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Dirichlet boundary value problems for uniformly elliptic equations in modified local generalized Sobolev–Morrey spaces

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Abstract. In this paper, we study the boundedness of the sublinear operators, generated by Calderón–Zygmund operators in local generalized Morrey spaces. By using these results we prove the solvability of the Dirichlet boundary value problem for a polyharmonic equation in modified local generalized Sobolev–Morrey spaces. We obtain a priori estimates for the solutions of the Dirichlet boundary value problems for the uniformly elliptic equations in modified local generalized Sobolev–Morrey spaces defined on bounded smooth domains.

Keywords: solvability of Dirichlet problem, modified local generalized Sobolev–Morrey spaces, a priori estimates, uniformly elliptic equations.

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1 Introduction

The classical Morrey spaces $L_{p,\lambda}$ are originally introduced in order to study the local behavior of solutions to elliptic partial differential equations. In fact, the better inclusion between the Morrey and Hölder spaces permits to obtain regularity of the solution to elliptic boundary value problems. For the properties and applications of the classical Morrey spaces we refer the readers to [30, 34].

In [8] Chiarenza and Frasca showed boundness of the Hardy–Littlewood maximal operator in $L_{p,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity in these spaces of some classical integral operators. The results in [8] allow us to study the regularity of the solutions of of elliptic/parabolic equations and systems in $L_{p,\lambda}$ (see [9, 11, 12, 33, 35–37] and the references therein). In [31] Mizuhara extended the Morrey's concept of integral average over a ball with a certain growth, taking a weight function $\varphi(x,r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of r^λ. So he put the beginning of the study of the generalized Morrey spaces $M_{p,\varphi}$, $p > 1$ with φ belonging to various classes of weight functions. In [32] Nakai proved boundedness of the maximal and

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Calderón–Zygmund operators in $M_{p,\varphi}$ imposing suitable integral and doubling conditions on φ. Taking a weight $w(x,t) = \varphi(x,t)^p t^n$ the conditions of Mizuhara–Nakai become

$$\int_r^\infty \varphi(x,\tau)^p \frac{d\tau}{\tau} \le C\varphi(x,r)^p, \qquad C^{-1} \le \frac{\varphi(x,t)}{\varphi(x,r)} \le C, \qquad \forall r \le t \le 2r,$$

where the constants do not depend on t, r and $x \in \mathbb{R}^n$.

In series of works, the first author studies the continuity in generalized Morrey spaces of sublinear operators generated by various integral operators as Calderón–Zygmund, Riesz and others (see [4,21,23]). The following theorem obtained in [21,23] extends the results of Nakai to the generalized Morrey spaces with weight $w(x,t) = \varphi(x,t)t^n$ (for the definition of the spaces see Section 2).

Theorem A ([23, Theorem 6.2]). *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \varphi_1(x,\tau)\frac{d\tau}{\tau} \le C\varphi_2(x,r), \tag{1.1}$$

where C does not depend on x and r. Then the Calderón–Zygmund operators are bound from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{p,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M_{1,\varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{p,\varphi_2}(\mathbb{R}^n)$.

This result is extended on spaces with weaker condition on the weight pair (φ_1, φ_2) (see [4]). A further development of the generalized Morrey spaces can be found in the works [4,24] and the references therein. In [4,24], Guliyev et al. obtained a weaker than (1.1) condition on the pair (φ_1, φ_2) which is optimal and ensure the boundedness of the classical integral operators from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{p,\varphi_2}(\mathbb{R}^n)$. Precisely, if

$$\int_r^\infty \frac{\operatorname{ess\,sup}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\varphi_2(x,r), \tag{1.2}$$

then the Calderón–Zygmund operators are bound from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{p,\varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M_{1,\varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{p,\varphi_2}(\mathbb{R}^n)$.

We use this integral inequality to obtain the Calderón–Zygmund type estimate for the $M_{p,\varphi}$-regularity of the solution. These results allow us to study the regularity of the solutions of various linear elliptic and parabolic boundary value problems in $M_{p,\varphi}$ (see [27,28,38]).

Later these results are extended on the local generalized Morrey spaces, which is obtained the boundedness of the Calderón–Zygmund operators from one local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi_1}(\mathbb{R}^n)$ to another $LM^{\{x_0\}}_{p,\varphi_2}(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ (see [25,26]), if the pair functions (φ_1, φ_2) satisfy the following condition

$$\int_r^\infty \frac{\operatorname{ess\,sup}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\varphi_2(x_0,r), \tag{1.3}$$

where C does not depend on r.

In this paper we study the boundedness of the sublinear operators, generated by Calderón–Zygmund operators in local generalized Morrey spaces. By using these results we obtain the regularity of the solutions of higher order uniformly elliptic boundary value problem in modified local generalized Sobolev–Morrey spaces defined on bounded smooth domains.

The paper is organized as follows. In Section 2 we give some definitions and some estimates of the Green function and the Poisson kernels. In Section 3 we prove the boundedness of the sublinear operators, generated by Calderón–Zygmund operators in the local generalized

Morrey spaces. Further, we obtain the regularity estimates for the solvability of the Dirichlet boundary value problem for polyharmonic equation in modified local generalized Sobolev–Morrey spaces. In Section 4 we prove a priori estimates for the solutions of the Dirichlet boundary value problems for the uniformly elliptic equations in modified local generalized Sobolev–Morrey spaces defined on bounded smooth domains.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Definitions and statement of the problem

Definition 2.1. Let $\varphi : \Omega \times \mathbb{R}+ \to \mathbb{R}+$ be a measurable function and $1 \le p < \infty$. For any domain Ω the generalized Morrey space $M_{p,\varphi}(\Omega)$ (the weak generalized Morrey space $WM_{p,\varphi}(\Omega)$) consists of all $f \in L_p^{\rm loc}(\Omega)$ such that

$$\|f\|_{M_{p,\varphi}(\Omega)} = \sup_{x\in\Omega,\ 0<r<d} \frac{1}{\varphi(x,r)} \frac{1}{|B(x,r)|^{\frac{1}{p}}} \|f\|_{L_p(\Omega(x,r))} < \infty,$$
$$\left(\|f\|_{WM_{p,\varphi}(\Omega)} = \sup_{x\in\Omega,\ 0<r<d} \frac{1}{\varphi(x,r)} \frac{1}{|B(x,r)|^{\frac{1}{p}}} \|f\|_{WL_p(\Omega(x,r))} < \infty\right)$$

where $d = \sup_{x,y\in\Omega} |x-y|$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\Omega(x,r) = \Omega \bigcap B(x,r)$.

In the case $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$, $M_{p,\varphi} = L_{p,\lambda}$, where $0 < \lambda < n$. If $\lambda = 0$, then $L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = n$, then $L_{p,n}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. In the case $\lambda < 0$ or $\lambda > n$, $L_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Definition 2.2. Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,d)$ and $1 \le p < \infty$. Fixed $x_0 \in \Omega$, we denote by $LM_{p,\varphi}^{\{x_0\}}(\Omega)$ $(WLM_{p,\varphi}^{\{x_0\}}(\Omega))$ the *local* generalized Morrey space (the weak *local* generalized Morrey space), the space of all functions $f \in L_p^{\rm loc}(\Omega)$ with finite quasinorm

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}(\Omega)} = \sup_{0<r<d} \frac{1}{\varphi(x_0,r)} \frac{1}{|B(x_0,r)|^{\frac{1}{p}}} \|f\|_{L_p(\Omega(x_0,r))}$$
$$\left(\|f\|_{WLM_{p,\varphi}^{\{x_0\}}(\Omega)} = \sup_{0<r<d} \frac{1}{\varphi(x_0,r)} \frac{1}{|B(x_0,r)|^{\frac{1}{p}}} \|f\|_{WL_p(\Omega(x_0,r))}\right).$$

Definition 2.3. Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,d)$ and $1 \le p < \infty$. We denote by $\widetilde{M}_{p,\varphi}(\Omega)$ $\big(\widetilde{M}_{p,\varphi}(\Omega)\big)$ the modified generalized Morrey space (the modified weak generalized Morrey space), the space of all functions $f \in L_p(\Omega)$ with finite norm

$$\|f\|_{\widetilde{M}_{p,\varphi}(\Omega)} = \|f\|_{M_{p,\varphi}(\Omega)} + \|f\|_{L_p(\Omega)}$$
$$\left(\|f\|_{W\widetilde{M}_{p,\varphi}(\Omega)} = \|f\|_{WM_{p,\varphi}(\Omega)} + \|f\|_{WL_p(\Omega)}\right).$$

Definition 2.4. Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,d)$ and $1 \le p < \infty$. Fixed $x_0 \in \Omega$, we denote by $\widetilde{LM}_{p,\varphi}^{\{x_0\}}(\Omega)$ $\big(\widetilde{LM}_{p,\varphi}^{\{x_0\}}(\Omega)\big)$ the modified local generalized Morrey space (the modified weak local generalized Morrey space), the space of all functions $f \in L_p(\Omega)$

with finite norm

$$\|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi}(\Omega)} = \|f\|_{LM^{\{x_0\}}_{p,\varphi}(\Omega)} + \|f\|_{L_p(\Omega)}$$
$$\Big(\|f\|_{W\widetilde{LM}^{\{x_0\}}_{p,\varphi}(\Omega)} = \|f\|_{WLM^{\{x_0\}}_{p,\varphi}(\Omega)} + \|f\|_{WL_p(\Omega)}\Big).$$

Definition 2.5. The modified generalized Sobolev–Morrey space $W^{2m}_{p,\varphi}(\Omega)$ consist of all functions $u \in W^{2m}_p(\Omega)$ with distributional derivatives $D^s_u \in \widetilde{M}_{p,\varphi}(\Omega)$, $0 \le |s| \le 2m$, endowed with the norm

$$\|u\|_{W^{2m}_{p,\varphi}(\Omega)} = \sum_{0\le|s|\le 2m} \|D^s u\|_{\widetilde{M}_{p,\varphi}(\Omega)}.$$

The modified local generalized Sobolev–Morrey space $W^{2m,\{x_0\}}_{p,\varphi}(\Omega)$ consist of all functions $u \in W^{2m}_p(\Omega)$ with distributional derivatives $D^s_u \in \widetilde{LM}^{\{x_0\}}_{p,\varphi}(\Omega)$, $0 \le |s| \le 2m$, endowed with the norm

$$\|u\|_{W^{2m,\{x_0\}}_{p,\varphi}(\Omega)} = \sum_{0\le|s|\le 2m} \|D^s u\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi}(\Omega)}.$$

The space $W^{2m,\{x_0\}}_{p,\varphi}(\Omega) \cap \mathring{W}^1_p(\Omega)$ consists of all functions $u \in \mathring{W}^1_p(\Omega)$ with $D^s_u \in LM^{\{x_0\}}_{p,\varphi}(\Omega)$, $0 \le |s| \le 2m$ and is endowed by the same norm. Recall that $\mathring{W}^1_p(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm in $W^1_p(\Omega)$.

At first we consider the Dirichlet boundary value problem for polyharmonic equation

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}} = g & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \ge 2$ is a bounded domain with sufficiently smooth boundary.

For the solutions of the problem (2.1) we give some estimates for the Green function and the Poisson kernels. Later we obtain a priori estimates for solvability of problem (2.1) in the local generalized Morrey spaces.

Let $G_m(x,y)$ be the Green function and $K_j(x,y)$, $j = \overline{0, m-1}$ be the Poisson kernels of problem (2.1). Then the solution of problem (2.1) can be written as

$$u(x) = \int_\Omega G_m(x,y) f(y) dy + \sum_{j=0}^{m-1} \int_{\partial\Omega} K_j(x,y) g(y) d\sigma_y$$

for correspondingly f and g. For example, when $m = 2$ and $n = 2$ we will used that there is a constant $C(\Omega)$ such that

$$|G_2(x,y)| \le C(\Omega) d(x) d(y) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}, \tag{2.2}$$

which was proved in [29], where d is the distance of x to the boundary $\partial\Omega$

$$d(x) = \inf_{\tilde{x}\in\partial\Omega} |x - \tilde{x}|. \tag{2.3}$$

However, we would like to mention that for G_m and K_j estimates are the optimal tools for deriving regularity results in spaces that involve to behavior at the boundary. Coming back to the $m = n = 2$ it follows from (2.2) that the solution of problem (2.1) satisfies the following estimates for appropriate f at $g = 0$

$$\|ud^{-2}\|_{L_\infty(\Omega)} \le C(\Omega)\, \|f\|_{L_1(\Omega)},$$
$$\|u\|_{L_\infty(\Omega)} \le C(\Omega)\, \|fd^2\|_{L_1(\Omega)}.$$

We also derive estimates for derivative of kernels. We will focus on estimate that contain growth rates near the boundary. These estimates are optimal. Indeed, when we consider $G_m(x,y)$ for $\Omega = B(x,r)$ a ball in $\mathbb{R}^n$ the growth rates near the boundary are sharp (see [18]). For $m = 1$ or $m \ge 2$ and $\Omega = B(x,r)$ it is known that the Green function is positive and can even be estimated from below by a positive function with the same singular behavior (see [19]). Let us remind that for $m \ge 2$ the Green function in general is not positive. For general domains the optimal behavior in absolute values is captured in our estimates. Sharp estimates for K_{m-1} and K_{m-2} in the case of a ball can be found in [20]. In [5] Barbatis considered the pointwise estimates for the Green function of higher order parabolic problems on domains and derived pointwise estimates for the kernel. For higher order parabolic systems the classical estimates obtained by Eidelman [17] were not considered in domains with boundary. For a survey on spectral theory of higher order elliptic operators, including some estimates for the corresponding kernels, we refer to [14].

Let G a function on $\Omega \times \Omega$ and $\alpha, \beta \in \mathbb{N}^n$. Derivatives of G are denoted by

$$D_x^\alpha D_y^\beta f(x,y) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdot \ldots \cdot \partial x_n^{\alpha_n}} \frac{\partial^{|\beta|}}{\partial y_1^{\beta_1} \partial y_2^{\beta_2} \cdot \ldots \cdot \partial y_n^{\beta_n}} G(x,y),$$

where $|\alpha| = \sum_{k=1}^n \alpha_k$, $|\beta| = \sum_{k=1}^n \beta_k$.

For completeness we will give some estimates for $G_m(x,y)$ and $K_j(x,y)$ depending on the distance to the boundary and auxiliary results with proof. We will do by estimating the j-th derivative through an integration of the $(j+1)$-th derivative along a path to the boundary. The dependence on the distance to the boundary $d(x)$ will appear closing a path which length is proportional to $d(x)$. The path will be constructed in Lemma 2.10.

Theorem 2.6 ([15,29]). *Let $G_m(x,y)$ be the Green function of problem (2.1). Then for every $x, y \in \Omega$ the following estimates hold:*

1. *if $2m - n > 0$, then*

$$|G_m(x,y)| \le d^{m-\frac{n}{2}}(x) \cdot d^{m-\frac{n}{2}}(y) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}^{\frac{n}{2}};$$

2. *if $2m - n = 0$, then*

$$|G_m(x,y)| \le \log\left(1 + \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}^m\right);$$

3. *if $2m - n < 0$, then*

$$|G_m(x,y)| \le |x-y|^{2m-n} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}^m.$$

Theorem 2.7 ([15,29]). *Let $K_j(x,y)$, $j = \overline{0, m-1}$ be the Poisson kernels of problem (2.1). Then for every $x \in \Omega$, $y \in \partial\Omega$ the following estimates hold:*

$$|K_j(x,y)| \le \frac{d^m(x)}{|x-y|^{n-j+m-1}}. \tag{2.4}$$

Remark 2.8. If $n-1<j\le m-1$, then from (2.4) we get the inequality

$$|K_j(x,y)|\le d^{1+j-n}(x) \tag{2.5}$$

on $\Omega\times\partial\Omega$.

Remark 2.9. The estimates in Theorem 2.7 hold for any uniformly elliptic operator of order $2m$.

In [19] the estimates in Theorem 2.6 are given for the case that $\Omega=B(x,r)$ in $\mathbb{R}^n$. In there the authors use an explicit formula for the Green's function, given in [6].

For general domains one cannot expect an explicit formula for the Green's functions and the Poisson kernels. We will use the estimates for $G_m(x,y)$ and $K_j(x,y)$ given in [29]. In [29] for sufficiently regular domains Ω some estimates for the Green's function and Poisson kernels was proved.

The following lemma is valid.

Lemma 2.10. *Let $x\in\Omega$ and $y\in\overline{\Omega}$. There exists a curve $\gamma_x^y:[0,1]\to\Omega$ with $\gamma_x^y(0)=x$, $\gamma_x^y(1)\in\partial\Omega$ and*

1. $$|\gamma_x^y(t)-y|\ge\frac{1}{2}|x-y| \quad \textit{for every } t\in[0,1], \tag{2.6}$$

2. $$l\le(1+\pi)d(x), \quad \textit{where } l \textit{ is the legth of } \gamma_x^y. \tag{2.7}$$

Moreover, if $\widetilde{\gamma}_x^y:[0,l]\to\overline{\Omega}$ is the parametrization by arc length of γ_x^y, then the following inequalities hold

3. $$\frac{1}{5}s\le|x-\widetilde{\gamma}_x^y(s)|\le s \quad \textit{for } s\in[0,l]. \tag{2.8}$$

We proceed with the proof of Theorem 2.6 and start from the estimates in [29] of the m-th derivative of $G_m(x,y)$.

Integrating this function along the path γ_x^y of Lemma 2.10. We find the estimates of the $(m-1)$-th derivative of $G_m(x,y)$ in terms of the distance to the boundary. Iterating the procedure m times we find the results as stated in Theorem 2.6.

We use some auxiliary results which can easy obtain from [29]. From these results we get the following theorem.

Theorem 2.11 ([15, 29]). *Let $G_m(x,y)$ be the Green's function of problem (2.1), $k\in\mathbb{N}^n$. Then for every $x,y\in\Omega$, the following estimates hold.*

1. *For $|k|\ge m$: if $2m-n-|k|<0$, then*

$$\left|D_x^kG_m(x,y)\right|\le C|x-y|^{2m-n-|k|}\min\left\{1,\frac{d(y)}{|x-y|}\right\}^m;$$

if $2m-n-|k|=0$, then

$$\left|D_x^kG_m(x,y)\right|\le C\log\left(1+\frac{d^m(y)}{|x-y|^m}\right)\approx\log\left(2+\frac{d(y)}{|x-y|}\right)\min\left\{1,\frac{d(y)}{|x-y|}\right\}^m; \tag{2.9}$$

if $2m-n-|k|>0$, then

$$\left|D_x^kG_m(x,y)\right|\le Cd^{2m-n-|k|}(y)\min\left\{1,\frac{d(y)}{|x-y|}\right\}^{n+|k|-m}.$$

2. *For $|k| < m$: if $2m - n - |k| < 0$, then*

$$\left|D_x^k G_m(x,y)\right| \le C|x-y|^{2m-n-|k|} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{m-|k|} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{m}.$$

If $2m - n - |k| = 0$, then

$$\begin{aligned}\left|D_x^k G_m(x,y)\right| &\le C \log\left(1 + \frac{d^m(y) d^{m-|k|}(x)}{|x-y|^{2m-|k|}}\right) \\ &\approx \log\left(2 + \frac{d(y)}{|x-y|}\right) \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{m} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{m-|k|}.\end{aligned} \tag{2.10}$$

If $2m - n - |k| > 0$, and moreover

a) $m - \frac{n}{2} \le |k|$, *then*

$$\left|D_x^k G_m(x,y)\right| \le C\, d^{2m-n-|k|}(y) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{m-|k|} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{n-m+|k|};$$

b) $|k| < m - \frac{n}{2}$, *then*

$$|D_x^k G_m(x,y)| \le C\, d(y)^{m-\frac{n}{2}} d^{2m-\frac{n}{2}-|k|}(x) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}^{\frac{n}{2}}.$$

Proof. Let $x, y \in \Omega$. We use the estimates derivatives of $G_m(x,y)$ from [29]. The estimates for the lower order derivatives of $G_m(x,y)$ will be obtained by integrating the higher order derivatives along the path γ_x^y from Lemma 2.10. This lemma corresponds to one of the integration steps. For example, with $\alpha, \beta \in \mathbb{N}^n$ and if $\tilde{x} \in \partial\Omega$ the endpoint of γ_x^y, then we find

$$D_x^\alpha D_y^\beta G_m(x,y) = D_x^\alpha D_y^\beta G_m(\tilde{x},y) + \int_{\gamma_x^y} \nabla_z D_z^\alpha D_y^\beta G_m(z,y) dz. \tag{2.11}$$

If $|\alpha| \le m-1$, then the first term on the right hand side of (2.11) equals to zero and we get

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \le \int_0^l \left|\nabla_x D_x^\alpha D_y^\beta G_m(\overline{\gamma}_x^y(s), y)\right| ds. \tag{2.12}$$

If $|\beta| \le m-1$, then similarly by integrating with respect to y we find

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \le \int_0^l \left|\nabla_y D_y^\beta D_x^\alpha G_m(x, \overline{\gamma}_y^x(s)\right| ds. \tag{2.13}$$

We distinguish the cases as in the statement of the theorem.

Case 1. Let $|k| = r \ge m$ and $\beta \in \mathbb{N}^n$ with $|\beta| = m-1$. Then from $k = \alpha$ and using the estimates from [29], we get

$$|D_x^\alpha D_y^\beta G_m(x,y)| \le |x-y|^{m-n-r}.$$

Case 2. Let $|k| = r < m$. Also we using the estimates for $\left|D_y^\beta D_x^\alpha D_x^k G_m(x,y)\right|$ from [29] and then integrates m times with respect of y and $m-r$ times with respect to x.

Thus the theorem is proved. □

The proof of Theorem 2.7. The method of proof is similar to the one used in Theorem 2.6. A difference is that in this case there is no symmetry between x and y. The following lemma, that corresponds to one integration step is as follows.

Lemma 2.12. *Let $\nu_1, k \in \mathbb{N}$ with $k \geq 2$. If*

$$|\nabla_x H(x,y)| \lesssim |x-y|^{-k} d^{\nu_1}(x)$$

for $x \in \Omega$, $y \in \partial\Omega$ and $H(\widetilde{x}, y) = 0$ for every $\widetilde{x} \in \partial\Omega$ with $\widetilde{x} \neq y$, then the following inequality holds

$$|H(x,y)| \lesssim |x-y|^{-k} d^{\nu_1+1}(x)$$

for $x \in \Omega$, $y \in \partial\Omega$.

If we use previous auxiliary results, then we can easily prove Lemma 2.12.

The Lemma 2.12 allow us to prove the following theorem for which Theorem 2.7 is a special case.

Theorem 2.13. *([15, 29]) Let $K_j(x,y)$, $j = \overline{0, m-1}$ be the Poisson kernels of problem (2.1) and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m-1$. Then the following estimate*

$$\left|D_x^\alpha K_j(x,y)\right| \lesssim \frac{d^{m-|\alpha|}(x)}{|x-y|^{n-j+m-1}}$$

holds for $x \in \Omega$ and $y \in \partial\Omega$.

Remark 2.14. The estimates of $D_x^\alpha K_j(x,y)$ for $|\alpha| \geq m$ can be found from [29]. Following estimate is valid

$$\left|D_x^\alpha K_j(x,y)\right| \lesssim |x-y|^{-n+j-|\alpha|+1}.$$

3 Sublinear operators, generated by Calderón–Zygmund operators in local generalized Morrey spaces

Let Ω be an open bounded subset of $\mathbb{R}^n$. Suppose that T represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\Omega)$

$$|Tf(x)| \leq c_0 \int_\Omega \frac{|f(y)|dy}{|x-y|^n}, \qquad x \notin \operatorname{supp}(f), \tag{3.1}$$

where c_0 is independent of f and x.

The following local estimates for the sublinear operator satisfying condition (3.1) are valid.

Lemma 3.1. *Let $1 \leq p < \infty$, Ω be an open bounded subset of $\mathbb{R}^n$, $x_0 \in \Omega$, $0 < r \leq d$, $d = \sup_{x,y\in\Omega} |x-y| < \infty$. Let also T be a sublinear operator satisfying condition (3.1), and bounded from $L_p(\Omega)$ to $WL_p(\Omega)$, and bounded on $L_p(\Omega)$ for $p > 1$.*

(i) Then the inequality

$$\|Tf\|_{WL_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\, dt + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)} \tag{3.2}$$

holds for any $\Omega(x_0, r)$ and for any $f \in L_p(\Omega)$.

(ii) Moreover, for $p > 1$ the inequality

$$\|Tf\|_{L_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\, dt + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)} \tag{3.3}$$

holds for any $\Omega(x_0,r)$ and for any $f \in L_p(\Omega)$.

Proof. Let $1 \le p < \infty$. Since

$$r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\, dt \ge r^{\frac{n}{p}} \|f\|_{L_p(\Omega(x_0,r))} \int_r^d t^{-\frac{n}{p}-1}\, dt \\ \approx \|f\|_{L_p(\Omega(x_0,r))} (d^{\frac{n}{p}} - r^{\frac{n}{p}}), \qquad r \in (0,d),$$

we get that

$$\|f\|_{L_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\, dt + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)}, \qquad r \in (0,d). \tag{3.4}$$

(i). Assume that $1 \le p < \infty$. Let $r \in (0, d/2)$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{\Omega(x_0,2r)}$ and $f_2 = f\chi_{\Omega\setminus\Omega(x_0,2r)}$. Taking into account the linearity of T, we have

$$\|Tf\|_{WL_p(\Omega(x_0,r))} \le \|Tf_1\|_{WL_p(\Omega(x_0,r))} + \|Tf_2\|_{WL_p(\Omega(x_0,r))}. \tag{3.5}$$

Since $f_1 \in L_p(\Omega)$, in view of (3.4), the boundedness of T from $L_p(\Omega)$ to $WL_p(\Omega)$ implies that

$$\|Tf_1\|_{WL_p(\Omega(x_0,r))} \le \|Tf_1\|_{WL_p(\Omega)} \lesssim \|f_1\|_{L_p(\Omega)} \approx \|f\|_{L_p(\Omega(x_0,2r))} \\ \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\, dt + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)}, \tag{3.6}$$

where the constant is independent of f, x_0 and r.

We have

$$|Tf_2(x)| \lesssim \int_{\Omega\setminus\Omega(x_0,2r)} \frac{|f(y)|dy}{|x-y|^{n-1}}, \qquad x \in \Omega(x_0,r).$$

It's clear that $x \in \Omega(x_0,r)$, $y \in \Omega\setminus\Omega(x_0,2r)$ implies $(1/2)|x_0-y| \le |x-y| < (3/2)|x_0-y|$. Therefore we obtain that

$$\|Tf_2\|_{L_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{\Omega\setminus\Omega(x_0,2r)} \frac{|f(y)|dy}{|x_0-y|^{n-1}}.$$

By Fubini's theorem, we get that

$$\begin{aligned} \int_{\Omega\setminus\Omega(x_0,2r)} \frac{|f(y)|}{|x_0-y|^{n-1}}\, dy &\approx \int_{\Omega\setminus\Omega(x_0,2r)} |f(y)| \left(1 + \int_{|x_0-y|}^d \frac{ds}{s^n}\right) dy \\ &= \int_{\Omega\setminus\Omega(x_0,2r)} |f(y)|\, dy + \int_{\Omega\setminus\Omega(x_0,2r)} |f(y)| \left(\int_{|x_0-y|}^d \frac{ds}{s^n}\right) dy \\ &= \int_{\Omega\setminus\Omega(x_0,2r)} |f(y)|\, dy + \int_{2r}^d \left(\int_{2r\le|x_0-y|\le s} |f(y)|\, dy\right) \frac{ds}{s^n} \\ &\le \int_\Omega |f(y)|\, dy + \int_{2r}^d \left(\int_{\Omega(x_0,s)} |f(y)|\, dy\right) \frac{ds}{s^n}. \end{aligned}$$

Applying Hölder's inequality, we arrive at

$$\int_{\Omega\setminus\Omega(x_0,2r)} \frac{|f(y)|dy}{|x_0-y|^n} \lesssim \|f\|_{L_p(\Omega)} + \int_{2r}^d s^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,s))}\, ds.$$

Thus the inequality

$$\|Tf_2\|_{L_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d s^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,s))}\, ds + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)} \tag{3.7}$$

holds for all $r \in (0, d/2)$.

On the other hand, since

$$\|Tf_2\|_{WL_p(\Omega(x_0,r))} \leq \|Tf_2\|_{L_p(\Omega(x_0,r))}$$

using (3.7), we get that

$$\|Tf_2\|_{WL_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d s^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,s))}\, ds + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)} \tag{3.8}$$

holds true for all $r \in (0, d/2)$.

Finally, combining (3.6) and (3.8), we obtain that

$$\|Tf\|_{WL_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d s^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,s))}\, ds + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)}$$

holds for all $r \in (0, d/2)$ with a constant independent of f, x_0 and r.

Let now $r \in [d/2, d)$. Then, using $(L_p(\Omega), WL_p(\Omega))$-boundedness of T, we obtain

$$\|Tf\|_{WL_p(\Omega(x_0,r))} \leq \|Tf\|_{WL_p(\Omega)} \lesssim \|f\|_{L_p(\Omega)} \approx r^{\frac{n}{p}} \|f\|_{L_p(\Omega)},$$

and, inequality (3.2) holds.

(ii). Assume that $1 < p < \infty$. Let again $r \in (0, d/2)$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{\Omega(x_0,2r)}$ and $f_2 = f\chi_{\Omega\setminus\Omega(x_0,2r)}$. Taking into account the linearity of T, we have

$$\|Tf\|_{L_p(\Omega(x_0,r))} \leq \|Tf_1\|_{L_p(\Omega(x_0,r))} + \|Tf_2\|_{L_p(\Omega(x_0,r))}. \tag{3.9}$$

Since $f_1 \in L_p(\Omega)$, in view of (3.4), the boundedness of T on $L_p(\Omega)$ implies that

$$\begin{aligned}\|Tf_1\|_{L_p(\Omega(x_0,r))} &\leq \|Tf_1\|_{L_p(\Omega)} \lesssim \|f_1\|_{L_p(\Omega)} \approx \|f\|_{L_p(\Omega(x_0,2r))} \\ &\lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|f\|_{L_p(\Omega(x_0,t))}\, dt + r^{\frac{n}{p}} \|f\|_{L_p(\Omega)},\end{aligned} \tag{3.10}$$

where the constant is independent of f, x_0 and r.

Combining (3.9), (3.10) and (3.7), we get inequality (3.3) holds for all $r \in (0, d/2)$ with a constant independent of f, x_0 and r.

If $r \in [d/2, d)$, then, using the boundedness of T on $L_p(\Omega)$, we obtain that

$$\|Tf\|_{L_p(\Omega(x_0,r))} \leq \|Tf\|_{L_p(\Omega)} \lesssim \|f\|_{L_p(\Omega)} \approx r^{\frac{n}{p}} \|f\|_{L_p(\Omega)},$$

and, inequality (3.3) holds. □

Now we are going to use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^d g(s) w(s) ds, \qquad 0 < t \leq d < \infty,$$

where w is a fixed function non-negative and measurable on $(0, d)$.

The following theorem was proved in [25].

Theorem 3.2. *Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,d)$. The inequality*

$$\operatorname*{ess\,sup}_{0<t<d} v_2(t) H_w^* g(t) \le C \operatorname*{ess\,sup}_{0<t<d} v_1(t) g(t) \tag{3.11}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,d)$ if and only if

$$B := \operatorname*{ess\,sup}_{0<t<d} v_2(t) \int_t^d \frac{w(s)ds}{\operatorname{ess\,sup}_{s<\tau<d} v_1(\tau)} < \infty. \tag{3.12}$$

Moreover, if C^* is the minimal value of C in (3.11), then $C^* = B$.

Remark 3.3. In (3.11) and (3.12) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

Theorem 3.4. *Let $1 \le p < \infty$, Ω be an open bounded subset of $\mathbb{R}^n$, $x_0 \in \Omega$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^d \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\, \tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \le C\, \varphi_2(x_0,r), \tag{3.13}$$

where C does not depend on r. Let also T be a sublinear operator satisfying condition (3.1), and bounded from $L_p(\Omega)$ to $WL_p(\Omega)$, and bounded on $L_p(\Omega)$ for $p>1$. Then there exists $c = c(p,\varphi_1,\varphi_2,n) > 0$ such that

$$\|Tf\|_{W\widetilde{LM}^{\{x_0\}}_{p,\varphi_2}(\Omega)} \le c \|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}.$$

Moreover, for $p>1$ there exists $c = c(p,\varphi_1,\varphi_2,n) > 0$ such that

$$\|Tf\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_2}(\Omega)} \le c \|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}.$$

Proof. By Theorem 3.2 and Lemma 3.1 with $v_2(r) = \varphi_2(x_0,r)^{-1}$, $v_1(r) = \varphi_1(x_0,r)^{-1} r^{-\frac{n}{p}}$ and $w(r) = r^{-\frac{n}{p}}$ we have

$$\begin{aligned}
\|Tf\|_{W\widetilde{LM}^{\{x_0\}}_{p,\varphi_2}(\Omega)} &\lesssim \sup_{0<r<d} \varphi_1(x_0,r)^{-1} \int_r^d \|f\|_{WL_p(\Omega(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} + \|Tf\|_{WL_p(\Omega)} \\
&\lesssim \sup_{0<r<d} \varphi_1(x_0,r)^{-1}\, r^{-\frac{n}{p}}\, \|f\|_{L_p(\Omega(x_0,r))} + \|f\|_{L_p(\Omega)} \\
&= \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}(\Omega)} + \|f\|_{L_p(\Omega)} = \|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}
\end{aligned}$$

and for $1<p<\infty$

$$\begin{aligned}
\|Tf\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_2}(\Omega)} &\lesssim \sup_{0<r<d} \varphi_1(x_0,r)^{-1} \int_r^d \|f\|_{L_p(\Omega(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} + \|Tf\|_{L_p(\Omega)} \\
&\lesssim \sup_{0<r<d} \varphi_1(x_0,r)^{-1}\, r^{-\frac{n}{p}}\, \|f\|_{L_p(\Omega(x_0,r))} + \|f\|_{L_p(\Omega)} \\
&= \|f\|_{LM^{\{x_0\}}_{p,\varphi_1}(\Omega)} + \|f\|_{L_p(\Omega)} = \|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}.
\end{aligned}$$

□

From Theorem 3.4 we get the following corollary.

Corollary 3.5. *Let $1 \le p < \infty$, Ω be an open bounded subset of $\mathbb{R}^n$, $x_0 \in \Omega$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^d \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\, \tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \le C\, \varphi_2(x,r), \tag{3.14}$$

where C does not depend on x and r. Let also T be a sublinear operator satisfying condition (3.1), and bounded from $L_p(\Omega)$ to $WL_p(\Omega)$, and bounded on $L_p(\Omega)$ for $p > 1$. Then there exists $c = c(p, \varphi_1, \varphi_2, n) > 0$ such that

$$\|Tf\|_{W\widetilde{M}_{p,\varphi_2}(\Omega)} \le c\|f\|_{\widetilde{M}_{p,\varphi_1}(\Omega)}.$$

Moreover, for $p > 1$ there exists $c = c(p, \varphi_1, \varphi_2, n) > 0$ such that

$$\|Tf\|_{\widetilde{M}^{p,\varphi_2}(\Omega)} \le c\|f\|_{\widetilde{M}_{p,\varphi_1}(\Omega)}.$$

4 Dirichlet boundary value problem for polyharmonic equation in modified local generalized Sobolev–Morrey spaces

Now we will derive regularity estimates for solution of problem (2.1) when $g = 0$

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \frac{\partial^k u}{\partial n^k} = 0 & \text{on } \Omega, \end{cases} \tag{4.1}$$

where $0 \le k \le m-1$, $\Omega \subset \mathbb{R}^n$ is bounded.

We get the estimates of solution problem (4.1) in modified local generalized Sobolev–Morrey spaces

$$\|u\|_{W^{2m,\{x_0\}}_{p,\varphi_2}(\Omega)} \lesssim \|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}}(\Omega).$$

Note that

$$Kf(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \sum_{|\alpha|=2m} D^{\alpha}_{x_i} G_m(x-y) f(y) dy$$

is the Calderón–Zygmund operator. Here and later we take, that function f define in $\mathbb{R}^n$, also this function is continuity extended to exterior of domain Ω with zero. The function $D^m_{x_i} G_m(x,y) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and this function is homogeneous of order $m - n$. Hence consequence, that $D^{2m}_{x_i} G_m(x,y)$ homogeneous of order $2m - n$ and tends to zero on unit sphere (see [15]). Then from general theory giving in [7] consequence that K bounded operator on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. Moreover, maximal singularity operator

$$\widetilde{K}f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \sum_{|\alpha|=2m} D^{\alpha} G_m(x,y) f(y) dy \right|$$

also a bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.

From Theorem 3.4 we get the following corollary.

Corollary 4.1. *Let $1 < p < \infty$, Ω be an open bounded subset of $\mathbb{R}^n$, $x_0 \in \Omega$, and (φ_1, φ_2) satisfy the condition (3.13). Then operators M and K are bounded from $\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)$ to $\widetilde{LM}^{\{x_0\}}_{p,\varphi_2}(\Omega)$.*

From Corollary 3.5 we get the following.

Corollary 4.2. *Let $1 < p < \infty$, Ω be an open bounded subset of $\mathbb{R}^n$, and (φ_1, φ_2) satisfy the condition (3.14). Then operators M and K are bounded from $\widetilde{M}_{p,\varphi_1}(\Omega)$ to $\widetilde{M}_{p,\varphi_2}(\Omega)$.*

Theorem 4.3. *Let $1 < p < \infty$, $x_0 \in \Omega$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \subset C^2$, and (φ_1, φ_2) satisfy the condition (3.13). Let also $f \in \widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)$ and function u is a solution of problem (4.1). Then there is exist constant C which dependent only at n, φ and Ω such that*

$$\|u\|_{W^{2m,\{x_0\}}_{p,\varphi_2}(\Omega)} \leq C\|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}. \tag{4.2}$$

Proof. The proved consequence from the above estimates of the Green's function from [27]: the following inequalities

$$|u(x)| + \left|D_{x_i} u(x)\right| \lesssim Mf(x), \tag{4.3}$$

$$\left|D_{x_i x_j} u(x)\right| \lesssim Kf(x) + Mf(x) + |f(x)| \tag{4.4}$$

hold uniformly for any $x \in \Omega$.

With similarly ideas can be proved estimates

$$|u(x)| + \left|\sum_{|\alpha| \leq m} D^{\alpha}_{x_i} u(x)\right| \lesssim Mf(x), \tag{4.5}$$

$$\left|\sum_{|\alpha| \leq 2m} D^{\alpha} u(x)\right| \lesssim \widetilde{K}f(x) + Mf(x) + |f(x)|. \tag{4.6}$$

Now we passing to prove of Theorem 4.3. From Corollary 4.1 imply that the operators M and $\widetilde{K}$ are bounded in $LM^{\{x_0\}}_{p,\varphi}(\Omega)$. Therefore statement of Theorem 4.3 and estimate (4.2) the immediately consequence from inequalities (4.5), (4.6) and Corollary 4.1.

Theorem 4.3 is proved. □

From inequalities (4.5), (4.6) and Corollary 4.2 we get the following corollary.

Corollary 4.4. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \subset C^2$, and (φ_1, φ_2) satisfy the condition (3.14). Let also $f \in \widetilde{M}_{p,\varphi_1}(\Omega)$ and function u is a solution of problem (4.1). Then there is exist constant C which dependent only at n, φ and Ω such that*

$$\|u\|_{W^{2m}_{p,\varphi_2}(\Omega)} \leq C\|f\|_{\widetilde{M}_{p,\varphi_1}(\Omega)}.$$

5 Estimates of solutions any higher order uniformly elliptic equation with smooth coefficients in modified local generalized Sobolev–Morrey spaces

Consider the boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ B_j u = \psi_j & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

for $j = 0, \ldots, m-1$. The following assumptions hold.

1. The operator

$$Lu = \sum_{|\alpha| \leq 2m} a_{\alpha,j}(x) D^{\alpha} u$$

is uniformly elliptic: there exists a constant $\gamma > 0$, such that

$$\gamma^{-1}|\xi|^2 \leq \sum_{\alpha,j} a_{\alpha,j}(x)\xi_\alpha\xi_j \leq \gamma|\xi|^2, \quad \text{a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^n$$

$$a_{\alpha,j}(x) = a_{j,\alpha}(x)$$

2. The boundary operators

$$B_j = \sum_{|\beta| \leq m_j} b_{j\beta} D^\beta, \quad \text{for } j = 0, \ m-1$$

satisfy the complementing condition relative to L (see the complementing condition on page 663 of [17]).

3. Let $l_1 > \max_j(2m - m_j)$ and $l_0 = \max_j(2m - m_j)$. The coefficients $a_{\alpha j}$ belong to $C^{l_1+1}(\overline{\Omega})$ and $b_{j\beta}$ belong to $C^{l_1+1}(\partial\Omega)$.

4. The boundary $\partial\Omega$ is C^{l_1+2m+1}.

5. $f \in LM_{p,\varphi}^{\{x_0\}}(\Omega)$ with $1 < p < \infty$ and $\varphi : \Omega \times \mathbb{R}+ \to \mathbb{R}+$ measurable.

Theorem 5.1. *Let us consider the boundary value problem* (5.1) *and satisfy conditions 1–5 and also condition of Theorem 4.3. Then there is exist constant C which dependent only at n, φ and Ω such that*

$$\|u\|_{W^{2m,\{x_0\}}_{p,\varphi_2}} \leq C\|f\|_{\widetilde{LM}^{\{x_0\}}_{p,\varphi_1}(\Omega)}. \tag{5.2}$$

Theorem 5.1 similarly ideas of Theorem 4.3 is proved.

For this it will be enough to consider the Krasovsky work [29]. We will recall the theorem in [29] which gives the estimates of the Green's function and the Poisson kernels. The proved consequence from this estimates. As in proof of Theorem 4.3 we use estimates (4.5), (4.6) and Corollary 4.1. Therefore statement of theorem and estimate (5.2) the immediately consequence from inequalities (4.5), (4.6). Theorem 5.1 is proved.

From inequalities (4.5), (4.6) and Corollary 4.2 we get the following corollary.

Corollary 5.2. *Let us consider the boundary value problem* (5.1) *and satisfy conditions 1–5 and also condition of Corollary 4.4. Then there is exist constant C which dependent only at n, φ and Ω such that*

$$\|u\|_{W^{2m}_{p,\varphi_2}} \leq C\|f\|_{\widetilde{M}_{p,\varphi_1}(\Omega)}.$$

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