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Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians

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Abstract. In this article we extend the Sobolev spaces with variable exponents to include the fractional case, and we prove a compact embedding theorem of these spaces into variable exponent Lebesgue spaces. As an application we prove the existence and uniqueness of a solution for a nonlocal problem involving the fractional $p(x)$-Laplacian.

Keywords: variable exponents, Sobolev spaces, fractional Laplacian.

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1 Introduction

Our main goal in this paper is to extend Sobolev spaces with variable exponents to cover the fractional case.

For a bounded domain with Lipschitz boundary $\Omega \subset \mathbb{R}^n$ we consider two variable exponents, that is, we let $q : \overline{\Omega} \to (1, \infty)$ and $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)$ be two continuous functions. We assume that p is symmetric, $p(x, y) = p(y, x)$, and that both p and q are bounded away from 1 and ∞, that is, there exist $1 < q_- < q_+ < +\infty$ and $1 < p_- < p_+ < +\infty$ such that $q_- \le q(x) \le q_+$ for every $x \in \overline{\Omega}$ and $p_- \le p(x, y) \le p_+$ for every $(x, y) \in \overline{\Omega} \times \overline{\Omega}$.

We define the Banach space $L^{q(x)}(\Omega)$ as usual,

$$L^{q(x)}(\Omega) := \left\{ f : \Omega \to \mathbb{R} : \ \exists \ \lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{q(x)} dx < \infty \right\},$$

with its natural norm

$$\|f\|_{L^{q(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{q(x)} dx < 1 \right\}.$$

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Now for $0 < s < 1$ we introduce the variable exponent Sobolev fractional space as follows:

$$W = W^{s,q(x),p(x,y)}(\Omega) := \Big\{ f : \Omega \to \mathbb{R} : f \in L^{q(x)}(\Omega) : \\ \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}} < \infty, \text{for some } \lambda > 0 \Big\},$$

and we set

$$[f]^{s,p(x,y)}(\Omega) := \inf\Big\{ \lambda > 0 : \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}} < 1 \Big\}$$

as the variable exponent seminorm. It is easy to see that W is a Banach space with the norm

$$\|f\|_W := \|f\|_{L^{q(x)}(\Omega)} + [f]^{s,p(x,y)}(\Omega);$$

in fact, one just has to follow the arguments in [20] for the constant exponent case. For general theory of classical Sobolev spaces we refer the reader to [1, 5] and for the variable exponent case to [8].

Our main result is the following compact embedding theorem into variable exponent Lebesgue spaces. For an analogous theorem for the Sobolev trace embedding we refer to the companion paper [3].

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded domain and $s \in (0,1)$. Let $q(x)$, $p(x,y)$ be continuous variable exponents with $sp(x,y) < n$ for $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and $q(x) > p(x,x)$ for $x \in \overline{\Omega}$. Assume that $r : \overline{\Omega} \to (1,\infty)$ is a continuous function such that*

$$p^*(x) := \frac{np(x,x)}{n - sp(x,x)} > r(x) \ge r_- > 1,$$

for $x \in \overline{\Omega}$. Then, there exists a constant $C = C(n,s,p,q,r,\Omega)$ such that for every $f \in W$, it holds that

$$\|f\|_{L^{r(x)}(\Omega)} \le C\|f\|_W.$$

That is, the space $W^{s,q(x),p(x,y)}(\Omega)$ is continuously embedded in $L^{r(x)}(\Omega)$ for any $r \in (1,p^*)$. Moreover, this embedding is compact.

In addition, when one considers functions $f \in W$ that are compactly supported inside Ω, it holds that

$$\|f\|_{L^{r(x)}(\Omega)} \le C[f]^{s,p(x,y)}(\Omega).$$

Remark 1.2. Observe that if p is a continuous variable exponent in $\overline{\Omega}$ and we extend p to $\overline{\Omega} \times \overline{\Omega}$ as $p(x,y) := \frac{p(x)+p(y)}{2}$, then $p^*(x)$ is the classical Sobolev exponent associated with $p(x)$, see [8].

Remark 1.3. When $q(x) \ge r(x)$ for every $x \in \overline{\Omega}$ the main inequality in the previous theorem, $\|f\|_{L^{r(x)}(\Omega)} \le C\|f\|_W$, trivially holds. Hence our results are meaningful when $q(x) < r(x)$ for some points x inside Ω.

With the above theorem at hand one can readily deduce existence of solutions to some nonlocal problems. Let us consider the operator $\mathcal{L}$ given by

$$\mathcal{L}u(x) := p.v. \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{n+sp(x,y)}}\, dy. \tag{1.1}$$

This operator appears naturally associated with the space W. In the constant exponent case it is known as the fractional p-Laplacian, see [2, 4, 6, 7, 9–11, 13, 14, 17–19] and references therein. On the other hand, we remark that (1.1) is a fractional version of the well-known $p(x)$-Laplacian, given by $\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$, that is associated with the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. We refer for instance to [8,12,15,16].

Let $f \in L^{a(x)}(\Omega)$, $a(x) > 1$. We look for solutions to the problem

$$\begin{cases} \mathcal{L}u(x) + |u(x)|^{q(x)-2}u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Associated with this problem we have the following functional

$$\mathcal{F}(u) := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}p(x,y)}dxdy + \int_\Omega \frac{|u(x)|^{q(x)}}{q(x)}dx - \int_\Omega f(x)u(x)\,dx. \tag{1.3}$$

To take into account the boundary condition in (1.2) we consider the space W_0 that is the closure in W of compactly supported functions in Ω. In order to have a well defined trace on $\partial\Omega$, for simplicity, we just restrict ourselves to $sp_- > 1$, since then it is easy to see that $W \subset W^{\tilde{s},p_-}(\Omega) \subset W^{\tilde{s}-1/p_-,p_-}(\partial\Omega)$, with $\tilde{s}p_- > 1$, see [1, 20]. Concerning problem (1.2), we shall prove the following existence and uniqueness result.

Theorem 1.4. *Let $s \in (1/2, 1)$, and let $q(x)$ and $p(x,y)$ be continuous variable exponents as in Theorem 1.1 with $sp_- > 1$. Let $f \in L^{a(x)}(\Omega)$, with $1 < a_- \le a(x) \le a_+ < +\infty$ for every $x \in \overline{\Omega}$, such that*

$$\frac{np(x,x)}{n - sp(x,x)} > \frac{a(x)}{a(x)-1} > 1.$$

Then, there exists a unique minimizer of (1.3) in W_0 that is the unique weak solution to (1.2).

The rest of the paper is organized as follows: In Section 2 we collect previous results on fractional Sobolev embeddings; in Section 3 we prove our main result, Theorem 1.1, and finally in Section 4 we deal with the elliptic problem (1.2).

2 Preliminary results.

In this section we collect some results that will be used along this paper.

Theorem 2.1 (Hölder's inequality). *Let $p, q, r : \overline{\Omega} \to (1, \infty)$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. If $f \in L^{r(x)}$ and $g \in L^{q(x)}$, then $fg \in L^{p(x)}$ and*

$$\|fg\|_{L^{p(x)}} \le c\|f\|_{L^{r(x)}}\|f\|_{L^{q(x)}}.$$

For the constant exponent case we have a fractional Sobolev embedding theorem.

Theorem 2.2 (Sobolev embedding, [20]). *Let $s \in (0, 1)$ and $p \in [1, +\infty)$ such that $sp < n$. Then, there exists a positive constant $C = C(n, p, s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \le C \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \right)^{1/p},$$

where

$$p^* = p^*(n,s) = \frac{np}{(n-sp)}$$

is the so-called "fractional critical exponent".

Consequently, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p,p^*]$.

Using the previous result together with an extension property, we also have an embedding theorem in a domain.

Theorem 2.3 ([20])**.** *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < n$. Let $\Omega \subset \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n,p,s,\Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have*

$$\|f\|_{L^q(\Omega)} \le C\|f\|_{W^{s,p}(\Omega)}$$

for any $q \in [p,p^*]$; i.e., the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p,p^*]$.

If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1,p^*]$. Moreover, this embedding is compact for $q \in [1,p^*)$.

3 Fractional Sobolev spaces with variable exponents.

Proof of Theorem 1.1. Being p, q and r continuous, and Ω bounded, there exist two positive constants k_1 and k_2 such that

$$q(x) - p(x,x) \ge k_1 > 0 \tag{3.1}$$

and

$$\frac{np(x,x)}{n-sp(x,x)} - r(x) \ge k_2 > 0, \tag{3.2}$$

for every $x \in \overline{\Omega}$.

Let $t \in (0,s)$. Since p, q and r are continuous, using (3.1) and (3.2) we can find a constant $\epsilon = \epsilon(p,r,q,k_2,k_1,t)$ and a finite family of disjoint Lipschitz sets B_i such that

$$\Omega = \cup_{i=1}^N B_i \quad \text{and} \quad \operatorname{diam}(B_i) < \epsilon,$$

that verify that

$$\begin{aligned} \frac{np(z,y)}{n-tp(z,y)} - r(x) &\ge \frac{k_2}{2}, \\ q(x) &\ge p(z,y) + \frac{k_1}{2}, \end{aligned} \tag{3.3}$$

for every $x \in B_i$ and $(z,y) \in B_i \times B_i$.

Let

$$p_i := \inf_{(z,y)\in B_i\times B_i} (p(z,y) - \delta).$$

From (3.3) and the continuity of the involved exponents we can choose $\delta = \delta(k_2)$, with $p_- - 1 > \delta > 0$, such that

$$\frac{np_i}{n-tp_i} \ge \frac{k_2}{3} + r(x) \tag{3.4}$$

for each $x \in B_i$.

It holds that

(1) if we let $p_i^* = \frac{np_i}{n-tp_i}$, then $p_i^* \geq \frac{k_2}{3} + r(x)$ for every $x \in B_i$,

(2) $q(x) \geq p_i + \frac{k_1}{2}$ for every $x \in B_i$.

Hence we can apply Theorem 2.3 for constant exponents to obtain the existence of a constant $C = C(n, p_i, t, \epsilon, B_i)$ such that

$$\|f\|_{L^{p_i^*}(B_i)} \leq C\Big(\|f\|_{L^{p_i}(B_i)} + [f]^{t,p_i}(B_i)\Big). \tag{3.5}$$

Now we want to show that the following three statements hold.

(A) There exists a constant c_1 such that

$$\sum_{i=0}^{N} \|f\|_{L^{p_i^*}(B_i)} \geq c_1 \|f\|_{L^{r(x)}(\Omega)}.$$

(B) There exists a constant c_2 such that

$$c_2 \|f\|_{L^{q(x)}(\Omega)} \geq \sum_{i=0}^{N} \|f\|_{L^{p_i}(B_i)}.$$

(C) There exists a constant c_3 such that

$$c_3 [f]^{s,p(x,y)}(\Omega) \geq \sum_{i=0}^{N} [f]^{t,p_i}(B_i).$$

These three inequalities and (3.5) imply that

$$\begin{aligned}
\|f\|_{L^{r(x)}(\Omega)} &\leq C \sum_{i=0}^{N} \|f\|_{L^{p_i^*}(B_i)} \\
&\leq C \sum_{i=0}^{N} \Big(\|f\|_{L^{p_i}(B_i)} + [f]^{t,p_i}(B_i)\Big) \\
&\leq C\Big(\|f\|_{L^{q(x)}(\Omega)} + [f]^{s,p(x,y)}(\Omega)\Big) \\
&= C\|f\|_W,
\end{aligned}$$

as we wanted to show.

Let us start with (A). We have

$$|f(x)| = \sum_{i=0}^{N} |f(x)| \chi_{B_i}.$$

Hence

$$\|f\|_{L^{r(x)}(\Omega)} \leq \sum_{i=0}^{N} \|f\|_{L^{r(x)}(B_i)}, \tag{3.6}$$

and by item (1), for each i, $p_i^* > r(x)$ if $x \in B_i$. Then we take $a_i(x)$ such that

$$\frac{1}{r(x)} = \frac{1}{p_i*} + \frac{1}{a(x)}.$$

Using Theorem 2.1 we obtain

$$\begin{aligned}\|f\|_{L^{r(x)}(B_i)} &\le c\|f\|_{L^{p_i^*(x)}(B_i)}\|1\|_{L^{a_i(x)}(B_i)}\\ &= C\|f\|_{L^{p_i^*(x)}(B_i)}.\end{aligned}$$

Thus, recalling (3.6) we get (A).

To show (B) we argue in a similar way using that $q(x) > p_i$ for $x \in B_i$.

In order to prove (C) let us set

$$F(x,y) := \frac{|f(x)-f(y)|}{|x-y|^s},$$

and observe that

$$\begin{aligned}[f]^{t,p_i}(B_i) &= \left(\int_{B_i}\int_{B_i}\frac{|f(x)-f(y)|^{p_i}}{|x-y|^{n+tp_i+sp_i-sp_i}}\,dxdy\right)^{\frac{1}{p_i}}\\ &= \left(\int_{B_i}\int_{B_i}\left(\frac{|f(x)-f(y)|}{|x-y|^s}\right)^{p_i}\frac{dxdy}{|x-y|^{n+(t-s)p_i}}\right)^{\frac{1}{p_i}}\\ &= \|F\|_{L^{p_i}(\mu,B_i\times B_i)}\\ &\le C\|F\|_{L^{p(x,y)}(\mu,B_i\times B_i)}\|1\|_{L^{b_i(x,y)}(\mu,B_i\times B_i)}\\ &= C\|F\|_{L^{p(x,y)}(\mu,B_i\times B_i)},\end{aligned} \tag{3.7}$$

where we have used Theorem 2.1 with

$$\frac{1}{p_i} = \frac{1}{p(x,y)} + \frac{1}{b_i(x,y)},$$

but considering the measure in $B_i \times B_i$ given by

$$d\mu(x,y) = \frac{dxdy}{|x-y|^{n+(t-s)p_i}}.$$

Now our aim is to show that

$$\|F\|_{L^{p(x,y)}(\mu,B_i\times B_i)} \le C[f]^{s,p(x,y)}(B_i) \tag{3.8}$$

for every i. If this is true, then we immediately derive (C) from (3.7).

Let $\lambda > 0$ be such that

$$\int_{B_i}\int_{B_i}\frac{|f(x)-f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dxdy < 1.$$

Choose

$$k := \sup\left\{1, \sup_{(x,y)\in\Omega\times\Omega}|x-y|^{s-t}\right\} \quad \text{and} \quad \tilde{\lambda} := \lambda k.$$

Then

$$\begin{aligned}&\int_{B_i}\int_{B_i}\left(\frac{|f(x)-f(y)|}{(\tilde{\lambda}|x-y|^s)}\right)^{p(x,y)}\frac{dxdy}{|x-y|^{n+(t-s)p_i}}\\ &\quad= \int_{B_i}\int_{B_i}\frac{|x-y|^{(s-t)p_i}}{k^{p(x,y)}}\frac{|f(x)-f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dxdy\\ &\quad\le \int_{B_i}\int_{B_i}\frac{|f(x)-f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dxdy < 1.\end{aligned}$$

Therefore

$$\|F\|_{L^{p(x,y)}(\mu,B_i\times B_i)} \leq \lambda k,$$

which implies the inequality (3.8).

On the other hand, when we consider functions that are compactly supported inside Ω we can get rid of the term $\|f\|_{L^{q(x)}(\Omega)}$ and it holds that

$$\|f\|_{L^{q(x)}(\Omega)} \leq C[f]^{s,p(x,y)}(\Omega).$$

Finally, we recall that the previous embedding is compact since in the constant exponent case we have that for subcritical exponents the embedding is compact. Hence, for a bounded sequence in W, f_i, we can mimic the previous proof obtaining that for each B_i we can extract a convergent subsequence in $L^{r(x)}(B_i)$. $\square$

Remark 3.1. Our result is sharp in the following sense: if

$$p^*(x_0) := \frac{np(x_0,x_0)}{n - sp(x_0,x_0)} < r(x_0)$$

for some $x_0 \in \Omega$, then the embedding of W in $L^{r(x)}(\Omega)$ cannot hold for every $q(x)$. In fact, from our continuity conditions on p and r there is a small ball $B_\delta(x_0)$ such that

$$\max_{\overline{B}_\delta(x_0)\times\overline{B}_\delta(x_0)} \frac{np(x,y)}{n - sp(x,y)} < \min_{\overline{B}_\delta(x_0)} r(x).$$

Now, fix $q < \min_{\overline{B}_\delta(x_0)} r(x)$ (note that for $q(x) \geq r(x)$ we trivially have that W is embedded in $L^{r(x)}(\Omega)$). In this situation, with the same arguments that hold for the constant exponent case, one can find a sequence f_k supported inside $B_\delta(x_0)$ such that $\|f_k\|_W \leq C$ and $\|f_k\|_{L^{r(x)}B_\delta(x_0)} \to +\infty$. In fact, just consider a smooth, compactly supported function g and take $f_k(x) = k^a g(kx)$ with a such that $ap(x,y) - n + sp(x,y) \leq 0$ and $ar(x) - n > 0$ for $x, y \in \overline{B}_\delta(x_0)$.

Finally, we mention that the critical case

$$p^*(x) := \frac{np(x,x)}{n - sp(x,x)} \geq r(x)$$

with equality for some $x_0 \in \Omega$ is left open.

4 Equations with the fractional $p(x)$-Laplacian.

In this section we apply our previous results to solve the following problem. Let us consider the operator $\mathcal{L}$ given by

$$\mathcal{L}u(x) := p.v. \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{n+sp(x,y)}}\, dy.$$

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $f \in L^{a(x)}(\Omega)$ with $a_+ > a(x) > a_- > 1$ for each $x \in \overline{\Omega}$. We look for solutions to the problem

$$\begin{cases} \mathcal{L}u(x) + |u(x)|^{q(x)-2}u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

To this end we consider the following functional

$$\mathcal{F}(u) := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}p(x,y)} dxdy + \int_\Omega \frac{|u(x)|^{q(x)}}{q(x)} dx - \int_\Omega f(x)u(x)\,dx. \tag{4.2}$$

Let us first state the definition of a weak solution to our problem (4.1). Note that here we are using that p is symmetric, that is, we have $p(x,y) = p(y,x)$.

Definition 4.1. We call u a weak solution to (4.1) if $u \in W_0^{s,q(x),p(x,y)}(\Omega)$ and

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp(x,y)}}\,dxdy + \int_\Omega |u|^{q(x)-2}u(x)v(x)\,dx = \int_\Omega f(x)v(x)\,dx, \tag{4.3}$$

for every $v \in W_0^{s,q(x),p(x,y)}(\Omega)$.

Now our aim is to show that $\mathcal{F}$ has a unique minimizer in $W_0^{s,q(x),p(x,y)}(\Omega)$. This minimizer shall provide the unique weak solution to the problem (4.1).

Proof of Theorem 1.4. We just observe that we can apply the direct method of Calculus of Variations. Note that the functional $\mathcal{F}$ given in (4.2) is bounded below and strictly convex (this holds since for any x and y the function $t \mapsto t^{p(x,y)}$ is strictly convex).

From our previous results, $W_0^{s,q(x),p(x,y)}(\Omega)$ is compactly embedded in $L^{r(x)}(\Omega)$ for $r(x) < p^*(x)$, see Theorem 1.1. In particular, we have that $W_0^{s,q(x),p(x,y)}(\Omega)$ is compactly embedded in $L^{\frac{a(x)}{a(x)-1}}(\Omega)$.

Let us see that $\mathcal{F}$ is coercive. We have

$$\begin{aligned}
\mathcal{F}(u) &= \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}p(x,y)} dxdy + \int_\Omega \frac{|u(x)|^{q(x)}}{q(x)} dx - \int_\Omega f(x)u(x)\,dx \\
&\geq \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}p(x,y)} dxdy + \int_\Omega \frac{|u(x)|^{q(x)}}{q(x)} dx - \|f\|_{L^{a(x)}(\Omega)} \|u\|_{L^{\frac{a(x)}{a(x)-1}}(\Omega)} \\
&\geq \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}p(x,y)} dxdy + \int_\Omega \frac{|u(x)|^{q(x)}}{q(x)} dx - C\|u\|_W.
\end{aligned}$$

Now, let us assume that $\|u\|_W > 1$. Then we have

$$\begin{aligned}
\frac{\mathcal{F}(u)}{\|u\|_W} &\geq \frac{1}{\|u\|_W} \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}p(x,y)} dxdy + \int_\Omega \frac{|u(x)|^{q(x)}}{q(x)} dx \right) - C \\
&\geq \|u\|_W^{\min\{p_-,q_-\}-1} - C.
\end{aligned}$$

We next choose a sequence u_j such that $\|u_j\|_W \to \infty$ as $j \to \infty$. Then we have

$$\mathcal{F}(u_j) \geq \|u_j\|_W^{\min\{p_-,q_-\}} - C\|u_j\|_W \to \infty,$$

and we conclude that $\mathcal{F}$ is coercive. Therefore, there is a unique minimizer of $\mathcal{F}$.

Finally, let us check that when u is a minimizer to (4.2) then it is a weak solution to (4.1). Given $v \in W_0^{s,q(x),p(x,y)}(\Omega)$ we compute

$$\begin{aligned}
0 = \frac{d}{dt}\mathcal{F}(u+tv)\Big|_{t=0} &= \int_\Omega\int_\Omega \frac{d}{dt}\frac{|u(x)-u(y)+t(v(x)-v(y))|^{p(x,y)}}{p(x,y)|x-y|^{n+sp(x,y)}}\,dxdy\Big|_{t=0} \\
&\quad + \int_\Omega \frac{d}{dt}\frac{|u(x)+tv(x)|^{q(x)}}{q(x)}dx\Big|_{t=0} - \int_\Omega \frac{d}{dt}f(x)(u(x)+tv(x))\,dx\Big|_{t=0} \\
&= \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp(x,y)}}\,dxdy \\
&\quad + \int_\Omega |u(x)|^{q(x)-2}u(x)v(x)dx - \int_\Omega f(x)v(x),
\end{aligned}$$

as u is a minimizer of (4.2). Thus, we deduce that u is a weak solution to the problem (4.1).

The proof of the converse (that every weak solution is a minimizer of $\mathcal{F}$) is standard and we leave the details to the reader. □

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