Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **68**, 1–12; https://doi.org/10.14232/ejqtde.2017.1.68 www.math.u-szeged.hu/ejqtde/

The number of zeros of Abelian integrals for a perturbation of a hyper-elliptic Hamiltonian system with a nilpotent center and a cuspidal loop

Ali Atabaigi ✉

Razi University, Kermanshah, 67149-67346, Iran

Received 1 December 2015, appeared 26 October 2017
Communicated by John R. Graef

Abstract. In this paper we consider the number of isolated zeros of Abelian integrals associated to the perturbed system $\dot{x} = y$, $\dot{y} = -x^3(x-1)^2 + \varepsilon(\alpha + \beta x + \gamma x^3)y$, where $\varepsilon > 0$ is small and α, β, $\gamma \in \mathbb{R}$. The unperturbed system has a cuspidal loop and a nilpotent center. It is proved that three is the upper bound for the number of isolated zeros of Abelian integrals, and there exists some α, β and γ such that the Abelian integrals could have three zeros which means three limit cycles could bifurcate from the nilpotent center and period annulus. The proof is based on a Chebyshev criterion for Abelian integrals, asymptotic behaviors of Abelian integrals and some techniques from polynomial algebra.

Keywords: hyper-elliptic Hamiltonian system, Abelian integrals, Liénard equation, limit cycle.

2010 Mathematics Subject Classification: 34C07, 34C08, 37G15, 34M50.

1 Introduction

Consider the following polynomial Liénard equations of type (m, n), i.e.

$$\dot{x} = y, \qquad \dot{y} = -g(x) - \varepsilon f(x)y, \tag{1.1}$$

where $\varepsilon > 0$ is small, $f(x)$ and $g(x)$ are polynomials of degree m and n, respectively. For $\varepsilon = 0$ the above system reduces to

$$\dot{x} = y, \qquad \dot{y} = -g(x), \tag{1.2}$$

which is a Hamiltonian system with the Hamiltonian function

$$H(x, y) = \frac{1}{2}y^2 + G(x), \qquad G(x) = \int_0^x g(s)ds,$$

where G is a polynomial in x of degree $n+1$. The level curves are rational for $n = 0, 1$, elliptic for $n = 2, 3$ and hyper-elliptic for $n \geq 4$. Suppose the unperturbed system (1.2) has a family of

✉ Email: aliaelmi@gmail.com (a.atabeigi@razi.ac.ir)

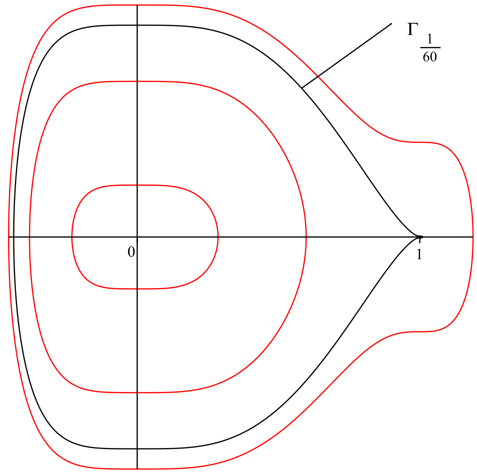

Figure 1.1: The level curves of $H(x,y) = h$.

periodic orbits Γ_h defined by $H(x,y) = h$. Then for the associated perturbed system (1.1) there exists an Abelian integral or so-called first-order Melnikov function of the following form:

$$I(h,\delta) = -\oint_{\Gamma_h} f(x)ydx.$$

According to the Poincaré–Pontryagin–Andronov theorem, it is known that the total number of isolated zeros (counting their multiplicities) of the Abelian integral $I(h,\delta)$ is an upper bound for the number of limit cycles bifurcated from the periodic annulus of the unperturbed system (1.2). The second part of Hilbert's 16th problem for system (1.1), asks for an upper bound for the number of limit cycles in terms of m and n and their relative distributions.

Some of the previous works that focus on hyper-elliptic case are as follows: Asheghi et al. in [1] studied the Chebyshev's property of a 3-dimensional vector space of Abelian integrals by integrating the 1-form $(\alpha_0 + \alpha_1 x + \alpha_2 x^2)ydx$ over the compact level curves of a hyper-elliptic Hamiltonian of degree 7. Wang in [8] investigated the number of different phase portraits of hyper-elliptic Hamiltonian system of degree five and obtained 40 different phase portraits. Kazemi et al. in [5] studied the zeros of Abelian integrals obtained by integrating 1-form $(a + bx + cx^3 + x^4)ydx$ over the compact level curves of $(1.1)|_{\varepsilon=0}$ with $g(x) = -x(x-\frac{1}{2})(x-1)^3$ and proved that the upper bound of the number of isolated zeros of Abelian integral is three. Wang in [7] studied the zeros of Abelian integrals obtained by integrating 1-form $(\alpha + \beta x + \gamma x^2)ydx$ over the compact level curves of the hyper-elliptic Hamiltonian of degree five $H(x,y) = \frac{y^2}{2} + \frac{1}{4}x^4 - \frac{1}{5}x^5$ and proved that the upper bound of the number of isolated zeros of Abelian integral is two.

In this paper, we provide a study of the zeros of Abelian integrals obtained by integrating the 1-form $y(\alpha + \beta x + \gamma x^3)dx$ over the compact level curves of the following Hamiltonian system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^3(x-1)^2, \end{cases} \tag{1.3}$$

which has a nilpotent center at $(0,0)$, a cusp point at $(1,0)$ and a cuspidal loop $\Gamma_{\frac{1}{60}}$ (see Fig. 1.1).

Inside and outside $\Gamma_{\frac{1}{60}}$ all orbits Γ_h are closed,

$$\Gamma_h : \{(x,y)|H(x,y) = h\},$$

with $H(x,y) = \frac{1}{2}y^2 + A(x)$ where $A(x) = \frac{1}{4}x^4 - \frac{2}{5}x^5 + \frac{1}{6}x^6$, and $h \in (0, \frac{1}{60}) \cup (\frac{1}{60}, +\infty)$. When $h \to 0^+$, Γ_h shrinks t the center $(0,0)$, and when $h \to \frac{1}{60}$, then Γ_h tends to $\Gamma_{\frac{1}{60}}$ from the inside and outside (see Fig. 1.1).
We intend to study a perturbation of (1.3) of the from:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^3(x-1)^2 + \varepsilon(\alpha + \beta x + \gamma x^3)y, \end{cases} \tag{1.4}$$

which is a Liénard system of type $(3,5)$. Here, $0 < |\varepsilon| \ll 1$, α, β and γ are arbitrary real parameters. According to classification in [8] system (1.3) is the case (14) ($\alpha > 0$).

Associated to the given perturbation we have the following Abelian integral

$$I(h,\delta) = \oint_{\Gamma_h} (\alpha + \beta x + \gamma x^3) y dx = \alpha I_0(h) + \beta I_1(h) + \gamma I_3(h), \qquad h \in \left(0, \tfrac{1}{60}\right) \tag{1.5}$$

where $I_k(h) = \oint_{\Gamma_h} x^k y dx,\ k = 0, 1, 3$, and $\delta = (\alpha, \beta, \gamma)$ is the parameter vector.

A limit cycle is an isolated periodic orbit in the set of periodic orbits. The Abelian integral $I(h,\delta)$ is a suitable tool for studying limit cycles of system (1.4). We recall that a limit cycle of system (1.4) corresponds to an isolated zero of the Abelian integral $I(h,\delta)$.

The rest of the paper is presented in two sections. In Section 2, we show that the Abelian integral (1.5) has the Chebyshev property with accuracy one for $h \in (0, \frac{1}{60})$. Hence, by the criterion introduced in [6] we get that the upper bound for the number of isolated zeros of $I(h,\delta)$ in any compact subinterval of $(0, \frac{1}{60})$ is three. In Section 3, we calculate the asymptotic expansions of Abelian integral $I(h,\delta)$ near the nilpotent center and the cuspidal loop, using that we get that there exists some α, β, γ such that the Abelian integral $I(h,\delta)$ can have three isolated zeros in $(0, \frac{1}{6})$, which means that the system (1.4) can have three limit cycles.

2 Bifurcation of limit cycles from the period annulus

In this section we study the maximum number of limit cycles which bifurcate from the period annulus of system (1.3) for $h \in (0, \frac{1}{60})$. We use an algebraic criterion given in [6] to study the related Abelian integral $I(h,\delta)$ of system (1.4). But first we give the following definitions.

Definition 2.1. The base functions $\{I_i(h,\delta),\ i = 0, 2, \ldots, n-1\}$ in the Abelian integral $I(h,\delta)$ is said to be a Chebyshev system with accuracy k, if the number of zeros of any nontrivial linear combination

$$\alpha_0 I_0(h) + \alpha_1 I_1(h) + \cdots + \alpha_{n-1} I_{n-1}(h),$$

counted with multiplicity is at most $n + k - 1$.

Definition 2.2. Let $f_0, f_1, \ldots, f_{k-1}$ be analytic functions on an open interval L of $\mathbb{R}$. The *continuous Wronskian* of $(f_0, f_1, \ldots, f_{k-1})$ at $x \in L$ is

$$W[f_0, f_1, \ldots, f_{k-1}](x) = \det\left(f_j^{(i)}(x)\right)_{0 \le i,j \le k-1} = \begin{vmatrix} f_0(x) & \cdots & f_{k-1}(x) \\ f_0'(x) & \cdots & f_{k-1}'(x) \\ & \vdots & \\ f_0^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

Consider a Hamiltonian function with the following special form

$$H(x,y) = A(x) + B(x)y^{2m},$$

which is analytic in some open subset of the plane and has a local minimum at the origin. We fix that $H(0,0) = 0$, then $(0,0)$ is the center critical point of the associated vector field. So, there exists a period annulus $\mathcal{P}$ foliated by the set of ovals $\Gamma_h \subset \{H(x,y) = h\}$ surrounding the origin. The period annulus can be parametrized by the energy levels $h \in (0, h_0)$ for some $h_0 \in (0, +\infty]$. In the sequel, we denote the projection of $\mathcal{P}$ on the x-axis by (x_ℓ, x_r). It is easy to verify that, under the above assumptions, $xA'(x) > 0$ for any $x \in (x_\ell, x_r) \setminus \{0\}$ and $B(0) > 0$. Thus by implicit function theorem, there exists a smooth unique analytic function $z(x)$ with $x_\ell < z(x) < 0$ such that $A(x) = A(z(x))$ for $0 < x < x_r$. Theorem A in [6] is as follows.

Theorem 2.3. *Let us consider the Abelian integrals*

$$I_i(h,\delta) = \int_{\Gamma_h} f_i(x) y^{2s-1} dx, \qquad i = 0, 1, \ldots, n-1,$$

where, for each $h \in (0, h_0)$, Γ_h is the oval surrounding the origin inside the level curve $\{A(x) + B(x)y^{2m} = h\}$. f_i are analytic functions on (x_l, x_r) and $s \in \mathbb{N}$. For $i = 0, 1, \ldots, n-1$, define

$$\omega_i(x) := \frac{f_i(x)}{A'(x)(B(x))^{\frac{2s-1}{2m}}}, \qquad m_i(x) = \omega_i(x) - \omega_i(z(x)).$$

If the following conditions are verified:

(i) $W[m_0, m_1, m_2, \ldots, m_i](x)$ is non-vanishing on $(0, x_r)$ for $i = 0, 1, \ldots, n-2$,

(ii) $W[m_0, m_1, m_2, \ldots, m_{n-1}](x)$ has k zeros on $(0, x_r)$ counted with multiplicities, and

(iii) $s > m(n + k - 2)$,

then the base functions $\{I_i(h,\delta),\ i = 0, 1, \ldots, n-1\}$ form a Chebyshev system with accuracy k on $(0, h_0)$. Here $W[m_0, m_1, m_2, \ldots, m_k](x)$ denotes the continuous Wronskian of the functions $\{m_0, m_1, m_2, \ldots, m_k\}$ at $x \in (0, x_r)$.

The efficiency of Theorem 2.3 comes from the fact that finding an upper bound for the number of zeros of Abelian integrals $I(h,\delta)$ follows just from some pure algebraic computations.

In the sequel, we will apply Theorem 2.3 to show that the Abelian integral (1.5)

$$I(h,\delta) = \oint_{\Gamma_h} (\alpha + \beta x + \gamma x^3) y dx = \alpha I_0(h) + \beta I_1(h) + \gamma I_3(h),$$

has Chebyshev property with accuracy one in the interval $(0, \frac{1}{60})$. Using the notation in Theorem 2.3 we have

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{4}x^4 - \frac{2}{5}x^5 + \frac{1}{6}x^6 =: \frac{1}{2}y^2 + A(x),$$

and $s = 1,\ n = 3$. The period annulus inside $\Gamma_{\frac{1}{6}}$ is foliated by the level curves

$$\Gamma_h = \left\{(x,y) \in \mathbb{R}^2 \mid H(x,y) = h,\ 0 < h < \tfrac{1}{60}\right\},$$

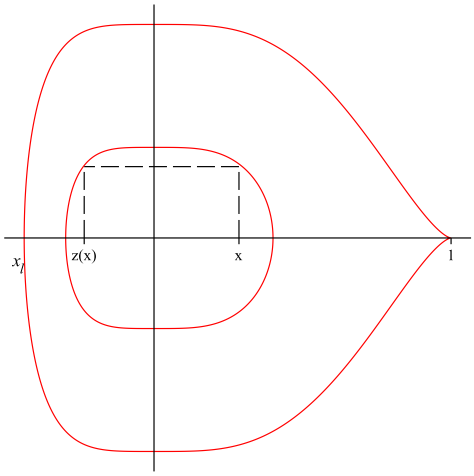

Figure 2.1: The involution of x and $z(x)$ defined by $A(x) = A(z(x))$.

whose images on the x-axis is an open interval $(x_l, 1)$, where

$$x_l = -\frac{1}{10}\sqrt[3]{28 + 10\sqrt{10}} + \frac{3}{5\sqrt[3]{28 + 10\sqrt{10}}} - \frac{1}{5} \approx -0.4370801776,$$

is the intersection point of $\Gamma_{\frac{1}{60}}$ with negative half x-axis, which satisfies

$$A(x) - A(1) = \frac{1}{60}\left(10\,x^3 + 6\,x^2 + 3\,x + 1\right)(x-1)^3 = 0. \tag{2.1}$$

It is easy to check that $xA'(x) > 0$ for all $x \in (x_l, 1) \setminus \{0\}$. Therefore, there exists an analytic function $z(x)$ with $x_l < z(x) < 0$ such that $A(x) = A(z(x))$ as $0 < x < 1$, see Fig. 2.1.

To apply Theorem 2.3, we notice that $I_k(h) = \oint_{\Gamma_h} x^k y dx$, hence $m = 1$, $n = 3$ and $s = 1$, so that the hypothesis (c) ($s > m(n + k - 2)$) of Theorem 2.3 is not fulfilled (note that, as we shall see, in our case $k = 1$). However it is possible to overcome this problem using the following result (see Lemma 4.1 in [2]), and obtain new Abelian integrals for which the corresponding s is large enough to verify the inequality.

Lemma 2.4. *Let γ_h be an oval inside the level curve $\{A(x) + B(x)y^2 = h\}$ and we consider a function F such that F/A' is analytic at $x = 0$. Then, for any $k \in \mathbb{N}$,*

$$\int_{\gamma_h} F(x) y^{k-2} dx = \int_{\gamma_h} G(x) y^k dx,$$

where $G(x) = \frac{2}{k}\left(\frac{BF}{A'}\right)'(x) - \left(\frac{B'F}{A'}\right)(x)$.

Here we have to promote the power s to three such that the condition $s > n - 1$ holds. On the oval Γ_h we have

$$\begin{aligned} I_i(h, \delta) &= \frac{1}{h}\oint_{\Gamma_h}\left(A(x) + \frac{y^2}{2}\right) x^i y dx \\ &= \frac{1}{2h}\left(\oint_{\Gamma_h} 2x^i A(x) y dx + \oint_{\Gamma_h} x^i y^3 dx\right), \qquad i = 0, 1, 2, 3. \end{aligned} \tag{2.2}$$

Now we apply Lemma 2.4 with $k = 3$ and $F(x) = 2x^i A(x)$ to the first integral above to get

$$\oint_{\Gamma_h} 2x^i A(x) y dx = \oint_{\Gamma_h} G_i(x) y^3 dx,$$

where $G_i(x) = \frac{d}{3dx}(\frac{2x^i A(x)}{A'(x)}) = \frac{g_i}{90(x-1)^3}$, and

$$g_i = 10(1+i)\,x^{i+3} - (30+34i)\,x^{i+2} + (33+39i)\,x^{i+1} - (15+15i)\,x^i.$$

By (2.2) we obtain

$$\begin{aligned} I_i(h,\delta) &= \frac{1}{2h}\oint_{\Gamma_h}\Big(x^i + G_i(x)\Big)\,y^3 dx = \frac{1}{4h^2}\oint_{\Gamma_h}(2A(x)+y^2)(x^i+G_i(x))y^3 dx \\ &= \frac{1}{4h^2}\left(\oint_{\Gamma_h} 2(x^i+G_i(x))A(x)y^3 dx + \oint_{\Gamma_h}(x^i+G_i(x))y^5 dx\right). \end{aligned} \tag{2.3}$$

Again we apply Lemma 2.4 with $k = 5$ and $F(x) = 2(x^i + G_i(x))A(x)$ to the first integral above to get

$$\oint_{\Gamma_h} 2(x^i+G_i(x))A(x)y^3 dx = \oint_{\Gamma_h}\tilde{G}_i(x)y^5 dx,$$

where $\tilde{G}_i(x) = \frac{d}{5dx}(\frac{2(x^i+G_i(x))A(x)}{A'(x)}) = \frac{\tilde{g}_i}{13500\,(x-1)^6}$, and

$$\begin{aligned} \tilde{g}_i = &\left(1000 + 1100\,i + 100\,i^2\right)x^{i+6} - \left(6000 + 7000\,i + 680\,i^2\right)x^{i+5} \\ &+ \left(15270 + 18674\,i + 1936\,i^2\right)x^{i+4} + \left(-21276 - 26784\,i - 2952\,i^2\right)x^{i+3} \\ &+ \left(21840\,i + 17271 + 2541\,i^2\right)x^{i+2} - \left(7830 + 9630\,i + 1170\,i^2\right)x^{i+1} \\ &+ \left(1575 + 1800\,i + 225\,i^2\right)x^i. \end{aligned}$$

From (2.3) we obtain

$$4h^2 I_i(h,\delta) = \oint_{\Gamma_h} f_i(x)y^5 dx \equiv \tilde{I}_i(h,\delta), \tag{2.4}$$

where $f_i(x) = x^i + G_i(x) + \tilde{G}_i(x)$. It is clear that $\{\tilde{I}_0, \tilde{I}_1, \tilde{I}_3\}$ is a Chebyshev system with accuracy one on $(0, \frac{1}{60})$ if and only if $\{I_0, I_1, I_3\}$ is a Chebyshev system with accuracy one on the same interval. As $s = 3$, $n = 3$ and the condition $s > n-1$ is satisfied, now, we can apply Theorem 2.3 to study the Chebyshev property of $\{\tilde{I}_0, \tilde{I}_1, \tilde{I}_3\}$ in the interval $(0, \frac{1}{60})$. For $i = 0, 1, 3$, let

$$\omega_i(x) = \frac{f_i(x)}{A'(x)}, \qquad m_i(x) = \omega_i(x) - \omega_i(z(x)).$$

We know that for $x_l < z < 0 < x < 1$, $A(x) = A(z(x))$ is equivalent to $\frac{1}{60}(x-z)q(x,z) = 0$, where

$$\begin{aligned} q(x,z) = &\,10\,x^5 - 24\,x^4 + 10\,zx^4 + 15\,x^3 - 24\,zx^3 + 10\,z^2x^3 \\ &+ 15\,zx^2 - 24\,z^2x^2 + 10\,x^2z^3 + 15\,z^2x - 24\,xz^3 \\ &+ 10\,xz^4 + 15\,z^3 - 24\,z^4 + 10\,z^5. \end{aligned}$$

So, $A(x) = A(z(x))$ is equivalent to $q(x, z(x)) = 0$. We need to prove that $\{m_0, m_1, m_3\}$ satisfy hypothesis (i)–(iii) of Theorem 2.3 with $k = 1$. To do this we prove the following lemma.

Lemma 2.5.

(i) $W[m_0](x) \neq 0$ *for all* $x \in (0,1)$*;*

(ii) $W[m_0, m_1](x) \neq 0$ *for all* $x \in (0,1)$*;*

(iii) $W[m_0, m_1, m_3](x)$ *has one zero in* $(0,1)$ *counted with multiplicities.*

Proof. Using Maple 15 we compute the above three Wronskians. We find out that

$$W[m_0](x) = \frac{(x-z)\,p_1(x,z)}{13500\,(x-1)^8\,x^3\,(z-1)^8\,z^3},$$
$$W[m_0,m_1](x) = \frac{(x-z)^3\,p_2(x,z)}{91125000\,(z-1)^{16}\,z^6\,(x-1)^{16}\,x^6\Delta(x,z)},$$
$$W[m_0,m_1,m_3](x) = \frac{(x-z)^4\,p_3(x,z)}{307546875000\,(z-1)^{23}\,z^8\,(x-1)^{23}\,x^8\,(\Delta(x,z))^3},$$

where $z = z(x)$ is defined by the equation $q(x,z(x)) = 0,\ x_l < z < 0 < x < 1$, implicitly. And $p_1(x,z), p_2(x,z)$ and $p_3(x,z)$ are polynomials in (x,z). Moreover the resultant between

$$\Delta(x,z) = 10\,x^4 - 24\,x^3 + 20\,zx^3 + 15\,x^2 - 48\,zx^2 + 30\,z^2x^2 + 30\,zx - 72\,z^2x + 40\,xz^3 + 45\,z^2 - 96\,z^3 + 50\,z^4,$$

and $q(x,z)$ with respect to z is

$$12960000\,x^6\,(x-1)^2\,(10\,x^3+6\,x^2+3\,x+1)^2\,(10\,x^2-24\,x+15)^3,$$

which has no roots in the interval $(0,1)$. This proves that the functions $W[m_0, m_1]$ and $W[m_0, m_1, m_3]$ are well defined.

Therefore by Theorem 2.3, we need to check if $p_i(x,z) \neq 0$ for all (x,z) satisfying $q(x,z) = 0$ and $x_l < z < 0 < x < 1$, for $i = 1,2,3$.

Using Maple 15, we calculate the resultant $r_1(x)$ between $p_1(x,z)$ and $q(x,z)$ with respect to z, to obtain

$$r_1(x) = x^6(x-1)^{14}k_1(x),$$

where $k_1(x)$ is a polynomial in x of degree 80. Applying Sturm's Theorem, we know that $k_1(x) \neq 0$ in $(0,1)$, thus there exist no points $(x,z) \in (0,1) \times (x_l, 0)$ such that satisfy $p_1(x,z) = 0$ and $q(x,z) = 0$, simultaneously, which implies that $W[m_0](x) \neq 0$ for $x \in (0,1)$.

Next we consider the resultant $r_2(x)$ between $p_2(x,z)$ and $q(x,z)$ with respect to z, and obtain

$$r_2(x) = x^{16}(x-1)^{28}k_2(x),$$

where $k_1(x)$ is a polynomial in x of degree 128. Applying Sturm's theorem, we get that $k_2(x) \neq 0$ in $(0,1)$, which implies that $W[m_0, m_1](x) \neq 0$ for $x \in (0,1)$.

Finally, we compute the resultant $r_3(x)$ between $p_3(x,z)$ and $q(x,z)$ to get

$$r_3(x) = x^{34}(x-1)^{44}k_3(x),$$

where $k_3(x)$ is a polynomial in x of degree 198. By applying Sturm's theorem, we get that $k_3(x)$ has only one root in $(0,1)$. Using the algorithm of real root isolation to $k_3(x)$ and using function realroot with accuracy $\frac{1}{1000}$ in Maple 15, we get that the root is located in the closed subinterval $[\frac{57}{64}, \frac{913}{1024}]$.

On the other hand, we get the resultant $r_3(z)$ between $p_3(x,z)$ and $q(x,z)$ with respect to x as follows

$$r_3(z) = z^{34}(z-1)^{44}k_3(z),$$

where $k_3(z)$ is a polynomial in z of degree 198. By applying Sturm's theorem, we get that $k_3(z)$ has only one root in $(x_l, 0)$. Again, using algorithm of real root isolation to $k_3(z)$ and using function realroot with accuracy $\frac{1}{1000}$ in Maple 15, we can prove that the root is in the closed subinterval $[-\frac{223}{512}, -\frac{445}{1024}]$.

Therefore, there exists a unique $x^* \in (0,1)$, with $\frac{57}{64} \le x^* \le \frac{913}{1024}$, so that $W[m_0, m_1, m_3](x^*) = 0$. We will now show that x^* is simple root. Let us denote $W[m_0, m_1, m_3](x)$ by $W_0(x, z(x))$ and calculate its derivative, that is

$$\frac{dW_0}{dx} = \frac{\partial W_0}{\partial x} + \frac{\partial W_0}{\partial z} \times \frac{dz}{dx} = \frac{(x-z)^3 p_4(x,z)}{(x-1)^{16}x^9(z-1)^{16}z^9(\Delta(x,z))^5},$$

where $p_4(x,z)$ is a polynomial in (x,z). The resultant with respect to z between $q(x,z)$ and $p_4(x,z)$ is

$$r_4(x) = x^{52}(x-1)^{66}\left(10x^2 - 24x + 15\right)^3\left(10x^3 + 6x^2 + 3x + 1\right)^4 k_4(x),$$

where $k_4(x)$ is a polynomial of degree 220 in x. By applying Sturm's theorem, we find that $k_4(x)$ has no zeros in $[\frac{57}{64}, \frac{913}{1024}]$. Therefore, $W[m_0, m_1, m_3](x)$ has exactly one simple root in the interval $(0,1)$. This ends the proof. □

So far we have proved the following.

Theorem 2.6. *The collection $\{I_0(h), I_1(h), I_3(h)\}$ is a Chebyshev system with accuracy one on the interval $\left(0, \frac{1}{60}\right)$. Hence, if the Abelian integral $I(h,\delta)$ is not identical to zero, then for all values of parameters (α, β, γ) it has at most three zeros, counting multiplicities, in any compact subinterval of $\left(0, \frac{1}{60}\right)$, and the number of limit cycles bifurcating from the periodic annulus is at most three.*

3 Asymptotic expansions of Abelian integral $I(h,\delta)$

In this section we study the asymptotic expansion of Abelian integral $I(h,\delta)$ at $h = 0$ and $h = \frac{1}{6}$, respectively. Using these asymptotic expansions we prove the following theorem.

Theorem 3.1. *Consider the Abelian integral (1.5). If $\gamma \neq 0$, Then $I(h,\delta)$ can have*

(i) two zeros near the $h = 0$ for some (α, β);

(ii) three zeros near the $h = \frac{1}{6}$ for some (α, β);

(iii) three zeros in $(0, \frac{1}{6})$ for some (α, β).

Proof. **i)** We follow the idea given in [4] and [7] and adapt our notations according to [7]. To obtain the asymptotic expansion of Abelian integral $I(h,\delta)$ as $h \to 0^+$, we compute $I(h,\delta)$ near the nilpotent center $(0,0)$. Let us denote the intersection points of the oval Γ_h with the negative and positive half x-axis by $x_l(h)$ and $x_r(h)$, respectively. We know that $A(x) = \frac{1}{4}x^4\left(1 - \frac{8}{5}x + \frac{2}{3}x^2\right)$, introduce

$$A(x) = u^4, \quad \text{or} \quad x\sqrt[4]{1 - \frac{8}{5}x + \frac{2}{3}x^2} = \sqrt{2}u. \tag{3.1}$$

Let

$$\psi(x,u) = x\sqrt[4]{1 - \frac{8}{5}x + \frac{2}{3}x^2} - \sqrt{2}u. \tag{3.2}$$

Applying the implicit function theorem to $\psi(x,u) = 0$ at $(x,u) = (0,0)$, we know that there exists a unique analytic function $x = \varphi(u)$ and a small positive number $0 < \rho \ll 1$ such that $\psi(\varphi(u),u) = 0$ for $|u| < \rho$. It can be checked that $\varphi(u)$ has the following expression:

$$\varphi(u) = \sqrt{2}u + \frac{4}{5}u^2 + \frac{59}{75}\sqrt{2}u^3 + \frac{736}{375}u^4 + \frac{3433}{1250}\sqrt{2}u^5 + o(u^5). \tag{3.3}$$

Using transformation (3.1) the Abelian integral (1.5) is written to

$$\begin{aligned} I(h,\delta) &= 2\sqrt{2}\int_{x_l(h)}^{x_r(h)} (\alpha + \beta x + \gamma x^3)\sqrt{h - A(x)}dx \\ &= 2\sqrt{2}\int_{-h^{\frac{1}{4}}}^{h^{\frac{1}{4}}} (\alpha + \beta x + \gamma x^3)|_{x=\varphi(u)}\sqrt{h - u^4}\varphi'(u)du \\ &= 2\sqrt{2}\sum_{k=0}^{+\infty} a_k(\delta)E_k, \end{aligned} \tag{3.4}$$

where the first five coefficients are as follows

$$\begin{aligned} a_0 &= \sqrt{2}\alpha, \\ a_1 &= \frac{8}{5}\alpha + 2\beta, \\ a_2 &= \sqrt{2}\left(\frac{59}{25}\alpha + \frac{12}{5}\beta\right), \\ a_3 &= \frac{2944}{375}\alpha + \frac{568}{75}\beta + 4\gamma, \\ a_4 &= \frac{1}{250}\sqrt{2}\left(3433\alpha + 3240\beta + 2000\gamma\right), \end{aligned}$$

and

$$E_k = \int_{-h^{\frac{1}{4}}}^{h^{\frac{1}{4}}} u^k\sqrt{h - u^4}du, \qquad k = 0,1,2,\dots \tag{3.5}$$

Let $u = h^{\frac{1}{4}}s$, then we have

$$E_k = h^{\frac{3+k}{4}}\int_{-1}^{1} s^k\sqrt{1 - s^4}ds = h^{\frac{3+k}{4}}\int_0^1 [1 + (-1)^k]s^k\sqrt{1 - s^4}ds.$$

Using the change of variable $s^4 = \tau$, we obtain

$$E_k = \frac{1 + (-1)^k}{4}h^{\frac{3+k}{4}}\int_0^1 \tau^{\frac{k-3}{4}}(1 - \tau)^{\frac{1}{2}}d\tau = \frac{1 + (-1)^k}{4}h^{\frac{3+k}{4}}\beta\left(\frac{k+1}{4}, \frac{3}{2}\right),$$

where $\beta(a,b)$ is the following Beta-function for $a>0,\ b>0$,

$$\beta(a,b)=\int_0^1 \tau^{a-1}(1-\tau)^{b-1}d\tau.$$

It is obvious that $E_k=0$ when k is odd. From the relation between Beta-function and Gamma-function,

$$\beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

we compute the three elliptic integrals E_k for $k=0,2,4$ as follows

$$E_0=\frac{\sqrt{2}}{3\Gamma^2(\frac{3}{4})}\pi^{\frac{3}{2}}h^{\frac{3}{4}},\qquad E_2=\frac{2\sqrt{2}}{5\sqrt{\pi}}\Gamma^2(\tfrac{3}{4})h^{\frac{5}{4}},\qquad E_4=\frac{\sqrt{2}}{21\Gamma^2(\frac{3}{4})}\pi^{\frac{3}{2}}h^{\frac{7}{4}},\tag{3.6}$$

where $\Gamma(a)=\int_0^{+\infty}x^{a-1}e^{-x}dx$ is the Gamma-function and $\Gamma(\frac{3}{4})\approx 1.225416702$.

Substituting (3.6) into (3.4) we get

$$I(h,\delta)=h^{\frac{3}{4}}[c_0(\delta)+c_2(\delta)h^{\frac{1}{2}}+c_4(\delta)h+\cdots],\tag{3.7}$$

where

$$\begin{aligned}
c_0(\delta)&=\frac{4\sqrt{2}}{3\Gamma^2(\frac{3}{4})}\pi^{\frac{3}{2}}\alpha,\\
c_2(\delta)&=\frac{8\sqrt{2}}{5\sqrt{\pi}}\Gamma^2\left(\frac{3}{4}\right)\left(\frac{59}{25}\,\alpha+\frac{12}{5}\,\beta\right),\\
c_4(\delta)&=\frac{2\sqrt{2}}{2625\Gamma^2(\frac{3}{4})}\pi^{\frac{3}{2}}\left(3433\,\alpha+3240\,\beta+2000\gamma\right).
\end{aligned}$$

When $c_0(\delta)=c_2(\delta)=0$, we get a unique solution

$$(\alpha^*,\beta^*)=(0,0).\tag{3.8}$$

Substituting (3.8) into $c_4(\delta)$, we get $c_4(\alpha^*,\beta^*,\gamma)=\frac{4000\sqrt{2}}{2625\Gamma^2(\frac{3}{4})}\pi^{\frac{3}{2}}\gamma$ and as $\gamma\neq 0$,

$$\operatorname{rank}\frac{\partial(c_0,c_2)}{\partial(\alpha,\beta,\gamma)}(\alpha^*,\beta^*,\gamma)=2,$$

hence by Theorem 1.3 in [4], we get that, if $\gamma\neq 0$, then the Abelian integral (1.5) can have at least two zeros for some (α,β) near (α^*,β^*). Therefore the perturbed system (1.4) could have at least two limit cycles near the origin, for ε sufficiently small.

ii) For the expansion of $I(h,\delta)$ near $h=\frac{1}{6}$, we introduce the change of variables $x=X+1,\ y=Y$ and still denote X,Y by x,y, respectively. Therefore, system (1.4) becomes

$$\begin{aligned}
\dot{x}&=y,\\
\dot{y}&=-x^2(x+1)^3+\varepsilon\, q(x,y),
\end{aligned}\tag{3.9}$$

where

$$q(x,y)=\left(\alpha+\beta+\gamma+(\beta+3\,\gamma)x+3\,\gamma\, x^2+\gamma\, x^3\right)y.$$

For $\varepsilon = 0$ system (3.9) has the Hamiltonian

$$\bar{H}(x,y) = \frac{1}{2}y^2 + \frac{1}{6}x^6 + \frac{3}{5}x^5 + \frac{3}{4}x^4 + \frac{1}{3}x^3.$$

This system has a cusp critical point at $(0,0)$ and a cuspidal loop Γ_0. There are two families of periodic orbits of (3.9) near Γ_0 given by $\Gamma_h^{\pm} : \bar{H}(x,y) = h,\ 0 < \pm h \ll 1$. Then the two corresponding Abelian integrals are given by

$$I_{\pm}(h) = \oint_{\Gamma_h^{\pm}} q dx.$$

It was proved in [3] that

$$I_{-}(h) = \bar{c}_0 + B_{00}\bar{c}_1|h|^{\frac{5}{6}} + (\bar{c}_2 + t\bar{c}_1)h + B_{10}\bar{c}_3|h|^{\frac{7}{6}} - \frac{1}{11}B_{00}\bar{c}_4|h|^{\frac{11}{6}} + O(h^2), \tag{3.10}$$

for $0 < -h \ll 1$, and

$$I_{+}(h) = \bar{c}_0 - B_{00}^*\bar{c}_1|h|^{\frac{5}{6}} + (\bar{c}_2 + t^*\bar{c}_1)h - B_{10}^*\bar{c}_3|h|^{\frac{7}{6}} - \frac{1}{11}B_{00}^*\bar{c}_4|h|^{\frac{11}{6}} + O(h^2), \tag{3.11}$$

for $0 < h \ll 1$, where $B_{00} > 0,\ B_{00}^* < 0,\ B_{10} > 0$ and $B_{10}^* < 0$ are some constants and $t, t^* \in \mathbb{R}$ and

$$\begin{aligned}
\bar{c}_0 &= \oint_{\Gamma_0} q dx = -0.3893984390\,\alpha - 0.06298450076\,\beta - 0.02583505112\,\gamma,\\
\bar{c}_1 &= 2\sqrt{2}\,(\alpha + \beta + \gamma)\,\sqrt[3]{3},\\
\bar{c}_2 &= \oint_{\Gamma_0} (q_y - \alpha - \beta - \gamma)dt = 21.18278454\,\gamma + 14.41919762\,\beta,\\
\bar{c}_3 &= -\sqrt{2}3^{2/3}\,(-2\,\beta + 3\,\alpha)\,,\\
\bar{c}_4 &= -18\,\sqrt[3]{3}\left[\frac{1419}{320}\sqrt{2}\alpha - \frac{73}{20}\sqrt{2}\beta - \frac{2}{3}\sqrt{2}\gamma\right].
\end{aligned} \tag{3.12}$$

By (3.12), when $\bar{c}_0 = \bar{c}_1 = 0$, we get a unique solution

$$(\bar{\alpha}, \bar{\beta}) = (0.113810855\gamma, -1.113810856\gamma). \tag{3.13}$$

Substituting (3.13) into $\bar{c}_2$, we get $c_2(\bar{\alpha}, \bar{\beta}, \gamma) = 5.12252570\gamma$ and as $\gamma \neq 0$,

$$\operatorname{rank} \frac{\partial(\bar{c}_0, \bar{c}_1)}{\partial(\alpha, \beta, \gamma)}(\bar{\alpha}, \bar{\beta}, \gamma) = 2.$$

Thus, by Theorem 3.2 of [3] we know that there exists some (α, β) close to $(\bar{\alpha}, \bar{\beta})$ for which the Abelian integral (1.5) could have 3 zeros near $h = \frac{1}{6}$. This means that the perturbed system (1.4) could have 3 limit cycles near cuspidal loop, for ε sufficiently small.

There are two different distributions of 3 limit cycles near cuspidal loop: $(1,2)$ and $(2,1)$, where (i, j) denotes that i limit cycles are outside the loop while j limit cycles inside the loop.

iii) From the calculations made along the proof of parts i) and ii) when $c_0(\delta) = c_2(\delta) = 0$, we get a unique solution $(\alpha^*, \beta^*) = (0,0)$. Substituting this into $c_4(\delta)$ and $\bar{c}_0$, we get $c_4(\alpha^*, \beta^*, \gamma) = \frac{4000\sqrt{2}}{2625\Gamma^2(\frac{3}{4})}\pi^{\frac{3}{2}}\gamma$ and $\bar{c}_0(\alpha^*, \beta^*, \gamma) = -0.02583505112\gamma$. Hence as $\gamma \neq 0$,

$$\operatorname{rank} \frac{\partial(c_0, c_2)}{\partial(\alpha, \beta, \gamma)}(\alpha^*, \beta^*, \gamma) = 2, \qquad c_4(\alpha^*, \beta^*, \gamma)\bar{c}_0(\alpha^*, \beta^*, \gamma) < 0,$$

thus using Theorem 2.1 given in [10] we get the result. This ends the proof of the theorem. □

References

[1] R. Asheghi, A. Bakhshalizadeh, The Chebyshev's property of certain hyperelliptic integrals of the first kind, *Chaos Solitons Fractals* **78**(2015), 162–175. MR3394236; https://doi.org/10.1016/j.chaos.2015.07.020

[2] M. Grau, F. Mañosas, J. Villadelprat, A Chebyshev criterion for Abelian integrals, *Trans. Amer. Math. Soc.* **363**(2011), No. 1, 109–129. MR2719674; https://doi.org/10.1090/S0002-9947-2010-05007-X

[3] M. Han, H. Zang, J. Yang, Limit cycle bifurcations by perturbing a cuspidal loop in a Hamiltonian system, *J. Differential Equations* **246**(2009), No. 1, 129–163. MR2467018; https://doi.org/10.1016/j.jde.2008.06.039

[4] J. Jiang, M. Han, Melnikov function and limit cycle bifurcation from a nilpotent center, *Bull. Sci. Math.* **132**(2008), No. 3, 182–193. MR2406824; https://doi.org/10.1016/j.bulsci.2006.11.006

[5] R. Kazemi, H. Z. Zangeneh, Bifurcation of limit cycles in small perturbations of a hyper-elliptic Hamiltonian system with two nilpotent saddles, *J. Appl. Anal. Comput.* **2**(2012), No. 4, 395–413. MR3006401; https://doi.org/10.11948/2012029

[6] F. Mañosas, J. Villadelprat, Bounding the number of zeros of certain Abelian integrals, *J. Differential Equations.* **251**(2011), No. 6, 1656–1669. MR2813894; https://doi.org/10.1016/j.jde.2011.05.026

[7] J. Wang, Estimate of the number of zeros of Abelian integrals for a perturbation of hyper-elliptic Hamiltonian system with nilpotent center, *Chaos Solitons Fractals* **45**(2012), No. 9, 1140–1146. MR2979224; https://doi.org/10.1016/j.chaos.2012.05.011

[8] Y. Xiong, Bifurcation of limit cycles by perturbing a class of hyper-elliptic Hamiltonian systems of degree five, *J. Math. Anal. Appl.* **411**(2014), No. 2, 559–573. MR3128414; https://doi.org/10.1016/j.jmaa.2013.06.073

[9] Y. Xiong, H. Zhong, The number of limit cycles in a Z_2-equivariant Liénard system, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **23**(2013), Art. ID 1350085, 17 pp. MR3071624; https://doi.org/10.1142/S0218127406015210

[10] J. Yang, M. Han, Limit cycle bifurcations of some Liénard systems with a cuspidal loop and a homoclinic loop, *Chaos Solitons Fractals* **44**(2011), No. 4–5, 269–289 MR2795933; https://doi.org/10.1016/j.chaos.2011.02.008