

$$\frac{\tan\left(\frac{(b-a)\sqrt{\sqrt{p^2+4\lambda}-p}}{\sqrt{2}}\right)}{\sqrt{\sqrt{p^2+4\lambda}-p}}=\frac{\tanh\left(\frac{(b-a)\sqrt{\sqrt{p^2+4\lambda}+p}}{\sqrt{2}}\right)}{\sqrt{\sqrt{p^2+4\lambda}+p}}, \tag{2.5}$$

and λ_3^p is the least positive solution of this equation.

3 Variational approach

In this section we obtain the variational approach of problem (1.3)–(1.4) and some results which will be used in our main results.

First, we consider the Hilbert space $H:=H^2(I)\cap H_0^1(I)$, where:

$$H^2(I)=\{u\in L^2(I)\mid u',u''\in L^2(I)\},$$

and

$$H_0^1(I)=\{u\in L^2(I)\mid u'\in L^2(I),\ u(a)=u(b)=0\}.$$

We say that $u\in H$ is a weak solution of (1.3)–(1.4) if it satisfies:

$$\int_a^b u''(t)\,v''(t)\,dt+p\int_a^b u'(t)\,v'(t)\,dt+\int_a^b c(t)\,u(t)\,v(t)\,dt=\int_a^b h(t)\,v(t)\,dt,\quad \forall v\in H. \tag{3.1}$$

For a function $f\in C(I)$. Let us denote:

$$f_m:=\min_{t\in I}f(t),\qquad f^m:=\max_{t\in I}f(t)\quad\text{and}\quad f^{\pm}(t)=\max\{0,\pm f(t)\},\ t\in I.$$

If $p=0$ and $a=0$, $b=1$, we have the following result, see [17,18].

Proposition 3.1. *Let $c(t)\neq -k^4\pi^4$ for any $k\in\mathbb{N}$ and all $t\in[0,1]$. Let $p=0$, $a=0$ and $b=1$, then the problem (1.3)–(1.4) has a unique solution $u\in X$. Moreover, if $-\pi^4<c_m<0$, then*

$$\|u\|_{C([0,1])}\leq\frac{\pi}{2\,(\pi^4+c_m)}\,\|h\|_{C([0,1])}.$$

Now, we enunciate an equivalent result to this proposition, which refers to our case.

Proposition 3.2. *Let $c(t)\neq -k^4\left(\frac{\pi}{b-a}\right)^4-k^2\,p\left(\frac{\pi}{b-a}\right)^2$ for any $k\in\{1,2,3,\dots\}$ and all $t\in I$. Then the problem (1.3)–(1.4) has a unique solution $u\in X$.*

Moreover, if $-\left(\frac{\pi}{b-a}\right)^4-p\left(\frac{\pi}{b-a}\right)^2<c_m<0$, then

$$\|u\|_{C([0,1])}\leq\frac{\pi}{2\left(\left(\frac{\pi}{b-a}\right)^4+p\left(\frac{\pi}{b-a}\right)^2+c_m\right)}\,\|h\|_{C([0,1])}.$$

Proof. If $c(t) \neq -k^4 \left(\frac{\pi}{b-a}\right)^4 - p\,k^2 \left(\frac{\pi}{b-a}\right)^2$ for any $k \in \{1,2,3,\dots\}$ and $t \in I$, it means that, since $c \in C(I)$, either there exist $k \in \{1,2,3\dots\}$ such that

$$c(t) \in \left(-(k+1)^4 \left(\frac{\pi}{b-a}\right)^4 - p\,(k+1)^2 \left(\frac{\pi}{b-a}\right)^2,\ -k^4 \left(\frac{\pi}{b-a}\right)^4 - p\,k^2 \left(\frac{\pi}{b-a}\right)^2\right)$$

or that $c_m > -\left(\frac{\pi}{b-a}\right)^4 - p\,\left(\frac{\pi}{b-a}\right)^2$, i.e. there is no any eigenvalue of $T[0,p]$ between c_m and c^m. As a consequence, the existence of a unique solution of problem (1.3)–(1.4) is ensured. Now, let us see the boundedness.

We have the two following Wirtinger inequalities for every $u \in H$, (see [14, 17]):

$$\|u\|_{L^2(I)} \leq \frac{b-a}{\pi}\,\|u'\|_{L^2(I)} \leq \left(\frac{b-a}{\pi}\right)^2 \|u''\|_{L^2(I)}\,, \tag{3.2}$$

and,

$$\|u\|_{C(I)} \leq \frac{\sqrt{b-a}}{2}\,\|u'\|_{L^2(I)}\,. \tag{3.3}$$

Now, multiplying equation (1.3) by the unique solution $u \in X$ and integrating, we have:

$$\int_a^b u^{(4)}(t)\,u(t)\,dt - p\int_a^b u''(t)\,u(t)\,dt + \int_a^b c(t)\,u^2(t)\,dt = \int_a^b h(t)\,u(t)\,dt\,,$$

which is equivalent to:

$$\int_a^b u''^2(t)\,dt + p\int_a^b u'^2(t)\,dt = \int_a^b h(t)\,u(t)\,dt - \int_a^b c(t)\,u^2(t)\,dt\,.$$

Now, taking into account the inequalities (3.2), the Hölder inequality and that $c_m \leq 0$ we have:

$$\|u''\|^2_{L^2(I)} + p\,\|u'\|^2_{L^2(I)} \geq \left(\frac{\pi}{b-a}\right)^2 \|u'\|^2_{L^2(I)} + p\,\|u'\|^2_{L^2(I)}\,,$$

and,

$$\begin{aligned}\int_a^b h(t)\,u(t)\,dt - \int_a^b c(t)\,u^2(t)\,dt &\leq \|h\|_{C(I)}\int_a^b |u(t)|\,dt - c_m\,\|u\|^2_{L^2(I)}\\ &\leq \|h\|_{C(I)}\sqrt{b-a}\,\|u\|_{L^2(I)} - c_m\,\|u\|^2_{L^2(I)}\\ &\leq \|h\|_{C(I)}\sqrt{b-a}\frac{b-a}{\pi}\,\|u'\|_{L^2(I)} - c_m\left(\frac{b-a}{\pi}\right)^2\|u'\|^2_{L^2(I)}\,.\end{aligned}$$

So, combining the last two inequalities we arrive to:

$$\left(\left(\frac{\pi}{b-a}\right)^2 + p + c_m\left(\frac{b-a}{\pi}\right)^2\right)\|u'\|_{L^2(I)} \leq \|h\|_{C(I)}\sqrt{b-a}\frac{b-a}{\pi}\,,$$

which is equivalent to:

$$\|u'\|_{L_2(I)} \leq \frac{\pi}{\sqrt{b-a}}\frac{\|h\|_{C(I)}}{\left(\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + c_m\right)}\,,$$

and this combined with the inequality (3.3) gives our result. □

Remark 3.3. We note that previous inequality includes Proposition 3.1 as a particular case.

For an arbitrary nonnegative continuous function $r(t) \geq 0$ in I, we define the scalar product:

$$(u,v) = \int_a^b u''(t)\,v''(t)\,dt + p\int_a^b u'(t)\,v'(t)\,dt + \int_a^b r(t)\,u(t)\,v(t)\,dt\,, \quad u,v \in H\,, \tag{3.4}$$

and $\|u\| = (u,u)^{1/2}$ its associated norm.

We have the following inequality:

$$|u(t)-u(s)| = \left|\int_s^t u'(r)\,dr\right| \leq \sqrt{t-s}\;\|u'\|_{L^2(I)} \leq \sqrt{t-s}\,\frac{b-a}{\pi}\,\|u''\|_{L^2(I)} \leq \sqrt{t-s}\,\frac{b-a}{\pi}\,\|u\|\;.$$

Thus, we can affirm that the embedding of H into $C(I)$ is compact.

Let $f(t)$ and $h(t)$ be continuous functions on I, following the arguments shown in [9], using the Riesz Representation Theorem we can define $S_f\colon H \to H$ and $h^* \in H$ such that:

$$(S_f u,v) = \int_a^b f(t)\,u(t)\,v(t)\,dt\,, \quad (h^*,v) = \int_a^b h(t)\,v(t)\,dt\,, \quad u,v \in H\,. \tag{3.5}$$

Now, let us introduce some results which make a relation between this norm and the norms $\|\cdot\|_{C(I)}$ and $\|\cdot\|_{L^2(I)}$. Such a result generalizes [9, Lemma 7].

Lemma 3.4. *Let $u \in H$, $r \in C(I)$, $r \geq 0$ in I and $\|\cdot\|$ be the norm associated to the scalar product (3.4). Then:*

$$\|u\|_{C(I)} \leq \frac{1}{\sqrt{\delta_1}}\,\|u\|\;,$$

and,

$$\|u\|_{L^2(I)} \leq \frac{\|u\|}{\sqrt{\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\{r(t)\}}}\,,$$

where:

$$\delta_1 = \max\left\{\frac{4\,p}{b-a}, \frac{4\,\pi^2}{(b-a)^3}\right\}\,. \tag{3.6}$$

Proof. Using the inequalities (3.2)–(3.3), we have that the two following inequalities are satisfied:

$$\begin{aligned} p\,\|u\|^2_{C(I)} &\leq \frac{b-a}{4}\,p\int_a^b (u'(t))^2\,dt \\ &\leq \frac{b-a}{4}\left(\int_a^b (u''(t))^2\,dt + p\int_a^b (u'(t))^2\,dt + \int_a^b r(t)\,u^2(t)\,dt\right) \\ &= \frac{b-a}{4}\,\|u\|^2\,, \end{aligned}$$

$$\begin{aligned} \|u\|^2_{C(I)} &\leq \frac{b-a}{4}\int_a^b (u'(t))^2\,dt \leq \frac{(b-a)^3}{4\,\pi^2}\int_a^b (u''(t))^2\,dt \\ &\leq \frac{(b-a)^3}{4\,\pi^2}\left(\int_a^b (u''(t))^2\,dt + p\int_a^b (u'(t))^2\,dt + \int_a^b r(t)\,u^2(t)\,dt\right) \\ &= \frac{(b-a)^3}{4\,\pi^2}\,\|u\|^2\,. \end{aligned}$$

So, if $p \neq 0$,

$$\|u\|_{C(I)} \le \min\left\{\sqrt{\frac{b-a}{4\,p}}, \frac{\sqrt{(b-a)^3}}{2\,\pi}\right\} \|u\| = \frac{1}{\sqrt{\delta_1}}\,\|u\|\,,$$

moreover, if $p = 0$,

$$\|u\|_{C(I)} \le \frac{\sqrt{(b-a)^3}}{2\,\pi}\,\|u\| = \frac{1}{\sqrt{\delta_1}}\,\|u\|\,.$$

On another hand,

$$\begin{aligned}\|u\|^2_{L^2(I)} &= \frac{\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\{r(t)\}}{\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\{r(t)\}} \int_a^b u^2(t)\,dt \\ &\le \frac{\int_a^b (u''(t))^2\,dt + p\,\int_a^b (u'(t))^2\,dt + \int_a^b r(t)\,u^2(t)\,dt}{\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\{r(t)\}} \\ &= \frac{\|u\|^2}{\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\{r(t)\}}\,.\end{aligned}$$

□

From classical arguments, see [1], we obtain the following result, where we see that a weak solution of (3.1) in H under suitable conditions is indeed a classical solution of (1.3)–(1.4) in X.

Proposition 3.5. *Let $c, h \in C(I)$. If $u \in H$ is a weak solution of* (3.1)*, then u is a classical solution of* (1.3)–(1.4) *in X.*

Next result improves [9, Lemma 8].

Lemma 3.6. *Let $S_f\colon H \to H$ be the operator previously defined in* (3.5)*. Then:*

$$\|S_f\| \le \frac{1}{\delta_1}\int_a^b |f(t)|\,dt\,.$$

Proof. Using Lemma 3.4 we can deduce the following inequalities which prove the result:

$$\begin{aligned}\|S_f\| &= \sup_{\|u\|=1}\|S_f\,u\| = \sup_{\|u\|=1}\sup_{\|v\|=1}\left|\int_a^b f(t)\,u(t)\,v(t)\,dt\right| \le \sup_{\|u\|=1}\sup_{\|v\|=1}\int_a^b |f(t)|\,|u(t)|\,|v(t)|\,dt \\ &\le \sup_{\|u\|=1}\|u\|_{C(I)}\sup_{\|v\|=1}\|v\|_{C(I)}\int_a^b |f(t)|\,dt \le \frac{1}{\delta_1}\int_a^b |f(t)|\,dt\,.\end{aligned}$$

□

Repeating the previous argument, we have:

$$\|S_f(u_n - u_m)\| \le \frac{1}{\sqrt{\delta_1}}\int_a^b |f(t)|\,dt\,\|u_n - u_m\|_{C(I)}\,.$$

Thus, from the compact embedding of H into $C(I)$, we can affirm that $S_f : H \to H$ is a compact operator.

The proof of next result is analogous to [9, Lemma 9].

Lemma 3.7. *Let $h^* \in H$ be previously defined in* (3.5)*. Then:*

$$\|h^*\| \le \sqrt{\frac{b-a}{\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\{r(t)\}}}\,\|h\|_{C(I)}\,.$$

4 Strongly inverse positive (negative) character of $T[p,c]$ in X

This section is devoted to prove maximum and anti-maximum principles for the problem (1.3)–(1.4). These results generalize those obtained in [9, 10] for $p = 0$. The proofs follow similar arguments to the ones given in such articles. We point out that on them there is no reference to spectral theory.

Moreover, we also generalize Corollary 2.6, in the sense that we allow c to pass throw the given eigenvalues.

The first result ensures the existence of a unique solution of the problem under certain hypotheses and gives sufficient conditions to affirm that the operator (1.3) is strongly inverse positive in X.

Theorem 4.1. *Let $c\,,h \in C(I)$ be such that*

$$\int_a^b c^-(t)\,dt < \delta_1\,,$$

where δ_1 has been defined in (3.6). Then problem (1.3)–(1.4) has a unique classical solution $u \in X$ and there exists $R > 0$ (depending on c and p) such that

$$\|u\|_{C(I)} \le R\ \|h\|_{C(I)}\,.$$

Moreover, if $c(t) \le -\lambda_2^p$, for every $t \in I$, then $T[p,c]$ is strongly inverse positive in X.

Proof. First, we decompose $c(t) = c^+(t) - c^-(t)$. And, we write the problem (1.3)–(1.4) as follows

$$\begin{aligned} &u^{(4)}(t) - p\,u''(t) + c^+(t)\,u(t) = c^-(t)\,u(t) + h(t)\,, \quad t \in I,\\ &u(a) = u(b) = u''(a) = u''(b) = 0\,. \end{aligned}$$

If we denote $r(t) := c^+(t)$ and $f(t) := c^-(t)$, we have that the weak formulation of problem (1.3) is given in the following way

$$u = S_{c^-}\,u + h^*\,, \quad u \in H \tag{4.1}$$

with the scalar product $(\cdot,\cdot)$ previously defined in (3.4).

Using Lemma 3.6 we have

$$\|S_{c^-}\| \le \frac{1}{\delta_1}\int_a^b \left|c^-(t)\right|\,dt = \frac{1}{\delta_1}\int_a^b c^-(t)\,dt < \frac{1}{\delta_1}\,\delta_1 = 1\,.$$

Hence, S_{c^-} is a contractive operator and there exists a unique weak solution $u \in H$. From Proposition 3.5, $u \in X$ is a classical solution of (1.3) in X.

Now, using (4.1) we obtain:

$$\|u\| = \|S_{c^-}\,u + h^*\| \le \|S_{c^-}\|\ \|u\| + \|h^*\|\ ,$$

then

$$\|u\| \le \frac{1}{1-\|S_{c^-}\|}\ \|h^*\|\ .$$

By another hand, using Lemmas 3.4 and 3.7

$$\begin{aligned}\|u\|_{C(I)} &\le \frac{1}{\sqrt{\delta_1}}\ \|u\| \le \frac{1}{\sqrt{\delta_1}(1-\|S_{c^-}\|)}\ \|h^*\| \\ &\le \frac{\sqrt{b-a}}{\sqrt{\delta_1}(1-\|S_{c^-}\|)\sqrt{\left(\frac{\pi}{b-a}\right)^4 + p\ \left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\{c^+(t)\}}}\ \|h\|_{C(I)} \\ &=: R\ \|h\|_{C(I)}\,. \end{aligned} \tag{4.2}$$

Moreover, from Lemma 3.6, we know that

$$R \le \frac{\sqrt{b-a}}{\frac{1}{\sqrt{\delta_1}}\left(\delta_1 - \int_a^b c^-(t)\,dt\right)\sqrt{\left(\frac{\pi}{b-a}\right)^4 + p\ \left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\{c^+(t)\}}}\,. \tag{4.3}$$

The proof behind here is just the same as in the particular case of $p=0$, which is collected in [9, Theorem 4]. □

Now, we introduce a result which also gives us sufficient conditions to ensure the existence of solution of our problem and, moreover, it warrants that the operator $T[p,c]$ is strongly inverse negative in X.

Theorem 4.2. *Let $c\,, h \in C(I)$ be such that*

$$-16\left(\frac{\pi}{b-a}\right)^4 - 4\,p\,\left(\frac{\pi}{b-a}\right)^2 < c_m < -\left(\frac{\pi}{b-a}\right)^4 - p\,\left(\frac{\pi}{b-a}\right)^2,$$

and

$$\int_a^b (c(t)-c_m)\ dt < \delta_1\,\delta_2\,,$$

where δ_1 is defined on (3.6) and

$$\delta_2 = \min\left\{-1-\frac{c_m}{\left(\frac{\pi}{b-a}\right)^4 + p\ \left(\frac{\pi}{b-a}\right)^2},\ 1+\frac{c_m}{16\left(\frac{\pi}{b-a}\right)^4 + 4p\ \left(\frac{\pi}{b-a}\right)^2}\right\}.$$

Then problem (1.3)–(1.4) has a unique classical solution on X and there exists $R>0$ (depending on c and p) such that

$$\|u\|_{C(I)} \le R\ \|h\|_{C(I)}\,.$$

Moreover, if $c_m \ge -\lambda_3^p$, then $T[p,c]$ is strongly inverse negative in X.

Proof. We rewrite the problem (1.3)–(1.4) in the following way

$$\begin{aligned} u^{(4)}(t) - p\,u''(t) + c_m\,u(t) &= (c_m - c(t))\,u(t) + h(t)\,, \quad t\in I\,, \\ u(a) = u(b) = u''(a) &= u''(b) = 0\,. \end{aligned} \tag{4.4}$$

In this case, we consider $r(t)\equiv 0$ and we have that the weak formulation is equivalent to

$$u + S_{c_m}\,u = S_{c_m - c}\,u + h^*\,. \tag{4.5}$$

Since $c_m \in \left(-16\left(\frac{\pi}{b-a}\right)^4 - 4\,p\,\left(\frac{\pi}{b-a}\right)^2, -\left(\frac{\pi}{b-a}\right)^4 - p\,\left(\frac{\pi}{b-a}\right)^2\right)$, we have that $I+S_{c_m}$ is invertible in H . Then we can write

$$u = (I+S_{c_m})^{-1}\,(S_{c_m-c}\,u + h^*)\,. \tag{4.6}$$

Moreover, $\left\|(I+S_{c_m})^{-1}\right\| = \frac{1}{\delta_2}$, where δ_2 is the distance from -1 to the spectrum of S_{c_m}, see [15], i.e.

$$\delta_2 = \min\left\{-1 - \frac{c_m}{\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2}, 1 + \frac{c_m}{16\left(\frac{\pi}{b-a}\right)^4 + 4\,p\,\left(\frac{\pi}{b-a}\right)^2}\right\}.$$

Since $\int_a^b (c(t)-c_m)\,dt < \delta_1\,\delta_2$ and using Lemma 3.6, we have

$$\|S_{c_m-c}\| \le \frac{1}{\delta_1}\int_a^b (c(t)-c_m)\,dt < \frac{\delta_1\,\delta_2}{\delta_1} = \delta_2\,,$$

so,

$$\left\|(I+S_{c_m})^{-1}\,S_{c_m-c}\right\| \le \left\|(I+S_{c_m})^{-1}\right\|\,\|S_{c_m-c}\| < \frac{\delta_2}{\delta_2} = 1\,.$$

So, as in Theorem 4.1, we can use the contractive character of operator $(I+S_{c_m})^{-1}\,S_{c_m-c}$ to ensure that there exists a unique weak solution of (1.3)–(1.4) $u \in H$, from Proposition 3.5, it is also a classical solution $u \in X$.

Now, using (4.6), we have

$$\|u\| \le \left\|(I+S_{c_m})^{-1}\right\|\,\|S_{c_m-c}\,u + h\| = \frac{1}{\delta_2}\,\|S_{c_m-c}\|\,\|u\| + \frac{1}{\delta_2}\,\|h^*\|\,.$$

As a consequence, we deduce that

$$\|u\| \le \frac{\|h^*\|}{\delta_2 - \|S_{c_m-c}\|}\,,$$

so, combining this inequality with Lemmas 3.4 and 3.7, we have

$$\begin{aligned}\|u\|_{C(I)} &\le \frac{1}{\sqrt{\delta_1}}\,\|u\| \le \frac{\|h^*\|}{\sqrt{\delta_1}(\delta_2 - \|S_{c_m-c}\|)}\\ &\le \frac{1}{\sqrt{\delta_1}(\delta_2 - \|S_{c_m-c}\|)}\sqrt{\frac{b-a}{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2}}\;\|h\|_{C(I)}\\ &=: R\;\|h\|_{C(I)}\,.\end{aligned} \tag{4.7}$$

Moreover, from Lemma 3.6

$$R \le \frac{1}{\frac{1}{\sqrt{\delta_1}}(\delta_1\delta_2 - \int_a^b (c(t)-c_m)\,dt)}\sqrt{\frac{b-a}{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2}}\,. \tag{4.8}$$

The proof of the fact that while $-\lambda_3^p \le c_m \le -\left(\frac{\pi}{b-a}\right)^4 - \left(\frac{\pi}{b-a}\right)^2$, $T[p,c]$ is strongly inverse negative is equal to [9, Theorem 5]. □

5 Maximum and anti-maximum principles for problem (1.3)–(1.4) with $h > 0$

In this section, even though we are not able to ensure the strongly inverse positive character of the operator $T[p,c]$ on X, we can ensure that, under the hypothesis that $h(t) > 0$ for every $t \in I$, then problem (1.3)–(1.4) has a unique positive (resp. negative) solution in X. The proofs follow similar steps to the ones given in [10].

Theorem 5.1. *Let $h \in C(I)$ be a function such that $0 < h_m \leq h^m$. Let $c \in C(I)$ be a function that satisfies one of the two following hypothesis:*

(1) $-\left(\frac{\pi}{b-a}\right)^4 - p\,\left(\frac{\pi}{b-a}\right)^2 < c_m \leq 0$ *and* $c^m \leq -\lambda_2^p + \frac{h_m}{h^m}\frac{2}{\pi}\left(\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + c_m\right)$.

(2) $\int_a^b c^-(t)\,dt < \delta_1$, *with* δ_1 *defined on Theorem 4.1, and*

$$c^m \leq -\lambda_2^p + \frac{h_m}{h^m}\frac{1}{\sqrt{\delta_1}}\left(\delta_1 - \int_a^b c^-(t)\,dt\right)\sqrt{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\left\{c^+(t), -\lambda_2^p\right\}}\,,$$

then problem (1.3)–(1.4) has a unique positive solution in X.

Proof. The existence of a unique solution follows from Proposition 3.2 on the first case and from Theorem 4.1 on the second one.

Now, let us see that this solution u is positive on (a,b).

Let us assume that $c^m > -\lambda_2^p$. If $c^m \leq -\lambda_2^p$, we can apply Corollary 2.6 or Theorem 4.1, respectively, to affirm that $T[p,c]$ is strongly inverse positive on X.

Let $d(t) := \min\left\{c(t), -\lambda_2^p\right\}$ be a continuous function such that $c_m \leq d(t) \leq -\lambda_2^p$. We transform the equation (1.3) in the following equivalent one:

$$T[p,d]\,u(t) = h(t) - (c(t) - d(t))\,u(t)\,,$$

and we consider the next recurrence formula

$$T[p,d]\,u_{n+1} = h(t) - (c(t) - d(t))\,u_n\,,\quad n = 0,1,2,\ldots$$

Since, $d(t) \leq -\lambda_2^p$ for every $t \in I$, $T[p,d]$ is a strongly inverse positive operator in X.

We choose $u_0 \equiv 0$ and we have $T[p,d]\,u_1 = h(t)$. Since $T[p,d]$ is strongly inverse positive, $u_1 > 0$ in (a,b), $u_1'(a) > 0$ and $u_1'(b) < 0$.

Now, using that $u_1 \in X$ is the unique solution of $T[p,d]\,u(t) = h(t)$, we deduce that

- if c satisfies (1), we can apply Proposition 3.2 to affirm

$$\|u_1\|_{C([0,1])} \leq \frac{\pi}{2\left(\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + c_m\right)}\,h^m\,;$$

- if c satisfies (2), we look at the proof of the Theorem 4.1 (equations (4.2) and (4.3)) to conclude that

$$\|u_1\|_{C(I)} \leq \frac{h^m}{\frac{1}{\sqrt{\delta_1}}\left(\delta_1 - \int_a^b c^-(t)\,dt\right)\sqrt{\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\left\{c^+(t), -\lambda_2^p\right\}}}\,.$$

Since $c(t) - d(t) \leq c^m + \lambda_2^p$, in both cases, using the hypotheses, we have

$$T[p,d]\,u_2 = h(t) - (c(t) - d(t))\,u_1 \geq h_m - \left(c^m + \lambda_2^p\right)\,\|u_1\|_{C(I)} \geq h_m - \frac{h_m}{h^m}\frac{1}{R}\,h^m\,R = 0\,,$$

where R is defined by:

$$R := \frac{\pi}{2\left(\left(\frac{\pi}{b-a}\right)^4 + \left(\frac{\pi}{b-a}\right)^2 + c_m\right)}$$

if (1) holds, and

$$R := \frac{1}{\frac{1}{\sqrt{\delta_1}}\left(\delta_1 - \int_a^b c^-(t)\,dt\right)\sqrt{\left(\frac{\pi}{b-a}\right)^4 + p\,\left(\frac{\pi}{b-a}\right)^2 + \min_{t\in I}\left\{c^+(t), -\lambda_2^p\right\}}},$$

when (2) is fulfilled.

From here, the proof is equal than in the case that $p = 0$, see [10, Theorem 4] □

Theorem 5.2. *Let $h \in C(I)$ be a function such that $0 < h_m \leq h^m$. Let $c \in C(I)$ be a function that satisfies*

$$\int_a^b (c(t) - c_m)\,dt < \delta_1\,\delta_2\,,$$

where δ_1 and δ_2 have been defined in Theorems 4.1 and 4.2, respectively, and

$$-\lambda_3^p - \frac{h_m}{h^m}\left(\frac{1}{\sqrt{\delta_1}}\left(\delta_1\,\delta_2 - \int_a^b (c(t) - c_m)\,dt\right)\right)\sqrt{\frac{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2}{b-a}} \leq c_m < -\left(\frac{\pi}{b-a}\right)^4 - \left(\frac{\pi}{b-a}\right)^2,$$

then problem (1.3)–(1.4) has a unique negative solution in X.

Proof. The existence of a unique solution, $u \in X$, is given by Theorem 4.2.

To see that $u < 0$ in (a,b), we assume that $c_m < -\lambda_3^p$. On the contrary, if $c_m \geq -\lambda_3^p$, using Theorem 4.2 we know that $T[p,c]$ is strongly inverse negative and the result follows directly.

Let $e(t) := \max\left\{c(t), -\lambda_3^p\right\}$ be a continuous function on I such that $-\lambda_3^p \leq e(t) \leq c^m$, and we write the equation (1.3) in an equivalent form:

$$L[p,e]\,u = h(t) - (c(t) - e(t))\;u\,,$$

which, from Theorem 4.2, is an strongly inverse negative operator on X, and we consider the recurrence formula:

$$L[p,e]\,u_{n+1} = h(t) - (c(t) - e(t))\;u_n\,,\quad n = 0,1,2,\ldots$$

As in the proof of Theorem 5.1, we choose $u_0 \equiv 0$ and we have $T[p,e]\,u_1 = h(t) > 0$, then $u_1 < 0$ on (a,b), $u_1'(a) < 0$ and $u_1'(b) > 0$.

Since, u_1 is the unique solution of problem $T[p,e]\,u = h(t)$ in X, using equations (4.7) and (4.8), we have:

$$\|u_1\|_{C(I)} \leq \frac{h^m}{\frac{1}{\sqrt{\delta_1}}\left(\delta_1\,\delta_2 - \int_a^b (c(t) - c_m)\,dt\right)}\sqrt{\frac{b-a}{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2}}\,.$$

Denoting:

$$R := \frac{1}{\frac{1}{\sqrt{\delta_1}}\left(\delta_1\,\delta_2 - \int_a^b (c(t) - c_m)\,dt\right)}\sqrt{\frac{b-a}{\left(\frac{\pi}{b-a}\right)^4 + p\left(\frac{\pi}{b-a}\right)^2}}\,,$$

and taking into account that $0 \geq c(t) - e(t) \geq c_m + \lambda_3^p$ and that $u_1 < 0$, we conclude:

$$\begin{aligned} T[p,e]u_2 &= h(t) - (c(t) - e(t))\,u_1 \geq h_m - (c_m + \lambda_3^p)u_{1m} \\ &= h_m + (c_m + \lambda_3^p)\,\|u_1\|_{C(I)} \geq h_m - \frac{h_m}{h^m}\cdot\frac{1}{R}h^m R = 0\,. \end{aligned}$$

The rest of the proof follows the same steps as [10, Theorem 5]. □

6 Particular case

To finish this paper, we present an example where we show the applicability of the previous results.

Let us consider a steel bridge of 1 km length (for steel, the Young's module is given by $E = 2.1 \cdot 10^{11}$ Pa$= 2.1 \cdot 10^{14}$ mPa).

We represent the transversal section of the roadway in Figure 6.1. By using the on-line calculator *skyciv.com* [19], we have that the moment of inertia is $I = 1.2575 \cdot 10^{-9}$ km^4. Thus, $E\,I = 264075 \frac{\text{km}^3\text{kg}}{\text{s}^2}$.

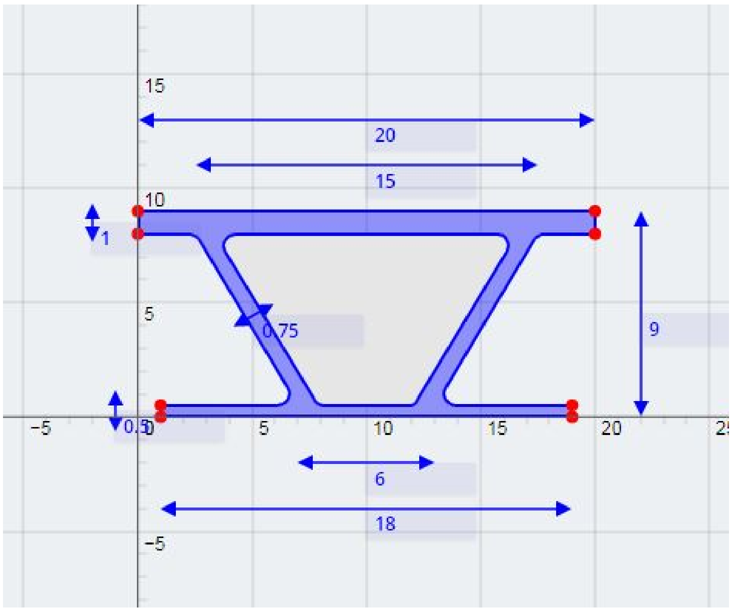

Figure 6.1: Transversal section of the roadway.

Moreover, let us assume that the strength tension of the cables is given by $T = 528150$ kN.

So, for $p = \frac{T}{E\,I} = 2$, $c(t) = \frac{f(t)}{264075}$ and $h(t) = \frac{q(t)}{264075} \geq 0$, we consider the problem which models our suspension bridge as we have mentioned in Remark 1.1:

$$\begin{cases} u^{(4)}(t) - 2\,u''(t) + c(t)\,u(t) = h(t)\,, & t \in [0,1]\,, \\ u(0) = u(1) = u''(0) = u''(1) = 0\,. \end{cases} \tag{6.1}$$

Then, by using the previous results, we can obtain conclusions on the dead load, $q(t)$, to ensure that the vertical oscillations of the roadway will be positive (downwards) for every live load, $f(t)$.

Remark 6.1. Realize that the units which appear in Problem (6.1) are not in the International System of Units (SI) because we use km instead of m.

Thus, the unit of strength will be kN instead of N and the unit of tension will be mPa instead of Pa.

Now, in order to apply the results, let us obtain the different eigenvalues given in Section 2 for $p = 2$.

- Clearly, from (2.3), $\bar{\lambda}_1 = \pi^4 + 2\,\pi^2$ and $\bar{\lambda}_1' = 16\,\pi^4 + 8\,\pi^2$ are the first and second positive eigenvalues of the related operator in X.

- $\bar{\lambda}_2 \approxeq -5.62^4$, the opposed of the least positive solution of (2.4) for $a = 0$, $b = 1$ and $p = 2$, is the biggest negative eigenvalue in X_1 and X_3.
- $\bar{\lambda}_3 \approxeq 4.018^4$, the least positive solution of (2.5) for $a = 0$, $b = 1$ and $p = 2$, is the least positive eigenvalue in U and V.

From Theorem 4.1, we conclude that if $\int_0^1 f^-(t)\,dt < 1056300\,\pi^2$, then the problem (6.1) has a unique solution. If, in addition, $f(t) \le 2.63424 \cdot 10^8 \frac{\text{kN}}{\text{km}}$, the vertical displacement of the bridge is downwards for every live load, $q(t)$.

Moreover, if we consider a constant and positive load, $q > 0$, from Theorem 5.1, we have:

(1) If $-264075\,\pi^2(\pi^2+2) < f_m \le 0$ and

$$f(t) \le 2.63434 \cdot 10^8 + 528150\,\pi \left(\pi^3 + 2 + \frac{f_m}{264075\,\pi^2}\right),$$

or,

(2) If $\int_0^1 f^-(t)\,dt < 1056300\,\pi^2$ and

$$f(t) \le 2.63434 \cdot 10^8 + \left(528150\,\pi + \int_0^1 \frac{f^-(t)}{2\,\pi}\,dt\right) \sqrt{\pi^4 + 2\,\pi^2 + \min_{t\in[0,1]} \left\{\frac{f^-(t)}{264075}, 5.62^4\right\}},$$

then (6.1) has a unique positive solution, which means that the vertical displacement is downwards.

For this example, the negative solution has no physical meaning. However, if we think in an abstract way on Problem (6.1), we can prove the existence of negative solutions as follows.

From Theorem 4.2, if $-2112600\,\pi^2(2\,\pi^2+1) < f_m < -264075\,\pi^2(\pi^2+2)$ and

$$\int_0^1 (f(t) - f_m)dt < 1056300\,\pi^2\delta_2, \tag{6.2}$$

where $\delta_2 = \min\left\{-1 - \frac{f_m}{264075\,\pi^2(\pi^2+2)}, 1 + \frac{c_m}{2112600\,\pi^2(2\,\pi^2+1)}\right\}$, then the problem (6.1) has a unique solution. If, in addition,

$$f(t) \ge -6.8823 \cdot 10^7 (> -2112600\,\pi^2(2\,\pi^2+1) \approxeq -4.32423 \cdot 10^8),$$

the unique solution is negative for every $q(t) \ge 0$.

Finally, if $q > 0$ is constant and (6.2) is fulfilled coupled with:

$$-6.8823 \cdot 10^7 - \sqrt{\pi^2+2}\left(528105\,\pi^2\delta_2 - \int_0^1 \frac{f(t) - f_m}{2}\,dt\right) \le f_m < -264075\,\pi^2(\pi^2+2),$$

then the problem (6.1) has a unique negative solution.

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