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On the non-hyperbolicity of a class of exponential polynomials

Gaspar Mora ✉

Universidad de Alicante, Departamento de Matemáticas, Facultad de Ciencias II
Campus de San Vicente del Raspeig, Ap. 99 E-03080, Alicante, Spain

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Abstract. In this paper we have constructed a class of non-hyperbolic exponential polynomials that contains all the partial sums of the Riemann zeta function. An exponential polynomial has been also defined to illustrate the complexity of the structure of the set defined by the closure of the real projections of its zeros. The sensitivity of this set, when the vector of delays is perturbed, has been analysed. These results have immediate implications in the theory of the neutral differential equations.

Keywords: zeros of exponential polynomials, functional-differential equations, stability.

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1 Introduction

We deal with exponential polynomials (EP for short) defined as

$$h(z,a,r) := 1 - \sum_{k=1}^{N} a_k e^{-zr_k}, \quad N \geq 1, \quad r_k > 0, \quad a_k \in \mathbb{R}, \quad z \in \mathbb{C}.$$

The vectors $a := (a_1, a_2, \dots, a_N)$, $r := (r_1, r_2, \dots, r_N)$ are known as vector of coefficients, and vector of delays, respectively. The closure of the real projections of the zeros of $h(z,a,r)$ is the set

$$R_{h(z,a,r)} = \overline{\{\Re z : h(z,a,r) = 0\}}.$$

In the Example 2.1 of this document we have constructed an EP $h(z,a,r)$ to illustrate, on one hand the complicate nature of $R_{h(z,a,r)}$, and on the other hand how the stability of $R_{h(z,a,r)}$ is modified when the vector of delays is perturbed. The main result of the paper is the Theorem 3.3. There we have defined a class, say $\mathcal{G}$, where any EP of $\mathcal{G}$ is non-hyperbolic, so any EP of $\mathcal{G}$ is not uniformly asymptotically stable (see, for instance, [1, Definitions 5.1, 5.2]), that contains to the family of EP having as components of the vectors of coefficients and delays the numbers

$$a_k := -1, \qquad r_k := \log(k+1), \qquad k = 1, 2, \dots, N.$$

✉ Email: gaspar.mora@ua.es

Therefore $\mathcal{G}$ contains all the partial sums of the Riemann zeta function,

$$\zeta_n(z) := \sum_{k=1}^{n} \frac{1}{k^z}, \qquad n = N+1.$$

The main result is based on the fact that the point 0 belongs to the sets $R_{g(z,a,r)}$ when $g(z,a,r) \in \mathcal{G}$. Indeed, we firstly prove that $0 \in R_{g(z,a,r)}$ when $g(z,a,r)$ is a partial sum $\zeta_n(z)$, $n \geq 2$. Then, by using a result of [1], the above property is also true for all the functions belonging to $\mathcal{G}$. That is, 0 is a point of $R_{g(z,a,r)}$ for any $g(z,a,r) \in \mathcal{G}$. Consequently, the non-hyperbolicity of any EP of $\mathcal{G}$ follows. Regarding the first question, it is important to stress that the vector of delays of $\zeta_n(z)$, for each $n \geq 2$, is $(\log 2, \log 3, \ldots, \log n)$ so its components are not commensurable nor rationally independent (RI for short) for any $n > 3$. Thus, *a priori*, we have a new difficulty to add to the problem of determining the structure of the sets $R_{g(z,a,r)}$ when $g(z,a,r)$ coincides with a partial sum $\zeta_n(z)$ for $n > 3$. Indeed, besides the case that the components of the vector of delays are commensurable (see [1, Lemma 2.4]), mostly of the known results about the zeros of exponential polynomials, apply when the vector of delays has RI components (see, for instance, [1,3,8,9,12–14] and [4, Chapter 3]).

The implications of the results of the present paper to the theory of functional difference equations and neutral functional differential equations are immediate. In effect, as we can see in [1], given the functional difference equation

$$x(t) - \sum_{k=1}^{N} a_k x(t - r_k) = 0, \tag{1.1}$$

for any continuous function $\phi : [-\rho, 0] \to \mathbb{R}$, where $\rho \geq \max\{r_k : 1 \leq k \leq N\}$, there exists a unique solution $x(\phi)$ of (1.1), for $t \geq -\rho$, which satisfies $x(\phi)(t) = \phi(t)$ for all $t \in [-\rho, 0]$. Therefore, by setting

$$(S(t)\phi)(u) := x(\phi)(t+u), \qquad u \in [-\rho, 0],$$

the set of operators

$$S(t) : \mathcal{C}([-\rho, 0], \mathbb{R}) \to \mathcal{C}([-\rho, 0], \mathbb{R}), \qquad t \geq 0,$$

is a strongly continuous semi-group of bounded linear operators on the space $\mathcal{C}([-\rho, 0], \mathbb{R})$ of continuous functions defined on $[-\rho, 0]$ and valued in $\mathbb{R}$. Moreover, if we define

$$\beta := \inf\left\{b : \text{there exists } A > 0 \text{ such that } |S(t)| \leq Ae^{bt}\right\},$$

then

$$\beta = \sup\left\{\Re z : 1 - \textstyle\sum_{k=1}^{N} a_k e^{-zr_k} = 0\right\}.$$

Therefore the location of the zeros of the EP $h(z,a,r)$ gives information about the order β of the semi-group $S(t)$.

The solution operator for the non-homogeneous neutral functional differential equation

$$\frac{d}{dt}\left(x(t) - \sum_{k=1}^{N} a_k x(t-r_k)\right) = b_0 x(t) - \sum_{k=1}^{N} b_k x(t-r_k), \qquad b_0,\, b_k \in \mathbb{R}, \tag{1.2}$$

which usually appears in models of distributed networks [10,11] and in the control of structures through delayed forcing depending on the acceleration [2], can be written as a sum of a completely continuous operator and the operator S described above (see [5] and [1, p. 436]. This gives information about the spectrum of the solution operator (see again [1, p. 436]).

Consequently, as it is noted in [12, Sect. 1], $h(z, a, r) = 0$ determines the essential spectrum of the solution operator of (1.2) (a precise description of this property can be found in [6, Part 3]).

The solutions of equation (1.2) satisfy a spectrum-determined growth condition (see, for instance, [7, Chapter 9, Corollary 3.1]), and the spectrum of the infinitesimal generator determines the stability of the zero solution, which can be sensitive to small changes in the delays (see again [12, Section 1]). Actually,

$$\sup \left\{ \Re z : 1 - \sum_{k=1}^{N} a_k e^{-zr_k} = 0 \right\},$$

could not be continuous with respect to the vector of delays $r := (r_1, \dots, r_N)$, like it is shown in [1, Example 2.1], and the same occurs in our example below. Therefore small changes in the delays can destabilise the equation, which is very important in control problems where usually there are slight delays in the application of the control action.

In the next section, we analyse the sensibility of certain exponential polynomial $h(z, a, r)$ with respect to the vector of delays.

2 Delay perturbations

As in [1], we introduce the notation

$$d(x, B) := \inf \{|x - y| : y \in B\}, \qquad \delta(A, B) := \sup \{d(x, B) : x \in A\},$$

and

$$\delta_H(A, B) := \max \{\delta(A, B), \delta(B, A)\} \quad \text{(Hausdorff distance)},$$

where $x \in \mathbb{R}$, and A, B are bounded subsets of $\mathbb{R}$.

From [1, Lemma 2.5], $R_{h(z,a,r)}$ is always lower semicontinuous in r at any vector r_0, that is,

$$\lim_{r \to r_0} \delta(R_{h(z,a,r_0)}, R_{h(z,a,r)}) = 0. \tag{2.1}$$

However, in general, (2.1) is not true if we substitute δ by the Hausdorff distance δ_H. Indeed, it occurs, for instance in an example of EP, given by Silkowski [15] and analysed in [1, Example 2.1], with a vector of delays r having two commensurable components. In our example, it is defined an EP $h(z, a, r)$ with a vector of delays having three components being RI two of them. As we prove below, the set $R_{h(z,a,r)}$ is the union of an isolated point and a closed interval.

Example 2.1. A study on the sensitivity of the exponential polynomial

$$h(z, a, r) := 1 - \frac{3}{27^z} - \frac{1}{64^z} - \frac{3}{216^z}. \tag{2.2}$$

The EP (2.2) is of the form $1 - \sum_{k=1}^{N} a_k e^{-zr_k}$, with $N = 3$ and

$$a = (3, 1, 3), \qquad r = (\log 27, \log 64, \log 216),$$

as vectors of coefficients and delays, respectively. Since $\log 216 = \log 27 + \frac{1}{2} \log 64$, the components of r are linearly dependent over the rationals, but $\log 27$, $\log 64$ are RI. Consider the sequence of vectors of delays $r_n := (\log 27, \log 64, \frac{1}{n} + \log 216)$, $n = 1, 2, \dots$ First we claim the components of r_n are RI for any fixed $n \geq 1$. Otherwise, for some β_1, $\beta_2 \in \mathbb{Q}$, we can write

$$1 = \beta_1 \log 2 + \beta_2 \log 3.$$

It means that the number e would be an algebraic number, which is a contradiction because e is transcendental. Consequently the claim follows. Now, let us define the sequence of exponential polynomials

$$h(z,a,r_n) := 1 - 3e^{-z\log 27} - e^{-z\log 64} - 3e^{-z(\log 216+\frac{1}{n})}, \qquad n \geq 1. \tag{2.3}$$

Our aim is now to find the sets $R_{h(z,a,r_n)}$. To do it we introduce a new sequence of EP

$$H(z,a,r_n) := h(-z,a,r_n), \qquad n = 1,2,\ldots$$

and then, by using the relation $R_{h(z,a,r_n)} = -R_{H(z,a,r_n)}$, we have enough to find the sets $R_{H(z,a,r_n)}$. By (2.3),

$$H(z,a,r_n) = 1 - 3e^{z\log 27} - e^{z\log 64} - 3e^{z(\log 216+\frac{1}{n})}, \qquad n \geq 1,$$

so, according to [13, Theorem 9], we firstly need to prove that the intermediate equations

$$64^x = 1 + 27^x 3 + 216^x 3e^{\frac{x}{n}}, \tag{2.4}$$

$$27^x 3 = 1 + 64^x + 216^x 3e^{\frac{x}{n}}, \tag{2.5}$$

do not have any real solution. Indeed, since $64^x \leq 1$ for any $x \leq 0$, we get

$$64^x < 1 + 27^x 3 + 216^x 3e^{\frac{x}{n}}, \quad \text{for all } n \geq 1.$$

If $x > 0$, since $64^x < 216^x 3e^{\frac{x}{n}}$, it follows

$$64^x < 1 + 27^x 3 + 216^x 3e^{\frac{x}{n}}, \quad \text{for all } n \geq 1.$$

Therefore the equation (2.4) has no real solution for any $n \geq 1$. Regarding the equation (2.5), by writing $27^x 3$ as 3^{3x+1}, if $x < -1/3$, one has $27^x 3 < 1$. Therefore

$$27^x 3 < 1 + 64^x + 216^x 3e^{\frac{x}{n}}, \quad \text{for all } n \geq 1.$$

If $x > 0$, since $27^x 3 < 216^x 3e^{\frac{x}{n}}$, we get

$$27^x 3 < 1 + 64^x + 216^x 3e^{\frac{x}{n}}, \quad \text{for all } n \geq 1.$$

Consequently (2.5) has no real solution whether $x \in \mathbb{R} \setminus [-1/3, 0]$ for all $n \geq 1$. It only remains to prove that the equation (2.5) has no real solution in the interval $[-1/3, 0]$, for all $n \geq 1$. Indeed, we write (2.5) as

$$27^x (1 - e^{\frac{x}{n}} \cdot 8^x) 3 = 1 + 64^x. \tag{2.6}$$

Then, for any $n \geq 1$, since $e^{\frac{-1}{3n}} \leq e^{\frac{x}{n}}$ for all $x \in [-1/3, 0]$, we have

$$27^x (1 - e^{\frac{x}{n}} \cdot 8^x) 3 \leq 27^x (1 - e^{\frac{-1}{3n}} \cdot 8^x) 3, \quad \text{for all } n \geq 1. \tag{2.7}$$

Now we claim that

$$27^x (1 - e^{\frac{-1}{3n}} \cdot 8^x) 3 < 1 + 64^x, \quad \text{if } x \in [-1/3, 0], \quad \text{for all } n \geq 1. \tag{2.8}$$

Indeed, by means of the change of variable $3x + 1 = u$, and defining $A_n := e^{\frac{-1}{3n}}$, the inequation (2.8) becomes

$$3^u \left(1 - \frac{A_n}{2} 2^u\right) < 1 + \frac{1}{4} 2^{2u}, \quad \text{if } u \in [0,1], \quad \text{for all } n \geq 1. \tag{2.9}$$

For each $n \geq 1$, it is not hard to check that the function

$$f_n(u) := 3^u \left(1 - \frac{A_n}{2} 2^u\right), \quad u \in [0,1],$$

attains its maximum value at a point, say u_n, satisfying $\frac{A_n}{2} 2^{u_n} = \frac{\log 3}{\log 6}$. Therefore we have

$$u_n = \frac{\log\left(\frac{\log 9}{A_n \log 6}\right)}{\log 2} = \frac{\frac{1}{3n} + \log\left(\frac{\log 9}{\log 6}\right)}{\log 2} \leq \frac{\frac{1}{3} + \log\left(\frac{\log 9}{\log 6}\right)}{\log 2}, \quad \text{for all } n \geq 1.$$

Then, by putting $c := \frac{\frac{1}{3}+\log\left(\frac{\log 9}{\log 6}\right)}{\log 2}$ and taking into account that $c \approx 0.7752037030$, it follows

$$f_n(u_n) = 3^{u_n}\left(1 - \frac{\log 3}{\log 6}\right) \leq 3^c \left(1 - \frac{\log 3}{\log 6}\right) \approx 0.9065920697, \quad \text{for all } n \geq 1,$$

which means that

$$f_n(u) < 1, \quad \text{for any } u \in [0,1], \quad \text{for all } n \geq 1.$$

On the other hand, the function

$$g(u) := 1 + \frac{1}{4} 2^{2u}, \qquad u \in [0,1],$$

is strictly increasing on $[0,1]$ and then its minimum value is $g(0) = \frac{5}{4}$. This proves (2.9), so (2.8) follows too. Then the claim follows. Therefore, taking into account (2.7), the equation (2.6) has no real solution in the interval $[-1/3, 0]$ for any $n \geq 1$. Consequently, (2.5) does not have any real solution. This implies, by virtue of [13, Theorem 9], that $R_{H(z,a,r_n)}$ has no gap for all $n \geq 1$ and then

$$R_{H(z,a,r_n)} = [\alpha_n, \beta_n] \quad \text{for all } n \geq 1,$$

where α_n, β_n, by [13, (4.1)], are the unique real solutions of the equations

$$1 = 27^x 3 + 64^x + 216^x 3 e^{\frac{x}{n}}, \qquad 216^x 3 e^{\frac{x}{n}} = 1 + 27^x 3 + 64^x,$$

respectively. Since for large enough n, α_n, β_n can be roughly taken as -0.47 and 0.22, respectively, the set $R_{H(z,a,r_n)} \approx [-0.47, 0.22]$. Therefore, noticing $h(z,a,r_n) = H(-z,a,r_n)$, for n sufficiently large

$$R_{h(z,a,r_n)} \approx [-0.22, 0.47]. \tag{2.10}$$

On the other hand, (2.2) can be written as a product, that is,

$$h(z,a,r) = h(z,a_1,r_1)h(z,a_2,r_2),$$

where

$$h(z,a_1,r_1) := 1 + \frac{1}{8^z}, \qquad h(z,a_2,r_2) := 1 - \frac{1}{8^z} - \frac{3}{27^z}.$$

Therefore $R_{h(z,a,r)} = R_{h(z,a_1,r_1)} \cup R_{h(z,a_2,r_2)}$. Since all the zeros of $h(z,a_1,r_1)$ are imaginary, $R_{h(z,a_1,r_1)} = \{0\}$. Regarding the set $R_{h(z,a_2,r_2)}$, we claim that

$$R_{h(z,a_2,r_2)} = [\alpha, \beta], \quad \text{for some } 0 < \alpha < \beta.$$

Indeed, as done earlier, we define $H(z,a_2,r_2) := h(-z,a_2,r_2)$. Then

$$H(z,a_2,r_2) = 1 - 8^z - 27^z 3 = 1 - e^{z \log 8} - 3e^{z \log 27},$$

so the vector $r_2 = (\log 8, \log 27)$ has RI components. It is immediate that the equation $8^x = 1 + 27^x 3$ has no real solution. Then, because of [13, Theorem 9], it follows that $R_{H(z,a_2,r_2)} = [\alpha', \beta']$ for some real numbers α', β'. From [13, (4.1)], α', β' are the unique real solutions of the equations

$$1 = 8^x + 27^x 3, \qquad 27^x 3 = 1 + 8^x,$$

respectively. An easy computation gives us the approximate values $\alpha' \approx -0.47$ and $\beta' \approx -0.17$, so $R_{H(z,a_2,r_2)} \approx [-0.47, -0.17]$. Noticing $H(z, a_2, r_2) := h(-z, a_2, r_2)$, we have

$$R_{h(z,a_2,r_2)} \approx [0.17, 0.47], \tag{2.11}$$

as claimed. Consequently

$$R_{h(z,a,r)} = R_{h(z,a_1,r_1)} \cup R_{h(z,a_2,r_2)} \approx \{0\} \cup [0.17, 0.47]. \tag{2.12}$$

However, from (2.10), $R_{h(z,a,r_n)} \approx [-0.22, 0.47]$ for n sufficiently large and then, taking into account (2.12), it is evident that

$$\lim_{n\to\infty} \delta_H(R_{h(z,a,r_n)}, R_{h(z,a,r)}) \neq 0.$$

This means that the continuity of $R_{h(z,a,r)}$ with respect to the Hausdorff metric at the vector $r = (\log 27, \log 64, \log 216)$ fails. In other words, the perturbation of the vector of delays has destabilised the closure of the set of the real part of the zeros of $h(z, a, r)$. However, $h(z, a, r)$ (see (2.2)) can be also written of the form

$$g(z, a, s) := 1 - \sum_{k=1}^{N} a_k e^{-z\gamma_k \cdot s}, \tag{2.13}$$

where

$$N = 3, \quad a = (3, 1, 3), \quad s = (\log 2, \log 3), \quad \gamma_1 = (0, 3), \quad \gamma_2 = (6, 0), \quad \gamma_3 = (3, 3),$$

and $\gamma_k \cdot s$ is the inner product in $\mathbb{R}^2$ of γ_k by s, for $k = 1, 2, 3$. Since the components of s are RI, by designating by $t = (t_1, t_2)$ a generic vector of delays, we could apply [1, Theorem 2.2] (*"If s is a fixed vector of $(\mathbb{R}_*^+)^M$, where $M > 1$ is an integer, $\mathbb{R}_*^+ := (0, +\infty)$ and the components of s are RI, then $R_{g(z,a,t)} \to R_{g(z,a,s)}$ in the Hausdorff metric as $t \to s$"*) obtaining

$$\lim_{t\to s} \delta_H(R_{g(z,a,t)}, R_{g(z,a,s)}) = 0,$$

which means that the perturbation of the vector of delays s does not destabilise the set $R_{g(z,a,s)}$. It is important to stress that, in spite of $R_{h(z,a,r)} = R_{g(z,a,r)}$, the vectors r, s, used in each representation of (2.2), are distinct: $r \in \mathbb{R}^3$ and it has components rationally dependent whereas $s \in \mathbb{R}^2$ with RI components.

3 The non-hyperbolicity of the class $\mathcal{G}$

Let us first recall that an EP $h(z, a, r)$ is said to be *hyperbolic at a vector* r_0 if $0 \notin R_{h(z,a,r_0)}$ (see, for instance, [1, Definition 5.1]). In this section we prove the non-hyperbolicity of a class of EP, denoted by $\mathcal{G}$, that contains all the partial sums of the Riemann zeta function. The functions of $\mathcal{G}$ will be of the form (2.13) and all them will be denoted as $g(z,a,s)$. Therefore we begin by expressing the partial sums $\zeta_n(z)$ under the form (2.13) with the peculiarity that the vectors of delays will have RI components. To do it, for each $n \geq 2$, it is enough to introduce vectors, say Γ_j, for $j = 1, \ldots, n-1$, as follows.

Given the integer $n \geq 2$, let k_n be the number of primes not exceeding n. For each $j = 1, \ldots, n-1$, the vector $\Gamma_j := (\Gamma_{jl})_{l=1,2,\ldots,k_n}$ of $\mathbb{R}^{k_n}$ has components Γ_{jl} defined as the unique non-negative integers such that each $j+1$ is expressed of a unique form as

$$j + 1 = 2^{\Gamma_{j1}} 3^{\Gamma_{j2}} \cdots p_{k_n}^{\Gamma_{jk_n}},$$

by virtue of the fundamental theorem of arithmetic. Then,

$$\Gamma_1 = (1, 0, \ldots, 0), \qquad \Gamma_2 = (0, 1, \ldots, 0), \qquad \Gamma_3 = (2, 0, \ldots, 0), \ldots \tag{3.1}$$

and so on. The vectors Γ_j allow us to write $\zeta_n(z)$ under the form

$$\zeta_n(z) = 1 + \sum_{j=1}^{n-1} e^{-z\Gamma_j \cdot p},$$

where $\Gamma_j \cdot p$ denotes the usual inner product in $\mathbb{R}^{k_n}$ of Γ_j by the vector p defined as

$$p := (\log 2, \log 3, \log 5, \ldots, \log p_{k_n}). \tag{3.2}$$

That is, the components of p are the logarithms of all the prime numbers not exceeding n, so p_{k_n} denotes the last prime such that $p_{k_n} \leq n$, and, consequently, p has RI components. We define the class

$$\mathcal{G} := \left\{ g(z,a,r) = 1 + \textstyle\sum_{j=1}^{n-1} e^{-z\Gamma_j \cdot r},\ n \geq 2,\ r \in (\mathbb{R}_*^+)^{k_n} \text{ with RI components} \right\},$$

where $\mathbb{R}_*^+ := (0, +\infty)$. Therefore $\mathcal{G}$ contains all the partial sums of the Riemann zeta function $\zeta_n(z)$, $n \geq 2$.

In order to facilitate the reading of the manuscript, we state two results that will be used below.

Theorem 3.1 (Sufficiency of [3, Theorem 2]). *Let $G_n(z) := \zeta_n(-z)$, $n \geq 2$, and $G_n^*(z) := G_n(z) - p_{k_n}^z$, where p_{k_n} is the last prime not exceeding $n > 2$. If a real number c is such that the vertical line $x = c$ intersects the level curve $|G_n^*(z)| = p_{k_n}^c$, then $c \in R_{G_n(z)} := \overline{\{\Re z : G_n(z) = 0\}}$.*

Corollary 3.2 ([1, Corollary 3.1]). *Let $g(z,a,r)$ be the EP defined as*

$$g(z,a,r) := a_0 + \sum_{j=1}^{N} a_j e^{-z\gamma_j \cdot r}, \tag{3.3}$$

where N, M are positive integers, $a_j \in \mathbb{R}$ for all $0 \leq j \leq N$, the vectors $\gamma_j \in \mathbb{R}^M$, $1 \leq j \leq N$, have components which are non-negative integers and r is a vector of $\mathbb{R}^M$ with positive components. Then, the following statements are equivalent:

(i) $0 \in R_{g(z,a,r_0)}$ *for some r_0 with RI components;*

(ii) $0 \in R_{g(z,a,r)}$ *for all r with RI components.*

The main result of the paper is the following.

Theorem 3.3. *Any EP of $\mathcal{G}$ is non-hyperbolic.*

Proof. We first claim that $0 \in R_{\zeta_n(z)} := \overline{\{\Re z : \zeta_n(z) = 0\}}$ for all $n \geq 2$. Indeed, for $n = 2$, $\zeta_2(z) = 1 + \frac{1}{2^z}$, whose zeros, by direct computation, are $z_k = \frac{(2k+1)\pi i}{\log 2}$, $k \in \mathbb{Z}$. Then $R_{\zeta_2(z)} = \{0\}$ and the claim follows for $n = 2$. Assume $n > 2$. We consider the analytic variety $|G_n^*(z)| = 1$, corresponding to the value $c = 0$ in Theorem 3.1. Noticing the definition of $G_n^*(z)$ (see again Theorem 3.1), we have

$$G_n^*(z) = \sum_{m=1,\, m\neq p_{k_n}}^{n} m^z = \sum_{m=1,\, m\neq p_{k_n}}^{n} m^x m^{iy} = \sum_{m=1,\, m\neq p_{k_n}}^{n} m^x e^{iy\log m}$$

$$= \sum_{m=1,\, m\neq p_{k_n}}^{n} m^x(\cos(y\log m) + i\sin(y\log m)).$$

Then, by taking the square of the modulus of $G_n^*(z)$, and from the elementary trigonometric formulas

$$\cos^2 A + \sin^2 A = 1, \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B, \qquad A, B \in \mathbb{R},$$

the Cartesian equation of $|G_n^*(z)| = 1$ is

$$\sum_{m=1,\, m\neq p_{k_n}}^{n} m^{2x} + 2\cdot 1^x \sum_{m=2, m\neq p_{k_n}}^{n} m^x \cos\left(y\log\left(\tfrac{m}{1}\right)\right)$$
$$+ \ldots + 2(n-1)^x \sum_{m=n, m\neq p_{k_n}}^{n} m^x \cos\left(y\log\left(\tfrac{m}{n-1}\right)\right) = 1. \quad (3.4)$$

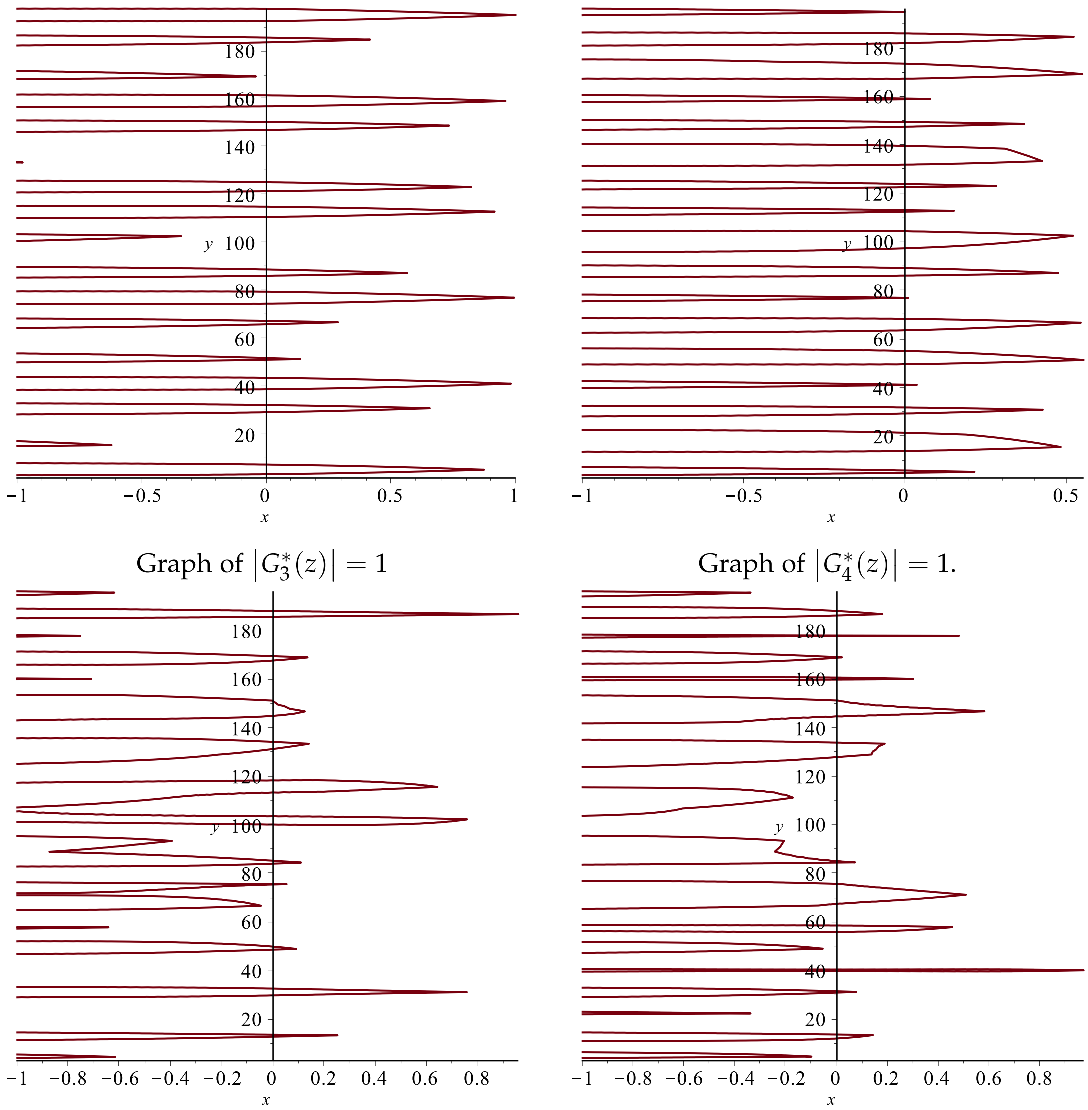
We can see in Figure 3.1 the graph of the analytic variety $|G_n^*(z)| = 1$ for some values of n, when $\Re(z) \in [-1, 1]$ and $\Im(z) \in [0, 200]$.

In equation (3.4), by canceling 1 and dividing by 2^{x+1}, for $x \to -\infty$, we can see that the horizontal lines

$$y = (2k+1)\frac{\pi}{2\log 2}, \qquad k \in \mathbb{Z}, \quad (3.5)$$

are asymptotes of the infinitely many arc-connected components of the analytic variety $|G_n^*(z)| = 1$. On the other hand, since the left-hand side of (3.4) tends to ∞ when $x \to \infty$, the range of x is upper bounded by a number, say $b_{n,0}^+$. Therefore the domain of the variable x is the interval $(-\infty, b_{n,0}^+)$, eventually could be $(-\infty, b_{n,0}^+]$ (in the first case the line $x = b_{n,0}^+$ is an asymptote, in the second one $x = b_{n,0}^+$ intersects $|G_n^*(z)| = 1$). Anyway, given $x \in (-\infty, b_{n,0}^+)$ there is at least a point of $|G_n^*(z)| = 1$ with abscissa x, and if $x > b_{n,0}^+$ there is no point of the variety. If there is a point of $|G_n^*(z)| = 1$ with abscissa $b_{n,0}^+$ we then say that $b_{n,0}^+$ is *accessible*. Consequently, each arc-connected component of $|G_n^*(z)| = 1$ is an open curve on the left, directed to $-\infty$, bounded by two asymptotes of the family (3.5), and closed on the right, where it is bounded by the line $x = b_{n,0}^+$. Thus if z_0 is a zero of $G_n^*(z)$ with $\Re z_0 \geq 0$, whose existence is assured by [4, Chapter 3, Theorem 3.19], z_0 is an interior point of $|G_n^*(z)| = 1$ because $|G_n^*(z_0)| = 0 < 1$ (the variety $|G_n^*(z)| = 1$ produces two open sets of interior and exterior points: $\{z : |G_n^*(z)| < 1\}$ and $\{z : |G_n^*(z)| > 1\}$, respectively). Therefore there exists a point z_1 of $|G_n^*(z)| = 1$ with $\Re z_0 < \Re z_1$. It means that $b_{n,0}^+ \geq \Re z_1 > \Re z_0 \geq 0$, so

$$b_{n,0}^+ > 0, \quad \text{for all } n > 2. \quad (3.6)$$



Graph of $|G_3^*(z)| = 1$ Graph of $|G_4^*(z)| = 1$.

Graph of $|G_5^*(z)| = 1$ Graph of $|G_6^*(z)| = 1$.

Figure 3.1: Graph of the analytic variety $|G_n^*(z)| = 1$, for some values of n.

Then, since the domain of x in $|G_n^*(z)| = 1$ is the interval $(-\infty, b_{n,0}^+)$ (eventually it could be $(-\infty, b_{n,0}^+]$), from (3.6), it follows that the line $x = 0$ intersects $|G_n^*(z)| = 1$. Consequently, by Theorem 3.1, $0 \in R_{G_n(z)}$, for all $n > 2$. Since $G_n(z) := \zeta_n(-z)$, it follows that $R_{\zeta_n(z)} = -R_{G_n(z)}$ and then $0 \in R_{\zeta_n(z)}$, for all $n > 2$. As we have just proved that $0 \in R_{\zeta_2(z)}$, we get

$$0 \in R_{\zeta_n(z)} \quad \text{for all } n \geq 2. \tag{3.7}$$

Since, for each $n \geq 2$, $\zeta_n(z)$ is an EP of the form (3.3) with $N = n - 1$, $M = k_n$, $\gamma_j = \Gamma_j$ defined in (3.1), $r = p$ defined in (3.2) and $a_j = 1$ for all $0 \leq j \leq N$, by applying Corollary 3.2, we deduce that $0 \in R_{g(z,a,r)}$ for any $g(z, a, r) \in \mathcal{G}$. Consequently, from [1, Definition 5.1], it follows that any EP of $\mathcal{G}$ is non-hyperbolic. The proof is now complete. $\square$

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