Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **62**, 1–9; doi: 10.14232/ejqtde.2017.1.62 http://www.math.u-szeged.hu/ejqtde/

Global well-posedness to the incompressible Navier–Stokes equations with damping

Xin Zhong ✉

School of Mathematics and Statistics, Southwest University
Chongqing 400715, People's Republic of China

Received 9 April 2017, appeared 4 September 2017
Communicated by Maria Alessandra Ragusa

Abstract. We study the Cauchy problem of the 3D incompressible Navier–Stokes equations with nonlinear damping term $\alpha|\mathbf{u}|^{\beta-1}\mathbf{u}$ ($\alpha > 0$ and $\beta \geq 1$). It is shown that the strong solution exists globally for any $\beta \geq 1$.

Keywords: Navier–Stokes equations, global well-posedness, damping.

2010 Mathematics Subject Classification: 35Q35, 35B65, 76N10.

1 Introduction

We are concerned with the following incompressible Navier–Stokes equations with damping in $\mathbb{R}^3$:

$$\begin{cases} \partial_t \mathbf{u} - \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \alpha|\mathbf{u}|^{\beta-1}\mathbf{u} + \nabla P = \mathbf{0}, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \\ \lim\limits_{|x|\to\infty} |\mathbf{u}(t,x)| = 0, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u^1(t,x), u^2(t,x), u^3(t,x))$ is the velocity field, $P(t,x)$ is a scalar pressure. $t \geq 0$ is the time, $x \in \mathbb{R}^3$ is the spatial coordinate. In the damping term, $\alpha > 0$ and $\beta \geq 1$ are two constants. The prescribed function $\mathbf{u}_0(x)$ is the initial velocity field with $\operatorname{div}\mathbf{u}_0 = 0$, while the constant $\mu > 0$ represents the viscosity coefficient of the flow.

This model comes from porous media flow, friction effects, or some dissipative mechanisms, mainly as a limiting system from compressible flows (see [1] for the physical background). The problem (1.1) was studied firstly by Cai and Jiu [1], they showed the existence of a global weak solution for any $\beta \geq 1$ and global strong solutions for $\beta \geq \frac{7}{2}$. Moreover, the uniqueness is shown for any $\frac{7}{2} \leq \beta \leq 5$. In [8], Zhang et al. proved for $\beta > 3$ and $\mathbf{u}_0 \in H^1 \cap L^{\beta+1}$ that the system (1.1) has a global strong solution and the strong solution is unique when $3 < \beta \leq 5$. Later, Zhou [9], Gala and Ragusa [3,4], improved the results in

✉ Email: xzhong1014@amss.ac.cn

[1,8]. He obtained that the strong solution exists globally for $\beta \geq 3$ and $\mathbf{u}_0 \in H^1$. Moreover, regularity criteria for (1.1) is also established for $1 \leq \beta < 3$ as follows: if $\mathbf{u}(t,x)$ satisfies

$$\mathbf{u} \in L^s(0,T;L^\gamma) \quad \text{with } \frac{2}{s} + \frac{3}{\gamma} \leq 1,\ 3 < \gamma < \infty, \tag{1.2}$$

or

$$\nabla\mathbf{u} \in L^{\tilde{s}}(0,T;L^{\tilde{\gamma}}) \quad \text{with } \frac{2}{\tilde{s}} + \frac{3}{\tilde{\gamma}} \leq 1,\ 3 < \tilde{\gamma} < \infty, \tag{1.3}$$

then the solution remains smooth on $[0,T]$. However, the global existence of strong solution to the problem (1.1) for $1 \leq \beta < 3$ is still unknown. In fact, this is the main motivation of this paper.

Before stating our main result, we first explain the notations and conventions used throughout this paper. We denote by

$$\int \cdot\, \mathrm{d}x = \int_{\mathbb{R}^3} \cdot\, \mathrm{d}x.$$

For $1 \leq p \leq \infty$ and integer $k \geq 0$, the standard Sobolev spaces are denoted by:

$$L^p = L^p(\mathbb{R}^3), \qquad H^k = H^{k,2}(\mathbb{R}^3).$$

Now we define precisely what we mean by strong solutions to the problem (1.1).

Definition 1.1 (Strong solutions)**.** A pair $(\mathbf{u}, P)$ is called a strong solution to (1.1) in $\mathbb{R}^3 \times (0,T)$ if (1.1) holds almost everywhere in $\mathbb{R}^3 \times (0,T)$ and

$$\mathbf{u} \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)).$$

Our main result reads as follows.

Theorem 1.2. *Suppose that $1 \leq \beta < 3$ and $\mathbf{u}_0 \in H^1(\mathbb{R}^3)$ with* $\operatorname{div}\mathbf{u}_0 = 0$*. Then there exists an absolute constant ε_0 independent of $\mathbf{u}_0, \mu, \alpha$, and β, such that if*

$$\mu^{-4}\|\mathbf{u}_0\|_{L^2}^2 \left(\|\nabla\mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} \right) \leq \varepsilon_0, \tag{1.4}$$

the problem (1.1) *has a unique global strong solution.*

Remark 1.3. Due to $1 \leq \beta < 3$, we get from Hölder's and Sobolev's inequalities that

$$\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} \leq \|\mathbf{u}_0\|_{L^2}^{\frac{5-\beta}{2}} \|\mathbf{u}_0\|_{L^6}^{\frac{3\beta-3}{2}} \leq C\|\mathbf{u}_0\|_{L^2}^{\frac{5-\beta}{2}} \|\nabla\mathbf{u}_0\|_{L^2}^{\frac{3\beta-3}{2}}.$$

Consequently, for the given initial velocity $\mathbf{u}_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div}\mathbf{u}_0 = 0$, it follows from (1.4) that the problem (1.1) has a unique global strong solution when the viscosity constant μ is sufficiently large or $\|\mathbf{u}_0\|_{L^2}\|\nabla\mathbf{u}_0\|_{L^2}$ is small enough.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. In Section 3, we show the local existence and uniqueness of solutions of the Cauchy problems (1.1). Finally, we give the proof of Theorem 1.2 in Section 4.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

We begin with the following Gronwall's inequality, which plays a central role in proving a priori estimates on strong solutions $(\mathbf{u}, P)$.

Lemma 2.1. *Suppose that h and r are integrable on (a, b) and nonnegative a.e. in (a, b). Further assume that $y \in C[a, b], y' \in L^1(a, b)$, and*

$$y'(t) \le h(t) + r(t)y(t) \quad \text{for a.e. } t \in (a, b).$$

Then

$$y(t) \le \left[y(a) + \int_a^t h(s) \exp\left(-\int_a^s r(\tau) d\tau \right) ds \right] \exp\left(\int_a^t r(s) ds \right), \qquad t \in [a, b].$$

Proof. See [7, pp. 12–13]. □

Next, the following Gagliardo–Nirenberg inequality will be used later.

Lemma 2.2. *Let $1 \le p, q, r \le \infty$, and j, m are arbitrary integers satisfying $0 \le j < m$. Assume that $v \in C_c^\infty(\mathbb{R}^n)$. Then*

$$\|D^j v\|_{L^p} \le C \|v\|_{L^q}^{1-a} \|D^m v\|_{L^r}^a,$$

where

$$-j + \frac{n}{p} = (1-a)\frac{n}{q} + a\left(-m + \frac{n}{r} \right),$$

and

$$a \in \begin{cases} \left[\frac{j}{m}, 1 \right), & \text{if } m - j - \frac{n}{r} \text{ is an nonnegative integer,} \\ \left[\frac{j}{m}, 1 \right], & \text{otherwise.} \end{cases}$$

The constant C depends only on n, m, j, q, r, a.

Proof. See [5, Theorem]. □

Finally, we need the following lemma to obtain the uniform bounds in the next section.

Lemma 2.3. *Let $g \in W^{1,1}(0, T)$ and $k \in L^1(0, T)$ satisfy*

$$\frac{dg}{dt} \le F(g) + k \quad \text{in } [0, T], \quad g(0) \le g_0,$$

where F is bounded on bounded sets from $\mathbb{R}$ into $\mathbb{R}$. Then for every $\varepsilon > 0$, there exists T_ε independent of g such that

$$g(t) \le g_0 + \varepsilon, \qquad \forall t \le T_\varepsilon.$$

Proof. See [6, Lemma 6]. □

3 Local existence and uniqueness of solutions

In this section, we shall prove the following local existence and uniqueness of strong solutions to the Cauchy problem (1.1).

Theorem 3.1. *Suppose that $1 \le \beta < 3$, $\mathbf{u}_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$. Then there exist a small positive time $T_0 > 0$ and a unique strong solution $(\mathbf{u}, P)$ to the Cauchy problem (1.1) in $\mathbb{R}^3 \times (0, T_0]$.*

3.1 A priori estimates

The main goal of this subsection is to derive the following key a priori estimates on $\Phi(t)$ defined by

$$\Phi(t) \triangleq \|\mathbf{u}(t)\|_{H^1}^2 + 1,$$

which are needed for the proof of Theorem 3.1.

Proposition 3.2. *Assume that* $\mathbf{u}_0 \in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} \mathbf{u}_0 = 0$. *Let* $(\mathbf{u}, P)$ *be a solution to the problem* (1.1) *on* $\mathbb{R}^3 \times (0, T]$. *Then there exist a small time* $T_0 \in (0, T]$ *and a positive constant* C *depending only on* μ, α, β, *and* $E_0 \triangleq \|\mathbf{u}_0\|_{H^1} + 1$ *such that*

$$\sup_{0<t\le T_0} \Phi(t) \le C. \tag{3.1}$$

Proof. Multiplying $(1.1)_1$ by $\mathbf{u}$ and integrating (by parts) the resulting equation over $\mathbb{R}^3$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int |\mathbf{u}|^2 dx + \alpha \int |\mathbf{u}|^{\beta+1} dx + \mu \int |\nabla \mathbf{u}|^2 dx = -\int (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} dx. \tag{3.2}$$

By the divergence theorem and $(1.1)_2$, we have

$$-\int (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} dx = -\int u^i \partial_i u^j u^j dx = \int u^i \partial_i u^j u^j dx = \int (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} dx.$$

Thus

$$-\int (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} dx = 0. \tag{3.3}$$

Inserting (3.3) into (3.2) and integrating with respect to t, we get

$$\|\mathbf{u}(t)\|_{L^2}^2 + \alpha \int_0^t \|\mathbf{u}\|_{L^{\beta+1}}^{\beta+1} ds + \mu \int_0^t \|\nabla \mathbf{u}\|_{L^2}^2 ds \le \|\mathbf{u}_0\|_{L^2}^2. \tag{3.4}$$

Multiplying $(1.1)_1$ by $\mathbf{u}_t$ and integrating (by parts) the resulting equation over $\mathbb{R}^3$, we obtain from Cauchy–Schwartz inequality that

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\int |\nabla \mathbf{u}|^2 dx + \frac{\alpha}{\beta+1}\frac{d}{dt}\int |\mathbf{u}|^{\beta+1} dx + \int |\mathbf{u}_t|^2 dx &= -\int (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_t dx \\ &\le \frac{1}{2}\int |\mathbf{u}_t|^2 dx + \frac{1}{2}\int |\mathbf{u}\cdot\nabla\mathbf{u}|^2 dx.\end{aligned}$$

Thus

$$\mu \frac{d}{dt}\int |\nabla \mathbf{u}|^2 dx + \frac{2\alpha}{\beta+1}\frac{d}{dt}\int |\mathbf{u}|^{\beta+1} dx + \int |\mathbf{u}_t|^2 dx \le \int |\mathbf{u}\cdot\nabla\mathbf{u}|^2 dx. \tag{3.5}$$

Similarly, multiplying $(1.1)_1$ by $-\Delta\mathbf{u}$ and integrating the resulting equations over $\mathbb{R}^3$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |\nabla \mathbf{u}|^2 dx + \alpha\beta \int |\mathbf{u}|^{\beta-1}|\nabla\mathbf{u}|^2 dx + \mu \int |\Delta \mathbf{u}|^2 dx &= \int (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta\mathbf{u} dx \\ &\le \frac{\mu}{2}\int |\Delta\mathbf{u}|^2 dx + \frac{1}{2\mu}\int |\mathbf{u}\cdot\nabla\mathbf{u}|^2 dx.\end{aligned}$$

Hence, we get

$$\mu \frac{d}{dt} \int |\nabla \mathbf{u}|^2 dx + 2\mu\alpha\beta \int |\mathbf{u}|^{\beta-1} |\nabla \mathbf{u}|^2 dx + \mu^2 \int |\Delta \mathbf{u}|^2 dx \le \int |\mathbf{u} \cdot \nabla \mathbf{u}|^2 dx, \tag{3.6}$$

which combined with (3.5) yields

$$\begin{aligned} &2\mu \frac{d}{dt} \int |\nabla \mathbf{u}|^2 dx + \frac{2\alpha}{\beta+1} \frac{d}{dt} \int |\mathbf{u}|^{\beta+1} dx + \int |\mathbf{u}_t|^2 dx + 2\mu\alpha\beta \int |\mathbf{u}|^{\beta-1} |\nabla \mathbf{u}|^2 dx + \mu^2 \int |\Delta \mathbf{u}|^2 dx \\ &\quad \le 2 \int |\mathbf{u} \cdot \nabla \mathbf{u}|^2 dx. \end{aligned} \tag{3.7}$$

Applying the Gagliardo–Nirenberg inequality and Sobolev's inequality, the right hand side of (3.7) can be bounded as

$$\begin{aligned} J &\triangleq 2 \int |\mathbf{u} \cdot \nabla \mathbf{u}|^2 dx \\ &\le C \|\mathbf{u}\|_{L^6} \|\Delta \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^2 \\ &\le C \|\Delta \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^3 \\ &\le \frac{\mu^2}{2} \|\Delta \mathbf{u}\|_{L^2}^2 + \frac{C}{\mu^2} \|\nabla \mathbf{u}\|_{L^2}^6. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we deduce that

$$\frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)} \frac{d}{dt} \int |\mathbf{u}|^{\beta+1} dx + \frac{1}{2\mu} \|\mathbf{u}_t\|_{L^2}^2 + \frac{\mu}{4} \|\Delta \mathbf{u}\|_{L^2}^2 \le \frac{C}{\mu^3} \|\nabla \mathbf{u}\|_{L^2}^4 \|\nabla \mathbf{u}\|_{L^2}^2. \tag{3.9}$$

Then integrating (3.9) with respect to t, we have

$$\|\nabla \mathbf{u}(t)\|_{L^2}^2 \le C + C \exp\left(C \int_0^t \Phi(s)^2 ds \right). \tag{3.10}$$

Combining (3.4) and (3.10), we deduce

$$\Phi(t) \le C + C \exp\left(C \int_0^t \Phi(s)^2 ds \right). \tag{3.11}$$

Let us define $\Psi(t) \triangleq \int_0^t \Phi(s)^2 ds$, then we infer from (3.11) that

$$\frac{d}{dt} \Psi(t) \le \left[C + C \exp\left(C \Psi(t) \right) \right]^2.$$

Hence, the desired (3.1) follows from this inequality and Lemma 2.3. This completes the proof of Proposition 3.2. □

3.2 Proof of Theorem 3.1

Since a priori estimates in higher norms have been derived, the local existence of strong solutions can be established by a standard Galerkin method (see for example [2]), and we omit the details. Thus we complete the proof of the existence part of Theorem 3.1.

The uniqueness part of Theorem 3.1 is an immediate consequence of the weak-strong unique result in [9, Theorem 3.1]. This finishes the proof of Theorem 3.1. □

4 Proof of Theorem 1.2

Throughout this section, we denote

$$C_0 \triangleq \|\mathbf{u}_0\|_{L^2}^2.$$

Sometimes we use $C(f)$ to emphasize the dependence on f. Let $(\mathbf{u}, P)$ be the strong solution to the problem (1.1) on $\mathbb{R}^3 \times (0, T)$, then one has the following results.

Lemma 4.1. *For any $t \in (0, T)$, there holds*

$$\|\mathbf{u}(t)\|_{L^2}^2 + \alpha \int_0^t \|\mathbf{u}\|_{L^{\beta+1}}^{\beta+1} ds + \mu \int_0^t \|\nabla \mathbf{u}\|_{L^2}^2 ds \le C_0. \tag{4.1}$$

Proof. This follows from (3.4). □

Lemma 4.2. *There exists an absolute constant C independent of $T, \mathbf{u}_0, \mu, \alpha,$ and β, such that for any $t \in (0, T)$, there holds*

$$\sup_{0\le s\le t} \|\nabla \mathbf{u}\|_{L^2}^2 \le \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)} \|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} + \frac{CC_0}{\mu^4} \sup_{0\le s\le t} \|\nabla \mathbf{u}\|_{L^2}^4. \tag{4.2}$$

Proof. We infer from (3.7) that

$$\begin{aligned}
&2\mu \frac{d}{dt} \int |\nabla \mathbf{u}|^2 dx + \frac{2\alpha}{\beta+1} \frac{d}{dt} \int |\mathbf{u}|^{\beta+1} dx + \int |\mathbf{u}_t|^2 dx + 2\mu\alpha\beta \int |\mathbf{u}|^{\beta-1} |\nabla \mathbf{u}|^2 dx + \mu^2 \int |\Delta \mathbf{u}|^2 dx \\
&\quad \le 2 \int |\mathbf{u} \cdot \nabla \mathbf{u}|^2 dx.
\end{aligned} \tag{4.3}$$

Then we obtain after integrating (4.3) with respect to t that

$$\begin{aligned}
&2\mu \sup_{0\le s\le t} \|\nabla \mathbf{u}\|_{L^2}^2 + \frac{2\alpha}{\beta+1} \sup_{0\le s\le t} \|\mathbf{u}\|_{L^{\beta+1}}^{\beta+1} + \mu^2 \int_0^t \|\Delta \mathbf{u}\|_{L^2}^2 ds \\
&\quad \le 2\mu \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{2\alpha}{\beta+1} \|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} + 2 \int_0^t \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2}^2 ds.
\end{aligned} \tag{4.4}$$

By virtue of the Gagliardo–Nirenberg and Sobolev's inequalities, one finds that

$$\begin{aligned}
2\|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2}^2 &\le C\|\mathbf{u}\|_{L^\infty}^2 \|\nabla \mathbf{u}\|_{L^2}^2 \\
&\le C\|\mathbf{u}\|_{L^6} \|\Delta \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^2 \\
&\le C\|\Delta \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^3 \\
&\le \frac{\mu^2}{2} \|\Delta \mathbf{u}\|_{L^2}^2 + C\mu^{-2} \|\nabla \mathbf{u}\|_{L^2}^6.
\end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.4) and employing (4.1), we derive that

$$\begin{aligned}
&2\mu \sup_{0\le s\le t} \|\nabla \mathbf{u}\|_{L^2}^2 + \frac{2\alpha}{\beta+1} \sup_{0\le s\le t} \|\mathbf{u}\|_{L^{\beta+1}}^{\beta+1} + \frac{\mu^2}{2} \int_0^t \|\Delta \mathbf{u}\|_{L^2}^2 ds \\
&\le 2\mu \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{2\alpha}{\beta+1} \|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} + C\mu^{-2} \int_0^t \|\nabla \mathbf{u}\|_{L^2}^6 ds \\
&\le 2\mu \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{2\alpha}{\beta+1} \|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} + C\mu^{-3} \sup_{0\le s\le t} \|\nabla \mathbf{u}\|_{L^2}^4 \int_0^t \mu \|\nabla \mathbf{u}\|_{L^2}^2 ds \\
&\le 2\mu \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{2\alpha}{\beta+1} \|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} + CC_0\mu^{-3} \sup_{0\le s\le t} \|\nabla \mathbf{u}\|_{L^2}^4.
\end{aligned}$$

This implies the desired (4.2) and finishes the proof of Lemma 4.2. □

Lemma 4.3. *There exists a positive constant* ε_0 *independent of* $T, \mathbf{u}_0, \mu, \alpha$, *and* β, *such that*

$$\sup_{0\le t\le T} \|\nabla \mathbf{u}\|_{L^2}^2 \le 2\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{2\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1}, \tag{4.6}$$

provided that

$$\mu^{-4}C_0\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1}\right) \le \varepsilon_0. \tag{4.7}$$

Proof. Define function $E(t)$ as follows

$$E(t) \triangleq \sup_{0\le s\le t} \|\nabla \mathbf{u}\|_{L^2}^2.$$

In view of the regularity of $\mathbf{u}$, one can easily check that $E(t)$ is a continuous function on $[0, T]$. By (4.2), there is an absolute constant M such that

$$E(t) \le \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} + M\mu^{-4}C_0E^2(t). \tag{4.8}$$

Now suppose that

$$M\mu^{-4}C_0\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1}\right) \le \frac{1}{8}, \tag{4.9}$$

and set

$$T_* \triangleq \max\left\{t\in[0,T] : E(s) \le 4\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{4\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1},\ \forall s\in[0,t]\right\}. \tag{4.10}$$

We claim that

$$T_* = T.$$

Otherwise, we have $T_* \in (0, T)$. By the continuity of $E(t)$, it follows from (4.8)–(4.10) that

$$\begin{aligned}
E(T_*) &\le \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} + M\mu^{-4}C_0E^2(T_*) \\
&\le \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} + M\mu^{-4}C_0E(T_*)\left(4\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{4\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1}\right) \\
&= \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} + 4M\mu^{-4}C_0\left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1}\right)E(T_*) \\
&\le \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} + \frac{1}{2}E(T_*),
\end{aligned}$$

and thus

$$E(T_*) \le 2\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{2\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1}.$$

This contradicts with (4.10).

Choosing $\varepsilon_0 = \frac{1}{8M}$, by virtue of the claim we showed in the above, we derive that

$$E(t) \le 2\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{2\alpha}{\mu(\beta+1)}\|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1}, \qquad 0 < t < T,$$

provided that (4.7) holds true. This gives the desired (4.6) and consequently completes the proof of Lemma 4.3. □

Now, we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let ε_0 be the constant stated in Lemma 4.3 and suppose that the initial velocity $\mathbf{u}_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$, and

$$\mu^{-4} C_0 \left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)} \|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} \right) \le \varepsilon_0.$$

According to Theorem 3.1, there is a unique local strong solution $(\mathbf{u}, P)$ to the system (1.1). Let T^* be the maximal existence time to the solution. We will show that $T^* = \infty$. Suppose, by contradiction, that $T^* < \infty$, then by (1.2), we deduce that for any (s, γ) with $\frac{2}{s} + \frac{3}{\gamma} \le 1$, $3 < \gamma < \infty$,

$$\int_0^{T^*} \|\mathbf{u}\|_{L^\gamma}^s dt = \infty,$$

which combined with the Sobolev inequality $\|\mathbf{u}\|_{L^6} \le C \|\nabla \mathbf{u}\|_{L^2}$ leads to

$$\int_0^{T^*} \|\nabla \mathbf{u}\|_{L^2}^4 dt = \infty. \tag{4.11}$$

By Lemma 4.3, for any $0 < T < T^*$, there holds

$$\sup_{0 \le t \le T} \|\nabla \mathbf{u}\|_{L^2}^2 \le 2 \|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{2\alpha}{\mu(\beta+1)} \|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1},$$

which together with Hölder's and Sobolev's inequalities implies that

$$\begin{aligned}
\int_0^{T^*} \|\nabla \mathbf{u}\|_{L^2}^4 dt &\le 4 \left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)} \|\mathbf{u}_0\|_{L^{\beta+1}}^{\beta+1} \right) T^* \\
&\le 4 \left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \frac{\alpha}{\mu(\beta+1)} \|\mathbf{u}_0\|_{L^2}^{\frac{5-\beta}{2}} \|\mathbf{u}_0\|_{L^6}^{\frac{3\beta-3}{2}} \right) T^* \\
&\le C(\mu, \alpha, \beta) \left(\|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\mathbf{u}_0\|_{L^2}^{\frac{5-\beta}{2}} \|\nabla \mathbf{u}_0\|_{L^2}^{\frac{3\beta-3}{2}} \right) T^* \\
&< +\infty,
\end{aligned}$$

contradicting to (4.11). This contradiction provides us that $T^* = \infty$, and thus we obtain the global strong solution. This finishes the proof of Theorem 1.2. □

Acknowledgements

The author would like to express his gratitude to the reviewers for careful reading and helpful suggestions which led to an improvement of the original manuscript. X. Zhong is supported by Fundamental Research Funds for the Central Universities (No. XDJK2017C050), China Postdoctoral Science Foundation (No. 2017M610579), and the Doctoral Fund of Southwest University (No. SWU116033).

References

[1] X. Cai, Q. Jiu, Weak and strong solutions for the incompressible Navier–Stokes equations with damping, *J. Math. Anal. Appl.* **343**(2008), 799–809. MR2401535; url

[2] P. Constantin, C. Foias, *Navier–Stokes equations*, University of Chicago Press, Chicago, IL, 1988. MR0972259

[3] S. Gala, M. A. Ragusa, A new regularity criterion for the Navier–Stokes equations in terms of the two components of the velocity, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 26, 1–9. MR3498744; url

[4] S. Gala, M. A. Ragusa, On the regularity criterion for the Navier–Stokes equations in terms of one directional derivative, *Asian-Eur. J. Math.* **10**(2017), Art. ID 1750012, 6 pp. MR3627667; url

[5] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa* **13**(1959), 115–162. MR0109940

[6] J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure, *SIAM J. Math. Anal.* **21**(1990), 1093–1117. MR1062395; url

[7] T. Tao, *Nonlinear dispersive equations. Local and global analysis*, American Mathematical Society, Providence, R.I., 2006. MR2233925; url

[8] Z. Zhang, X. Wu, M. Lu, On the uniqueness of strong solution to the incompressible Navier–Stokes equations with damping, *J. Math. Anal. Appl.* **377**(2011), 414–419. MR2754840; url

[9] Y. Zhou, Regularity and uniqueness for the 3D incompressible Navier–Stokes equations with damping, *Appl. Math. Lett.* **25**(2012), 1822–1825. MR2957760; url