Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **3**, 1–27; https://doi.org/10.14232/ejqtde.2018.1.3 www.math.u-szeged.hu/ejqtde/

Qualitative approximation of solutions to discrete Volterra equations

Janusz Migda[1] and **Małgorzata Migda**[✉2]

[1]Faculty of Mathematics and Computer Science, A. Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
[2]Institute of Mathematics, Poznań University of Technology, Piotrowo 3A, 60-965 Poznań, Poland

Received 11 November 2017, appeared 9 February 2018
Communicated by Stevo Stević

Abstract. We present a new approach to the theory of asymptotic properties of solutions to discrete Volterra equations of the form

$$\Delta^m x_n = b_n + \sum_{k=1}^{n} K(n,k) f(k, x_{\sigma(k)}).$$

Our method is based on using the iterated remainder operator and asymptotic difference pairs. This approach allows us to control the degree of approximation.

Keywords: Volterra discrete equation, difference pair, prescribed asymptotic behavior, asymptotically polynomial solution, bounded solution.

2010 Mathematics Subject Classification: 39A10, 39A22, 39A24.

1 Introduction

Let $\mathbb{N}$, $\mathbb{R}$ denote the set of positive integers and real numbers respectively. Let $m \in \mathbb{N}$. We consider the nonlinear discrete Volterra equations of non-convolution type

$$\Delta^m x_n = b_n + \sum_{k=1}^{n} K(n,k) f(k, x_{\sigma(k)}), \tag{E}$$

$$b_n \in \mathbb{R}, \quad f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \quad K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad \sigma : \mathbb{N} \to \mathbb{N}, \quad \sigma(k) \to \infty.$$

By a *solution* of (E) we mean a sequence $x : \mathbb{N} \to \mathbb{R}$ satisfying (E) for every large n. We say that x is a *full solution* of (E) if (E) is satisfied for every n. Moreover, if $p \in \mathbb{N}$ and (E) is satisfied for every $n \geq p$, then we say that x is a *p-solution*. In this paper we regard equation (E) as a generalization of the equation

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n. \tag{1.1}$$

✉Corresponding author. Email: malgorzata.migda@put.poznan.pl

Indeed, if $K(n,k) = 0$ for $k \neq n$, then denoting $a_n = K(n,n)$ we may rewrite (E) in the form (1.1). Hence the ordinary difference equation (1.1) is a special case of (E).

Volterra difference equations appeared as a discretization of Volterra integral and integro-differential equations. They also often arise during the mathematical modeling of some real life situations where the current state is determined by the whole previous history. Therefore, many papers have been devoted to these types of equations during the last few years. For example, the boundedness of solutions of such equations was studied in [6,12,17–22,25,39–41,44]. Some results on the boundedness and growth of solutions of related difference equations were proved also in [45–47]. The periodicity was investigated, e.g., in [1,9–11,16,22,37,43]. Several fundamental results on the stability of linear Volterra difference equations, of both convolution and non–convolution type, can be found in [7,8,15]; see also [2,5,23,24,26,40,48]. Some related results on dynamic equations can be found in [3] and [4].

In recent years the first author presented a new theory of the study of asymptotic properties of the solutions to difference equations. This theory is based mainly on the examination of the behavior of the iterated remainder operator and on the application of asymptotic difference pairs. This approach allows us to control the degree of approximation. The theory was formed in three stages:

(S1) the approximation of solutions with accuracy $\mathrm{o}(1)$, (papers [27,28]),

(S2) the approximation with accuracy $\mathrm{o}(n^s)$, $s \in (-\infty, 0]$, (papers [29,30,32,34,35]),

(S3) the approximation with accuracy determined by a certain asymptotic difference pair (papers [33,36]).

In papers [34,35] this new theory was applied to the study of neutral type equations. The application to the discrete Volterra equations was presented in [38] (stage (S1)) and in [37] (stage (S2)). In this paper we continue those investigations by applying asymptotic difference pairs and we generalize the main results from [27–31,33,37,38]. Moreover, we generalize some earlier results, for example, from [13,14,25,42,49].

The paper is organized as follows. In Section 2, we introduce notation and terminology. In Section 3, in Theorems 3.1 and 3.2, we obtain our main results. In Section 4, we present some consequences of our main results. These consequences concern asymptotically polynomial solutions. In the next section we use our results to investigate bounded solutions. In Section 6, we give some remarks. In particular, we present some tests that are helpful in verifying whether a given kernel K fulfills the assumptions of the main theorems. In the last section we present some applications.

2 Notation and terminology

In the paper we regard $\mathbb{N} \times \mathbb{R}$ as a metric subspace of the Euclidean plane $\mathbb{R}^2$. The space $\mathbb{R}^{\mathbb{N}}$ of all real sequences we denote also by SQ. Moreover

$$\mathrm{SQ}^* = \{x \in \mathrm{SQ} : \ x_n \neq 0 \ \text{ for any } n\}.$$

For integers p, q such that $0 \leq p \leq q$, we define

$$\mathbb{N}(p) = \{p, p+1, p+2, \dots\}, \qquad \mathbb{N}(p,q) = \{p, p+1, \dots, q\}.$$

We use the symbols
$$\mathrm{Sol}(\mathrm{E}), \quad \mathrm{Sol}_p(\mathrm{E})$$
to denote the set of all solutions of (E), and the set of all p-solutions of (E) respectively. If x, y in SQ, then
$$xy \quad \text{and} \quad |x|$$
denotes the sequences defined by $xy(n) = x_n y_n$ and $|x|(n) = |x_n|$ respectively. Moreover
$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|.$$
If there exists a positive constant λ such that $x_n \geq \lambda$ for any n, then we write
$$x \gg 0.$$
Let $a, b, w \in \mathrm{SQ}$, $p \in \mathbb{N}$, $t \in [1, \infty)$, $X \subset \mathrm{SQ}$. We will use the following notations
$$\mathrm{Fin}(p) = \{x \in \mathrm{SQ} : x_n = 0 \text{ for } n \geq p\}, \qquad \mathrm{Fin} = \bigcup_{p=1}^{\infty} \mathrm{Fin}(p).$$
$$\mathrm{o}(1) = \{x \in \mathrm{SQ} : \ x \text{ is convergent to zero}\}, \quad \mathrm{O}(1) = \{x \in \mathrm{SQ} : \ x \text{ is bounded}\},$$
$$\mathrm{o}(a) = \{ax : \ x \in \mathrm{o}(1)\} + \mathrm{Fin}, \quad \mathrm{O}(a) = \{ax : \ x \in \mathrm{O}(1)\} + \mathrm{Fin},$$
$$\mathrm{O}(w, \sigma) = \{y \in \mathrm{SQ} : y \circ \sigma \in \mathrm{O}(w)\},$$
$$\mathrm{A}(t) := \left\{a \in \mathrm{SQ} : \sum_{n=1}^{\infty} n^{t-1}|a_n| < \infty\right\}, \quad \mathrm{A}(\infty) = \bigcap_{t\in[1,\infty)} \mathrm{A}(t),$$
$$\Delta^{-m}b = \{y \in \mathrm{SQ} : \ \Delta^m y = b\}, \quad \Delta^{-m}X = \{y \in \mathrm{SQ} : \ \Delta^m y \in X\},$$
$$\mathrm{Pol}(m-1) = \mathrm{Ker}\Delta^m = \Delta^{-m}0.$$
Note that $\mathrm{Pol}(m-1)$ is the space of all polynomial sequences of degree less than m. Moreover for any $y \in \Delta^{-m}b$ we have
$$\Delta^{-m}b = y + \mathrm{Pol}(m-1).$$
Note also that
$$\bigcup_{\lambda\in(0,1)} \mathrm{O}(\lambda^n) \subset \mathrm{A}(\infty) \subset \bigcap_{s\in\mathbb{R}} \mathrm{o}(n^s).$$
For $L : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, $A \subset \mathrm{SQ}$, and $t \in [1, \infty]$ we define
$$L' \in \mathrm{SQ}, \quad L'(n) = \sum_{k=1}^{n} |L(n,k)|, \quad \mathrm{K}(A) = \left\{L \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}} : L' \in A\right\}, \quad \mathrm{K}(t) = \mathrm{K}(\mathrm{A}(t)).$$
For a subset Y of a metric space X and $\varepsilon > 0$ we define an *ε-framed* interior of Y by
$$\mathrm{Int}(Y, \varepsilon) = \{x \in X : \overline{\mathrm{B}}(x, \varepsilon) \subset Y\}$$
where $\overline{\mathrm{B}}(x, \varepsilon)$ denotes a closed ball of radius ε centered at x. We say that a subset U of X is a *uniform neighborhood* of a subset Z of X, if there exists a positive number ε such that $Z \subset \mathrm{Int}(U, \varepsilon)$. We say that a function $h : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is unbounded at a point $p \in [-\infty, \infty]$ if there exist sequences $x : \mathbb{N} \to \mathbb{R}$ and $n : \mathbb{N} \to \mathbb{N}$ such that
$$\lim_{k\to\infty} x_k = p, \quad \lim_{k\to\infty} n_k = \infty \quad \text{and} \quad \lim_{k\to\infty} h(n_k, x_k) = \infty.$$

Let $h : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, $x \in \mathrm{SQ}$. We will use the following notations

$$\mathrm{U}(h) = \{p \in [-\infty, \infty] : \ h \text{ is unbounded at } p\},$$

$$\mathrm{L}(x) = \{p \in [-\infty, \infty] : p \text{ is a limit point of } x\}.$$

Let $g : [0, \infty) \to [0, \infty)$ and $w \in \mathrm{SQ}^*$, we say that f is (g, w)-*dominated* if

$$|f(n,t)| \leq g(|tw_n^{-1}|) \quad \text{for } (n,t) \in \mathbb{N} \times \mathbb{R}. \tag{2.1}$$

We say that a function $g : [0, \infty) \to [0, \infty)$ is of *Bihari type* if

$$\int_1^\infty \frac{dt}{g(t)} = \infty. \tag{2.2}$$

2.1 Remainder operator

Let

$$\mathrm{S}(m) = \left\{ a \in \mathrm{SQ} : \ \text{the series } \sum_{i_1=1}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} a_{i_m} \text{ is convergent} \right\}.$$

For any $a \in \mathrm{S}(m)$ we define the sequence $r^m(a)$ by

$$r^m(a)(n) = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} a_{i_m}.$$

Then $\mathrm{S}(m)$ is a linear subspace of $\mathrm{o}(1)$, $r^m(a) \in \mathrm{o}(1)$ for any $a \in \mathrm{S}(m)$ and

$$r^m : \mathrm{S}(m) \to \mathrm{o}(1)$$

is a linear operator which we call the *remainder operator of order m*. The value $r^m(a)(n)$ we denote also by $r_n^m(a)$ or simply $r_n^m a$. If $a \in \mathrm{A}(m)$, then $a \in \mathrm{S}(m)$ and

$$r^m(a)(n) = \sum_{j=n}^{\infty} \binom{m-1+j-n}{m-1} a_j. \tag{2.3}$$

for any $n \in \mathbb{N}$. The following lemma is a consequence of [31, Lemma 3.1, Lemma 4.2, and Lemma 4.8].

Lemma 2.1. *Assume $a \in \mathrm{A}(m)$, $u \in \mathrm{O}(1)$, $k \in \{0, 1, \ldots, m\}$, and $p \in \mathbb{N}$. Then*

(a) $\mathrm{O}(a) \subset \mathrm{A}(m) \subset \mathrm{o}(n^{1-m})$, $|r^m(ua)| \leq \|u\| r^m|a|$, $\Delta r^m|a| \leq 0$,

(b) $|r_p^m a| \leq r_p^m|a| \leq \sum_{n=p}^\infty n^{m-1}|a_n|$, $r^k a \in \mathrm{A}(m-k)$,

(c) $\Delta^m r^m a = (-1)^m a$, $r^m \mathrm{Fin}(p) = \mathrm{Fin}(p) = \Delta^m \mathrm{Fin}(p)$.

For more information about the remainder operator see [31].

2.2 Asymptotic difference pairs

We say that a pair (A, Z) of linear subspaces of SQ is an *asymptotic difference pair* of order m or, simply, *m-pair* if

$$\mathrm{Fin} + Z \subset Z, \quad \mathrm{O}(1)A \subset A, \quad A \subset \Delta^m Z.$$

We say that an m-pair (A, Z) is *evanescent* if $Z \subset \mathrm{o}(1)$. If $A \subset \mathrm{SQ}$ and (A, A) is an m-pair, then we say that A is an m-space. We will use the following lemma.

Lemma 2.2. *Assume (A, Z) is an m-pair, $a, b, x \in \mathrm{SQ}$, and $W \subset \mathrm{SQ}$. Then*

(a) if $Z + W \subset W$ and $b - a \in A$, then $W \cap \Delta^{-m}b + Z = W \cap \Delta^{-m}a + Z$,

(b) if $a \in A$ and $\Delta^m x \in \mathrm{O}(a) + b$, then $x \in \Delta^{-m}b + Z$,

(c) if $Z \subset \mathrm{o}(1)$, then $A \subset \mathrm{A}(m)$ and $r^m A \subset Z$.

Proof. Let $y \in W \cap \Delta^{-m}a$. Then

$$\Delta^m y - b = a - b \in A \subset \Delta^m Z.$$

Hence $\Delta^m y - b = \Delta^m z$ for some $z \in Z$. Therefore $\Delta^m(y - z) = b$ and we obtain $y - z \in \Delta^{-m}b$. Moreover $y - z \in W + Z \subset W$. Hence $y - z \in W \cap \Delta^m b$. If $z_1 \in Z$, then

$$y + z_1 = y - z + z + z_1 \in W \cap \Delta^{-m}b + Z$$

and we obtain

$$W \cap \Delta^{-m}a + Z \subset W \cap \Delta^{-m}b + Z.$$

Since A is a linear space, the reverse inclusion follows by interchanging the letters a and b in the previous part of the proof. Hence we get (a). For the proof of (b) see [33, Lemma 3.7]. (c) is a consequence of [33, Remark 3.4]. □

Example 2.3. Assume $s \in \mathbb{R}$, $(s+1)(s+2)\dots(s+m) \neq 0$, and $t \in (-\infty, m-1]$. Then

$$(\mathrm{o}(n^s), \mathrm{o}(n^{s+m})), \quad (\mathrm{O}(n^s), \mathrm{O}(n^{s+m})), \quad (\mathrm{A}(m-t), \mathrm{o}(n^t))$$

are m-pairs.

Example 2.4. If $\lambda \in (1, \infty)$, then $\mathrm{o}(\lambda^n)$ and $\mathrm{O}(\lambda^n)$ are m-spaces.

Example 2.5. If $k \in \mathbb{N}(0, m-1)$, then $(\mathrm{A}(m-k), \Delta^{-k}\mathrm{o}(1))$ is an asymptotic m-pair.

Example 2.6. Assume $s \in (-\infty, -m)$, $t \in (-\infty, 0]$, and $u \in [1, \infty)$. Then

$$(\mathrm{o}(n^s), \mathrm{o}(n^{s+m})), \quad (\mathrm{O}(n^s), \mathrm{O}(n^{s+m})), \quad (\mathrm{A}(m-t), \mathrm{o}(n^t)), \quad (\mathrm{A}(m+u), \mathrm{A}(u))$$

are evanescent m-pairs.

Example 2.7. If $\lambda \in (0, 1)$, then $\mathrm{o}(\lambda^n)$, $\mathrm{O}(\lambda^n)$, and $\mathrm{A}(\infty)$ are evanescent m-spaces.

For more information about difference pairs see [33].

2.3 Fixed point lemma

We will use the following fundamental lemma.

Lemma 2.8. *Assume* $y \in \mathrm{SQ}$, $\rho \in \mathrm{o}(1)$, *and*

$$S = \{x \in \mathrm{SQ} : |x - y| \leq |\rho|\}.$$

Then the formula $d(x,y) = \sup_{n \in \mathbb{N}} |x_n - y_n|$ *defines a metric on* S *such that any continuous map* $H : S \to S$ *has a fixed point.*

Proof. The assertion is a consequence of [32, Theorem 3.3 and Theorem 3.1]. □

3 The set of solutions

In this section, in Theorems 3.1 and 3.2, we obtain our main results.

For a sequence $x \in \mathrm{SQ}$ we define sequences $F(x)$ and $G(x)$ by

$$F(x)(k) = f(k, x_{\sigma(k)}), \qquad G(x)(n) = \sum_{k=1}^{n} K(n,k) f(k, x_{\sigma(k)}). \tag{3.1}$$

Let $K \in \mathrm{K}(m)$ and $p \in \mathbb{N}$. We say that a sequence $y \in \mathrm{SQ}$ is (K, f, p)-*regular* if there exist a subset U of $\mathbb{R}$ and $M > 0$ such that

$$y(\mathbb{N}) \subset \mathrm{Int}(U, Mr_p^m K'), \quad |f(n,t)| \leq M \quad \text{for any} \quad (n,t) \in \mathbb{N} \times U, \tag{3.2}$$

and the restriction $f|\mathbb{N} \times U$ is continuous. We say that y is f-*regular* if there exist a uniform neighborhood U of $y(\mathbb{N})$ such that the restriction $f|\mathbb{N} \times U$ is continuous and bounded.

We say that a subset W of SQ is (f,σ)-*ordinary* if for any $y \in W$ the sequence $F(y)$ is bounded. If any $y \in W$ is f-regular, then we say that W is f-regular.

Theorem 3.1. *Assume* (A, Z) *is an* m*-pair,* $K \in \mathrm{K}(A)$*, and* $W \subset \mathrm{SQ}$*. It follows that*

- $(A1)$ *if* W *is* (f,σ)*-ordinary, then* $W \cap \mathrm{Sol}(\mathrm{E}) \subset \Delta^{-m} b + Z$.

Moreover, assume that the pair (A, Z) *is evanescent,* $y \in \Delta^{-m} b$*, and* $p \in \mathbb{N}$*. It follows that*

- $(A2)$ *if* y *is* (K, f, p)*-regular, then* $y \in \mathrm{Sol}_p(\mathrm{E}) + Z$,
- $(A3)$ *if* y *is* f*-regular, then* $y \in \mathrm{Sol}(\mathrm{E}) + Z$,
- $(A4)$ *if* W *is* f*-regular and* $Z + W \subset W$*, then* $W \cap \mathrm{Sol}(\mathrm{E}) + Z = W \cap \Delta^{-m} b + Z$,
- $(A5)$ *if* $Z + W \subset W$ *and* f *is continuous and bounded, then*

$$W \cap \mathrm{Sol}_1(\mathrm{E}) + Z = W \cap \mathrm{Sol}(\mathrm{E}) + Z = W \cap \Delta^{-m} b + Z.$$

Theorem 3.2. *Assume* (A, Z) *is an* m*-pair,* $K \in \mathrm{K}(A)$*,* $w \in \mathrm{SQ}^*$*,* $g : [0, \infty) \to [0, \infty)$*, and* f *is* (g, w)*-dominated. It follows that*

- $(B1)$ *if* g *is locally bounded, then* $\mathrm{O}(w, \sigma) \cap \mathrm{Sol}(\mathrm{E}) \subset \Delta^{-m} b + Z$,
- $(B2)$ *if* g *is nondecreasing,* $\sigma(n) \leq n$ *for large* n*,* $K \in \mathrm{K}(1)$*,* $b \in \mathrm{A}(1)$*,* $w^{-1} \in \mathrm{O}(n^{1-m})$*, and* g *is of Bihari type, then*

$$\mathrm{Sol}(\mathrm{E}) \subset \Delta^{-m} b + Z.$$

Moreover, assume that the pair (A, Z) *is evanescent,* $y \in \Delta^{-m}b$, $W \subset \mathrm{O}(w, \sigma)$, $L, M > 0$, $p \in \mathbb{N}$, f *is continuous, and* $Z + W \subset W$*. It follows that*

(B3) *if* $g[0, L] \subset [0, M]$ *and* $|y \circ \sigma| \leq L|w| - Mr_p^m K'$, *then* $y \in \mathrm{Sol}_p(\mathrm{E}) + Z$.

Moreover, assume that g *is locally bounded and* $|w| \gg 0$*. It follows that*

(B4a) $W \cap \Delta^{-m}b + Z = W \cap \mathrm{Sol}(\mathrm{E}) + Z$,

(B4b) $\mathrm{O}(w, \sigma) \cap \Delta^{-m}b + Z = \mathrm{O}(w, \sigma) \cap \mathrm{Sol}(\mathrm{E}) + Z$,

(B4c) *if* $w \circ \sigma \in \mathrm{O}(w)$, *then* $\mathrm{O}(w) \cap \Delta^{-m}b + Z = \mathrm{O}(w) \cap \mathrm{Sol}(\mathrm{E}) + Z$.

Moreover, assume that g *is bounded. It follows that*

(B5a) $W \cap \Delta^{-m}b + Z = W \cap \mathrm{Sol}(\mathrm{E}) + Z = W \cap \mathrm{Sol}_1(\mathrm{E}) + Z$,

(B5b) $\mathrm{O}(w, \sigma) \cap \Delta^{-m}b + Z = \mathrm{O}(w, \sigma) \cap \mathrm{Sol}(\mathrm{E}) + Z = \mathrm{O}(w, \sigma) \cap \mathrm{Sol}_1(\mathrm{E}) + Z$,

(B5c) *if* $w \circ \sigma \in \mathrm{O}(w)$, *then* $\mathrm{O}(w) \cap \Delta^{-m}b + Z = \mathrm{O}(w) \cap \mathrm{Sol}(\mathrm{E}) + Z = \mathrm{O}(w) \cap \mathrm{Sol}_1(\mathrm{E}) + Z$.

The following, final, theorem is a curiosity. It concerns all the solutions of equation (E); moreover there are no conditions placed on the function f. This theorem generalizes [33, Theorem 4.2].

Theorem 3.3. *Assume* (A, Z) *is an* m*-pair,* $K \in \mathrm{K}(A)$, *and* $x \in \mathrm{Sol}(\mathrm{E})$. *Then*

$$x \in \Delta^{-m}b + Z \quad \text{or} \quad \mathrm{L}(x) \cap \mathrm{U}(f) \neq \emptyset.$$

3.1 The proof of Theorem 3.1

(A1) Assume W is (f, σ)-ordinary and $x \in W \cap \mathrm{Sol}(\mathrm{E})$. Let $M = \|F(x)\|$. By (3.1), $|G(x)| \leq MK'$. Hence

$$\Delta^m x \in G(x) + b + \mathrm{Fin} \subset \mathrm{O}(K') + b + \mathrm{Fin} = \mathrm{O}(K') + b.$$

Moreover $K' \in A$. Therefore, using Lemma 2.2, we obtain $x \in \Delta^{-m}b + Z$.

(A2) Choose a positive constant M and a subset U of $\mathbb{R}$ such that (3.2) is satisfied and f is continuous on $\mathbb{N} \times U$. Let $a = K'$. Define $\rho \in \mathrm{SQ}$ and $S \subset \mathrm{SQ}$ by

$$\rho_n = \begin{cases} Mr_n^m a & \text{for } n \geq p, \\ 0 & \text{for } n < p, \end{cases} \qquad S = \{x \in \mathrm{SQ} : |x - y| \leq \rho\}. \tag{3.3}$$

Since the sequence $r^m|a|$ is nonincreasing, we have $\rho_n \leq \rho_p$ for any n. Assume $x \in S$. If $k \in \mathbb{N}$, then $|x_{\sigma(k)} - y_{\sigma(k)}| \leq \rho_{\sigma(k)} \leq \rho_p$ and we obtain

$$x_{\sigma(k)} \in \overline{\mathrm{B}}(y_{\sigma(k)}, \rho_p) \subset U.$$

Hence $|f(k, x_{\sigma(k)})| \leq M$. Therefore, for $n \in \mathbb{N}$, we get

$$|G(x)(n)| = \left| \sum_{k=1}^{n} K(n,k) f(k, x_{\sigma(k)}) \right| \leq \sum_{k=1}^{n} |K(n,k)||f(k, x_{\sigma(k)})| \leq Ma_n.$$

Thus, for any $x \in S$, we have $Gx \in \mathrm{O}(a) \subset A \subset \mathrm{A}(m)$. Let

$$H : S \to \mathrm{SQ}, \qquad H(x)(n) = \begin{cases} y_n & \text{for } n < p \\ y_n + (-1)^m r_n^m Gx & \text{for } n \geq p. \end{cases} \tag{3.4}$$

If $x \in S$ and $n \geq p$, then

$$|H(x)(n) - y_n| = |r_n^m Gx| \leq r_n^m |Gx| \leq M r_n^m a = \rho_n.$$

Hence $HS \subset S$. Let $\varepsilon > 0$. Choose $q \in \mathbb{N}$ and $\beta > 0$ such that

$$M \sum_{n=q}^{\infty} n^{m-1} a_n < \varepsilon \quad \text{and} \quad \beta \sum_{n=p}^{q} n^{m-1} a_n < \varepsilon. \tag{3.5}$$

Let

$$D = \{(n,t) \in \mathbb{N} \times \mathbb{R} : n \in \mathbb{N}(p,q) \quad \text{and} \quad |t - y_{\sigma(n)}| \leq \rho_n\}.$$

Then D is a compact subset of $\mathbb{R}^2$. Hence f is uniformly continuous on D and there exists $\delta > 0$ such that if $(n,s),(n,t) \in D$ and $|s - t| < \delta$, then

$$|f(n,s) - f(n,t)| < \beta.$$

Let $x, z \in S$, $\|x - z\| < \delta$. Using Lemma 2.1 we obtain

$$\begin{aligned} \|Hx - Hz\| &= \|r^m(Gx - Gz)\| = \sup_{n \geq p} |r_n^m (Gx - Gz)| \leq \sup_{n \geq p} r_n^m |Gx - Gz| \\ &= r_p^m |Gx - Gz| \leq \sum_{n=p}^{\infty} n^{m-1} |G(x)(n) - G(z)(n)| \\ &\leq \sum_{n=p}^{q} n^{m-1} |G(x)(n) - G(z)(n)| + \sum_{n=q}^{\infty} n^{m-1} |G(x)(n) - G(z)(n)| \\ &\leq \beta \sum_{n=p}^{q} n^{m-1} a_n + \sum_{n=q}^{\infty} n^{m-1} |G(x)(n)| + \sum_{n=q}^{\infty} n^{m-1} |G(z)(n)| \\ &\leq \varepsilon + M \sum_{n=q}^{\infty} n^{m-1} a_n + M \sum_{n=q}^{\infty} n^{m-1} a_n \leq 3\varepsilon. \end{aligned}$$

Hence the map $H : S \to S$ is continuous. By Lemma 2.8, there exists an $x \in S$ such that $Hx = x$. Then, for $n \geq p$, we get $x_n = y_n + (-1)^m r_n^m Gx$. Hence

$$x - y - (-1)^m r^m Gx \in \mathrm{Fin}(p). \tag{3.6}$$

Therefore, by Lemma 2.1,

$$\Delta^m x - b - Gx \in \Delta^m \mathrm{Fin}(p) = \mathrm{Fin}(p).$$

Thus $x \in \mathrm{Sol}_p(\mathrm{E})$. Moreover, $Gx \in \mathrm{O}(a) \subset A$. By (3.6), we have

$$y \in x + r^m A + \mathrm{Fin}(p) \subset x + Z \subset \mathrm{Sol}_p(\mathrm{E}) + Z.$$

(A3) Now, we assume that y is f-regular. Choose a uniform neighborhood U of $y(\mathbb{N})$ such that the restriction $f|\mathbb{N} \times U$ is continuous and bounded. There exists a positive constant c such that $y(\mathbb{N}) \subset \mathrm{Int}(U, c)$. Let

$$M = \sup\{|f(n,t)| : n \in \mathbb{N}, t \in U\}.$$

Since $r^m K' \in \mathrm{o}(1)$, there exists an index p such that $Mr_p^m K' \le c$. Then

$$y(\mathbb{N}) \subset \mathrm{Int}(U,c) \subset \mathrm{Int}(U, Mr_p^m K') \subset U.$$

This means that y is (K, f, p)-regular. By (A2), we get $y \in \mathrm{Sol}_p(\mathrm{E}) + Z \subset \mathrm{Sol}(\mathrm{E}) + Z$.
(A4) Now, we assume that W is f-regular and $Z + W \subset W$. Let

$$S = \mathrm{Sol}(\mathrm{E}), \qquad Y = \Delta^{-m} b.$$

Obviously, W is (f, σ)-ordinary. If $w \in W \cap S$, then, by (A1), $w = y + z$ for some $y \in Y$ and $z \in Z$. Hence $y = -z + w \in Z + W \subset W$. Therefore $w = y + z \in W \cap Y + Z$ and we obtain

$$W \cap S + Z \subset W \cap Y + Z.$$

If $w \in W \cap Y$, then, by (A3), $w = x + z$ for some $x \in S$ and $z \in Z$. Hence $x = -z + w \in Z + W \subset W$. Therefore $w = x + z \in W \cap S + Z$ and we obtain

$$W \cap Y + Z \subset W \cap S + Z.$$

(A5) Now we assume that f is continuous and bounded and $Z + W \subset W$. By (A4) we have

$$W \cap \mathrm{Sol}(\mathrm{E}) + Z = W \cap \Delta^{-m} b + Z.$$

Since $\mathrm{Sol}_1(\mathrm{E}) \subset \mathrm{Sol}(\mathrm{E})$, we get

$$W \cap \mathrm{Sol}_1(\mathrm{E}) + Z \subset W \cap \Delta^{-m} b + Z.$$

Let $M = \sup\{|f(n,t)| : (n,t) \in \mathbb{N} \times \mathbb{R}\}$ and let $U = \mathbb{R}$. Then for any $y \in \mathrm{SQ}$ we have

$$y(\mathbb{N}) \subset \mathbb{R} = \mathrm{Int}(U, Mr_1^m K').$$

Since f is continuous on $\mathbb{R}$, any $y \in \mathrm{SQ}$ is $(K, f, 1)$-regular. Hence, by (A2), we obtain

$$W \cap \Delta^{-m} b + Z \subset W \cap \mathrm{Sol}_1(\mathrm{E}) + Z.$$

3.2 The proof of Theorem 3.2

We will use the following three lemmas.

Lemma 3.4 ([35, Lemma 4.1]). *Assume $\alpha, u \in \mathrm{SQ}$ are nonnegative, $p \in \mathbb{N}$, $g : [0,\infty) \to [0,\infty)$,*

$$0 \le c < \beta, \quad g(c) > 0, \quad u_n \le c + \sum_{j=p}^{n-1} \alpha_j g(u_j) \quad \text{for } n \ge p, \quad \sum_{j=1}^{\infty} \alpha_j \le \int_c^\beta \frac{dt}{g(t)},$$

and g is nondecreasing. Then $u_n \le \beta$ for $n \ge p$.

Lemma 3.5 ([30, Lemma 7.3]). *If x is a sequence of real numbers, $m \in \mathbb{N}$ and $p \in \mathbb{N}(m)$ then there exists a positive constant $L = L(x, p, m)$ such that*

$$|x_n| \le n^{m-1} \left(L + \sum_{i=p}^{n-1} |\Delta^m x_i| \right) \quad \text{for } n \ge p.$$

Lemma 3.6. *Let* $w \in \mathrm{SQ}$

(1) *if* $|w| \gg 0$, *then* $\mathrm{O}(w) + \mathrm{O}(1) \subset \mathrm{O}(w)$, *and* $\mathrm{O}(w, \sigma) + \mathrm{O}(1) \subset \mathrm{O}(w, \sigma)$,

(2) *if* $y \in \mathrm{O}(w, \sigma)$, *then* $\mathrm{O}(y) \subset \mathrm{O}(w, \sigma)$,

(3) *if* $w \circ \sigma \in \mathrm{O}(w)$, *then* $\mathrm{O}(w) \subset \mathrm{O}(w, \sigma)$.

Proof. Let $y \in \mathrm{O}(w)$ and $u \in \mathrm{O}(1)$. Choose positive $\delta, L, M \in \mathbb{R}$ such that

$$|w_n| \geq \delta, \quad |u_n| \leq L, \quad \text{and} \quad |y_n| \leq M|w_n|$$

for any n. Then

$$|y_n + u_n| \leq |y_n| + |u_n| \leq M|w_n| + L = M|w_n| + L\delta^{-1}\delta \leq M|w_n| + L\delta^{-1}|w_n| = (M + L\delta^{-1})|w_n|$$

for any n. Hence $\mathrm{O}(w) + \mathrm{O}(1) \subset \mathrm{O}(w)$. Similarly $\mathrm{O}(w, \sigma) + \mathrm{O}(1) \subset \mathrm{O}(w, \sigma)$. Assume $y \in \mathrm{O}(w, \sigma)$ and $x \in \mathrm{O}(y)$. There exist positive constants M, P such that

$$|y(\sigma(n))| \leq M|w_n|, \quad |x_n| \leq P|y_n|$$

for large n. Then $|x(\sigma(n))| \leq P|y(\sigma(n))| \leq PM|w_n|$ for large n. Hence $x \in \mathrm{O}(w, \sigma)$ and we get (2). (3) is a consequence of (2). □

Now we start the proof of Theorem 3.2.

(B1) Assume g is locally bounded. Let P be a positive constant. For any $t \in [0, P]$ there exist a neighborhood U_t of t and a positive constant Q_t such that $|g(s)| \leq Q_t$ for any $s \in U_t$. By compactness of $[0, P]$ we can choose $t_1, t_2, \ldots, t_n$ such that $[0, P] \subset U_{t_1} \cup U_{t_2} \cup \cdots \cup U_{t_n}$. Then

$$g(s) \leq Q = \max\{Q_{t_1}, \ldots, Q_{t_n}\} \tag{3.7}$$

for any $s \in [0, P]$. Let $y \in \mathrm{O}(w, \sigma)$. Then $y \circ \sigma \in \mathrm{O}(w)$. Since $w \in \mathrm{SQ}^*$, there exists a positive constant P such that

$$|y_{\sigma(n)}| \leq P|w_n| \tag{3.8}$$

for any n. Using (2.1), (3.8), and (3.7) we get

$$|F(x)(n)| = |f(n, y_{\sigma(n)})| \leq g\left(\frac{|y_{\sigma(n)}|}{|w_n|}\right) \leq Q.$$

Hence the set $\mathrm{O}(w, \sigma)$ is (f, σ)-ordinary and, by Theorem 3.1 (A1), we obtain

$$\mathrm{O}(w, \sigma) \cap \mathrm{Sol(E)} \subset \Delta^{-m} b + Z.$$

(B2) Assume x is a solution of (E). Since $K \in \mathrm{K}(1)$, we have $K' \in \mathrm{A}(1)$. Hence

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |K(i, j)| = \sum_{i=1}^{\infty} \sum_{j=1}^{i} |K(i, j)| = \sum_{i=1}^{\infty} K'(i) < \infty.$$

Choose $M > 0$ such that $|w_n^{-1}| \leq M n^{1-m}$. For $j \in \mathbb{N}$ let

$$u_j = \left| x_{\sigma(j)} w_j^{-1} \right|, \qquad \alpha_j = M \sum_{i=j}^{\infty} |K(i, j)|.$$

Using the condition: $K(i,j) = 0$ for $i < j$ we obtain

$$\sum_{j=1}^{\infty} \alpha_j = M \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} |K(i,j)| \leq M \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |K(i,j)| = M \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |K(i,j)|$$
$$= M \sum_{i=1}^{\infty} \sum_{j=1}^{i} |K(i,j)| = M \sum_{i=1}^{\infty} K'(i) < \infty.$$

By Lemma 3.5, there exists a positive constant L such that

$$|x_{\sigma(n)}| \leq \sigma(n)^{m-1} \left(L + \sum_{i=p}^{\sigma(n)-1} |\Delta^m x_i| \right) \leq n^{m-1} \left(L + \sum_{i=p}^{n-1} |\Delta^m x_i| \right).$$

Let $c = ML + M \sum_{i=1}^{\infty} |b_i|$. Then

$$u_n = \left| x_{\sigma(n)} w_n^{-1} \right| \leq ML + M \sum_{i=1}^{n-1} |\Delta^m x_i| = ML + M \sum_{i=1}^{n-1} \left| b_i + \sum_{j=1}^{i} K(i,j) f\left(j, x_{\sigma(j)}\right) \right|$$
$$\leq ML + M \sum_{i=1}^{\infty} |b_i| + M \sum_{i=1}^{n-1} \sum_{j=1}^{i} |K(i,j)| g(u_j) = c + M \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} |K(i,j)| g(u_j)$$
$$= c + M \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} |K(i,j)| g(u_j) \leq c + M \sum_{j=1}^{n-1} \sum_{i=1}^{\infty} |K(i,j)| g(u_j)$$
$$= c + \sum_{j=1}^{n-1} \sum_{i=j}^{\infty} M |K(i,j)| g(u_j) = c + \sum_{j=1}^{n-1} \alpha_j g(u_j).$$

Hence, by Lemma 3.4, the sequence u is bounded. Therefore, there exists a constant $Q > 1$ such that $g(u_i) \leq Q$ for any i and we get

$$\left| f(i, x_{\sigma(i)}) \right| \leq g\left(\left| x_{\sigma(i)} w_i^{-1} \right| \right) = g(u_i) \leq Q$$

for any i. Hence

$$\left| \sum_{i=1}^{n} K(n,i) f(i, x_{\sigma(i)}) \right| \leq Q \sum_{i=1}^{n} |K(n,i)| = QK'_n.$$

For large n we have

$$\Delta^m x_n = b_n + \sum_{i=1}^{n} K(n,i) f(i, x_{\sigma(i)}).$$

Hence $\Delta^m x \in b + \mathrm{O}(K')$ and $K' \in A$. By Lemma 2.2, we have $x \in \Delta^{-m} b + Z$.

(B3) Let $a = K'$. Define ρ and S by (3.3). Let $x \in S$. Using the inequality

$$|y \circ \sigma| \leq L|w| - M r_p^m a, \quad \text{we get}$$

$$\left| \frac{x_{\sigma(n)}}{w_n} \right| = \left| \frac{x_{\sigma(n)} - y_{\sigma(n)} + y_{\sigma(n)}}{|w_n|} \right| \leq \frac{|x_{\sigma(n)} - y_{\sigma(n)}| + |y_{\sigma(n)}|}{|w_n|} \leq \frac{M r_p^m a + |y_{\sigma(n)}|}{w_n} \leq L$$

for any n. Using (2.1) and inclusion $g[0, L] \subset [0, M]$, we have

$$|F(x)(n)| = |f(n, x_{\sigma(n)})| \leq g\left(\frac{|x_{\sigma(n)}|}{w_n} \right) \leq M$$

for any n. Therefore

$$|G(x)(n)| = \left|\sum_{k=1}^{n} K(n,k)F(x)(k)\right| \le \sum_{k=1}^{n} M|K(n,k)| \le Ma_n.$$

Now, repeating the second part of the proof of Theorem 3.1 (A2), we obtain

$$y \in \mathrm{Sol}_p(\mathrm{E}) + Z.$$

(B4a) Now, we assume that g is locally bounded, $|w| \gg 0$, $W \subset \mathrm{O}(w,\sigma)$, and $Z + W \subset W$. Let $y \in W \cap \Delta^{-m}b$. Choose positive constants P, λ such that $|y \circ \sigma| \le P|w|$ and $|w| > \lambda$. Let

$$L_1 = P + 1 \quad \text{and} \quad \alpha = \inf\{L_1|w_n| - |y_{\sigma(n)}| : n \in \mathbb{N}\}.$$

Then

$$L_1|w_n| - |y_{\sigma(n)}| = P|w_n| - |y_{\sigma(n)}| + |w_n| \ge P|w_n| - |y_{\sigma(n)}| + \lambda \ge \lambda$$

for any n. Hence $\alpha \ge \lambda > 0$. Similarly as in (3.7) there exists a positive constant M_1 such that $g[0, L_1] \subset [0, M_1]$. Since $\lim_{n\to\infty} r_n^m|a| = 0$, there exists an index p such that

$$M_1 r_p^m |a| \le \alpha.$$

Then $M_1 r_p^m|a| \le L_1 w_n - |y_{\sigma(n)}|$ for any n. Hence, by (B3), $y \in \mathrm{Sol}_p(\mathrm{E}) + Z$ and we obtain

$$W \cap \Delta^{-m}b \subset \mathrm{Sol}(\mathrm{E}) + Z.$$

By (B1), we have $W \cap \mathrm{Sol}(\mathrm{E}) \subset \Delta^{-m}b + Z$. Using [33, Lemma 4.10] we obtain

$$W \cap \Delta^{-m}b + Z = W \cap \mathrm{Sol}(\mathrm{E}) + Z.$$

(B4b) Since $Z \subset \mathrm{o}(1)$, by Lemma 3.6 (1), we have $\mathrm{O}(w,\sigma) + Z \subset \mathrm{O}(w,\sigma)$. Hence, by (B4a), we get

$$\mathrm{O}(w,\sigma) \cap \Delta^{-m}b + Z = \mathrm{O}(w,\sigma) \cap \mathrm{Sol}(\mathrm{E}) + Z.$$

(B4c) By Lemma 3.6 (1) and (3) we have

$$\mathrm{O}(w) + Z \subset \mathrm{O}(w) \quad \text{and} \quad \mathrm{O}(w) \subset \mathrm{O}(w,\sigma).$$

Hence (B4c) is a consequence of (B4a).

(B5a) Since $\mathrm{Sol}_1(\mathrm{E}) \subset \mathrm{Sol}(\mathrm{E})$ we have

$$W \cap \mathrm{Sol}_1(\mathrm{E}) + Z \subset W \cap \mathrm{Sol}(\mathrm{E}) + Z. \tag{3.9}$$

Choose $M, \delta \in (0, \infty)$ such that $|g| \le M$ and $|w| \ge \delta$. Let $y \in W \cap \Delta^{-m}b$. Since $y \in \mathrm{O}(w,\sigma)$, there exists a positive P such that $|y \circ \sigma| \le P|w|$. Let

$$L = P + \delta^{-1} M r_1^m K'.$$

Then

$$|y \circ \sigma| \le P|w| = L|w| - \delta^{-1}|w| M r_1^m K' \le L|w| - M r_1^m K'.$$

Moreover $g[0, L] \subset [0, M]$. Hence, by (B3), $y \in \mathrm{Sol}_1(\mathrm{E}) + Z$ and we obtain

$$W \cap \Delta^{-m}b \subset \mathrm{Sol}_1(\mathrm{E}) + Z$$

Let $w \in W \cap \Delta^{-m}b$. Choose $x \in \mathrm{Sol}_1(\mathrm{E})$ and $z \in Z$ such that $w = x + z$. Then

$$x = w - z \in W + Z \subset W.$$

Hence $w \in W \cap \mathrm{Sol}_1(\mathrm{E}) + Z$ and we obtain

$$W \cap \Delta^{-m}b \subset W \cap \mathrm{Sol}_1(\mathrm{E}) + Z. \tag{3.10}$$

By (B4a) we have

$$W \cap \Delta^{-m}b + Z = W \cap \mathrm{Sol}(\mathrm{E}) + Z \tag{3.11}$$

Using (3.9), (3.10), and (3.11) we obtain (B5a).

(B5b) Analogously to the proof of (B4b), we can see that (B5b) is a consequence of (B5a).

(B5c) The assertion is a consequence of (B5a) and Lemma 3.6 (1) and (2).

3.3 The proof of Theorem 3.3

Assume

$$\mathrm{L}(x) \cap \mathrm{U}(f) = \emptyset. \tag{3.12}$$

We will show that the sequence $F(x)$ is bounded. If

$$\limsup_{n\to\infty} F(x)(n) = \limsup_{n\to\infty} f(n, x_{\sigma(n)}) = \infty,$$

then there exists an increasing sequence (n_k) of natural numbers such that

$$\lim_{k\to\infty} f(n_k, x_{\sigma(n_k)}) = \infty.$$

Let $y_k = x_{\sigma(n_k)}$ and let $p \in \mathrm{L}(y)$. There exists a subsequence (y_{k_i}) of (y_k) such that

$$\lim_{i\to\infty} y_{k_i} = p.$$

Then $\lim_{i\to\infty} f(n_{k_i}, y_{k_i}) = \infty$. Hence $p \in \mathrm{U}(f)$. Since $y_k = x_{\sigma(n_k)}$ and $\sigma(n) \to \infty$, we have $\mathrm{L}(y) \subset \mathrm{L}(x)$. Therefore $p \in \mathrm{L}(x)$ which contradicts (3.12). Analogously $\liminf F(x)(n) > -\infty$ and so $F(x)$ is bounded. Since $x \in \mathrm{Sol}(\mathrm{E})$ we have

$$\Delta^m x \in aF(x) + b + \mathrm{Fin} \subset \mathrm{O}(a) + b + \mathrm{Fin} = \mathrm{O}(a) + b$$

and, by Lemma 2.2 (b), we obtain $x \in \Delta^{-m}b + Z$.

4 Asymptotically polynomial solutions

In this section we apply our main results to investigate asymptotically polynomial solutions of equation (E). We assume that $g : [0,\infty) \to [0,\infty)$ and $w \in \mathrm{SQ}^*$.

Let $k \in \mathbb{N}(0,m)$. We say that a sequence φ is *asymptotically polynomial of type* (m,k) if

$$\varphi \in \mathrm{Pol}(m) + \mathrm{o}(n^k).$$

Moreover, if

$$\varphi \in \mathrm{Pol}(m) + \Delta^{-k}\mathrm{o}(1),$$

then we say that φ is *regularly asymptotically polynomial of type* (m,k). Note that, by [30, Lemma 3.1 (b)], we have

$$\Delta^{-k}\mathrm{o}(1) = \{x \in \mathrm{o}(n^k) : \ \Delta^p x \in \mathrm{o}(n^{k-p}) \ \text{ for any } \ p \in \mathbb{N}(0,k)\}.$$

Corollary 4.1. *Assume (A, Z) is an m-pair, $K \in \mathrm{K}(A)$, $b \in \mathrm{A}$, and x is an (f, σ)-ordinary solution of* (E). *Then*

$$x \in \mathrm{Pol}(m-1) + Z.$$

Proof. By Theorem 3.1 (A1), we have $x \in \Delta^{-m}b + Z$. Since $b - 0 \in A$, taking $W = \mathrm{SQ}$ in Lemma 2.2 (a), we obtain $\Delta^{-m}b + Z = \Delta^{-m}0 + Z = \mathrm{Pol}(m-1) + Z$. □

Note that if $k \in \mathbb{N}(0, m-1)$ and $Z \subset \mathrm{o}(n^k)$, then by Corollary 4.1, any (f, σ)-ordinary solution of (E) is asymptotically polynomial of type $(m-1, k)$.

Corollary 4.2. *Assume $s \in (-\infty, m-1]$, $K \in \mathrm{K}(m-s)$, $b \in \mathrm{A}(m-s)$, and x is an (f, σ)-ordinary solution of* (E). *Then*

$$x \in \mathrm{Pol}(m-1) + \mathrm{o}(n^s).$$

Proof. By Example 2.3, $(\mathrm{A}(m-s), \mathrm{o}(n^s))$ is an asymptotic m-pair. Hence the assertion is a consequence of Corollary 4.1. □

Corollary 4.3. *Assume $k \in \mathbb{N}(0, m-1)$, $K \in \mathrm{K}(m-k)$, and $b \in \mathrm{A}(m-k)$. Than any (f, σ)-ordinary solution x of* (E) *is regularly asymptotically polynomial of type $(m-1, k)$.*

Proof. By Example 2.5, $(\mathrm{A}(m-k), \Delta^{-k}\mathrm{o}(1))$ is an asymptotic m-pair. Hence, by Corollary 4.1 we obtain

$$x \in \mathrm{Pol}(m-1) + \Delta^{-k}\mathrm{o}(1).$$ □

Corollary 4.4. *Assume $s \in (-\infty, m-1]$, $K \in \mathrm{K}(m-s)$, $b \in \mathrm{A}(m-s)$. Then for any (f, σ)-ordinary solution x of* (E) *there exist a sequence $\varphi \in \mathrm{Pol}(m-1)$ and $z \in \mathrm{o}(n^s)$ such that $x = \varphi + z$ and $\Delta^p z_n = \mathrm{o}(n^{s-p})$ for any $p \in \mathbb{N}(1, m)$.*

Proof. By [33, Example 5.3], $(\mathrm{A}(m-s), r^m\mathrm{A}(m-s))$ is an m-pair. Hence, by Corollary 4.1, there exist a sequence $\varphi \in \mathrm{Pol}(m-1)$ and $z \in r^m\mathrm{A}(m-s)$ such that $x = \varphi + z$. By [30, Lemma 4.2], we have $\Delta^p z_n = \mathrm{o}(n^{s-p})$ for any $p \in \mathbb{N}(0, m)$. □

Corollary 4.5. *Assume $K \in \mathrm{K}(1)$, $b \in \mathrm{A}(1)$, and x is an (f, σ)-ordinary solution of* (E). *Then there exists a constant $\lambda \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} \frac{\Delta^{m-p-1}x_n}{n^p} = \frac{\lambda}{p!} \tag{4.1}$$

for any $p \in \mathbb{N}(0, m-1)$.

Proof. Taking $k = m-1$ in Corollary 4.3 we obtain

$$x \in \mathrm{Pol}(m-1) + \Delta^{-m+1}\mathrm{o}(1). \tag{4.2}$$

The existence of λ follows from [30, Lemma 3.8]. □

Note that if condition (4.1) is satisfied, then by (4.2), x is regularly asymptotically polynomial of type $(m-1, m-1)$.

Corollary 4.6. *Assume (A, Z) is an m-pair, $K \in \mathrm{K}(A)$, $b \in A$, g is locally bounded, and f is (g, w)-dominated. Then*

$$\mathrm{O}(w, \sigma) \cap \mathrm{Sol}(\mathrm{E}) \subset \mathrm{Pol}(m-1) + Z.$$

Proof. Note that $b - 0 \in A$. Let $W = \mathrm{SQ}$. By Lemma 2.2 (a), we have

$$\Delta^{-m}b + Z = W \cap \Delta^{-m}b + Z = W \cap \Delta^{-m}0 + Z = \mathrm{Pol}(m-1) + Z.$$

Hence the assertion is a consequence of Theorem 3.2 (B1). □

Corollary 4.7. *Assume $s \in (-\infty, m-1]$, $K \in \mathrm{K}(m-s)$, $b \in \mathrm{A}(m-s)$, g is locally bounded, and f is (g, w)-dominated. Then*

$$\mathrm{O}(w,\sigma) \cap \mathrm{Sol}(\mathrm{E}) \subset \mathrm{Pol}(m-1) + \mathrm{o}(n^s).$$

Proof. Since $(\mathrm{A}(m-s),\ \mathrm{o}(n^s))$ is an asymptotic m-pair, the assertion is a consequence of Corollary 4.6. □

Corollary 4.8. *Assume $s \in (-\infty, m-1]$, $K \in \mathrm{K}(m-s)$, $b \in \mathrm{A}(m-s)$, $k \in [s, m-1] \cap \mathbb{N}(0)$, $w_n = n^k$, $\sigma(n) = \mathrm{O}(n)$, g is locally bounded, and f is (g, w)-dominated. Then*

$$\mathrm{O}(n^k) \cap \mathrm{Sol}(\mathrm{E}) \subset \mathrm{Pol}(k) + \mathrm{o}(n^s).$$

Proof. Let $y \in \mathrm{O}(n^k) \cap \mathrm{Sol}(\mathrm{E})$. Choose positive constants Q and L such that

$$\sigma(n) \le Qn \quad \text{and} \quad |y_n| \le Ln^k$$

for large n. Then $|y_{\sigma(n)}| \le L\sigma(n)^k \le LQ^k n^k$. Hence $y \circ \sigma \in \mathrm{O}(n^k) = \mathrm{O}(w_n)$. Therefore $y \in \mathrm{O}(w, \sigma)$ and, by Corollary 4.7, we have $y \in \mathrm{Pol}(m-1) + \mathrm{o}(n^s)$. Choose $\varphi \in \mathrm{Pol}(m-1)$ and $z \in \mathrm{o}(n^s)$ such that $y = \varphi + z$. Then $\varphi = y - z \in \mathrm{O}(n^k)$ and we obtain $\varphi \in \mathrm{Pol}(k)$. □

Corollary 4.9. *Assume $k \in \mathbb{N}(0, m-1)$, $K \in \mathrm{K}(m-k)$, $b \in \mathrm{A}(m-k)$, g is locally bounded, and f is (g, w)-dominated. Then*

$$\mathrm{O}(w,\sigma) \cap \mathrm{Sol}(\mathrm{E}) \subset \mathrm{Pol}(m-1) + \Delta^{-k}\mathrm{o}(1).$$

Proof. Since $(\mathrm{A}(m-k),\ \Delta^{-m}\mathrm{o}(1))$ is an asymptotic m-pair and $b \in A$, we have

$$\Delta^{-k}b + \Delta^{-k}\mathrm{o}(1) = \mathrm{Pol}(m-1) + \Delta^{-k}\mathrm{o}(1).$$

Hence the assertion is a consequence of Corollary 4.6. □

Corollary 4.10. *Assume (A, Z) is an m-pair, $K \in \mathrm{K}(A)$, $b \in A$, $A \subset \mathrm{A}(1)$, g is nondecreasing, $\sigma(n) \le n$ for large n, $n^{m-1} = \mathrm{O}(w_n)$, f is (g, w)-dominated, and g is of Bihari type. Then*

$$\mathrm{Sol}(\mathrm{E}) \subset \mathrm{Pol}(m-1) + Z.$$

Proof. Since $\Delta^{-m}b + Z = \mathrm{Pol}(m-1) + Z$, the assertion is a consequence of Theorem 3.2 (B2). □

Corollary 4.11. *If (A, Z) is an evanescent m-pair, $K \in \mathrm{K}(A)$, $b \in A$, and $\varphi \in \mathrm{Pol}(m-1)$ is f-regular, then $\varphi \in \mathrm{Sol}(\mathrm{E}) + Z$.*

Proof. Note that $b \in A \subset \mathrm{A}(m)$. Let $z = (-1)^m r^m b$, and let $y = \varphi + z$. Then

$$\Delta^m y = \Delta^m \varphi + \Delta^m z = 0 + b = b.$$

Since φ is f-regular, there exists a subset U of $\mathbb{R}$ and a positive number ε such that

$$\varphi(\mathbb{N}) \subset \operatorname{Int}(U, \varepsilon)$$

and $f|\mathbb{N} \times U$ is continuous and bounded. Let $\mu \in (0, \varepsilon/2)$. Since $z_n = \mathrm{o}(1)$, there exists an index p such that $|z_n| \leq \mu$ for any $n \geq p$. Then

$$(\varphi + z)(\mathbb{N}(p)) \subset \operatorname{Int}(U, \mu).$$

Let

$$y^*(n) = \begin{cases} \varphi(n) & \text{for } n < p \\ (\varphi + z)(n) & \text{for } n \geq p \end{cases}, \qquad b^*(n) = \begin{cases} \Delta^m \varphi(n) & \text{for } n < p \\ b(n) & \text{for } n \geq p \end{cases}.$$

Then y^* is f-regular and $\Delta^m y^* = b^*$. Hence, by Theorem 3.1 (A3), there exists a solution x of the equation

$$\Delta^m x_n = b^*(n) + \sum_{k=1}^{n} K(n,k) f(k, x_{\sigma(k)})$$

such that $y^* \in x + Z$. Since $b^*(n) = b_n$ for $n \geq p$, we get $x \in \mathrm{Sol(E)}$. By the definition of y^* we have $\varphi + z - y^* \in \operatorname{Fin}(p)$. Hence

$$\varphi \in y^* - z + \operatorname{Fin}(p) \subset y^* + Z \subset x + Z + Z = x + Z. \qquad \square$$

5 Bounded solutions

In this section we apply our main results to investigate the bounded solutions of equation (E).

We say that a function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is *locally equibounded* if for every $t \in \mathbb{R}$ there exists a neighborhood U of t such that f is bounded on $\mathbb{N} \times U$. Obviously every bounded function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is locally equibounded.

Example 5.1. Let $f_1(n,t) = t$ and $f_2(n,t) = n$. Then f_1 is continuous, unbounded and locally equibounded, f_2 is continuous but not locally equibounded.

Example 5.2. Assume $g_1, \dots, g_k : \mathbb{R} \to \mathbb{R}$ are continuous, $\alpha_1, \dots, \alpha_k \in \mathrm{O}(1)$ and let

$$f(n,t) = \sum_{i=1}^{k} \alpha_i(n) g_i(t).$$

Then f is continuous and locally equibounded.

Lemma 5.3. *If f is locally equibounded, then* $\mathrm{O}(1)$ *is (f, σ)-ordinary.*

Proof. Let $x \in \mathrm{O}(1)$. Choose $a, b \in \mathbb{R}$ such that $x(\mathbb{N}) \subset [a,b]$. For any $t \in [a,b]$ there exist an open subset U_t of $\mathbb{R}$ and a positive constant M_t such that

$$|f(n,s)| \leq M_t$$

for any $s \in U_t$ and any $n \in \mathbb{N}$. There exists a finite subset $\{t_1, \dots t_n\}$ such that

$$[a,b] \subset U_{t_1} \cup \dots \cup U_{t_n}.$$

If $M = \max(M_{t_1}, \dots M_{t_n})$, then $|f(k, x_{\sigma(k)})| \leq M$ for any k. $\square$

In the next corollary we identify the set $\mathbb{R}$ with the space $\mathrm{Pol}(0)$ of constant sequences.

Corollary 5.4. *Assume (A,Z) is an m-pair, $K \in \mathrm{K}(A)$, $w \in \mathrm{O}(1)$, $b = \Delta^m w$, and f is locally equibounded. Then*

$$\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) \subset w + \mathbb{R} + Z.$$

Proof. Note that $\Delta^{-m} b = w + \mathrm{Pol}(m-1)$. Since the sequence w is bounded, we have

$$\mathrm{O}(1) \cap \Delta^{-m} b = \mathrm{O}(1) \cap (w + \mathrm{Pol}(m-1)) = w + \mathrm{Pol}(0) = w + \mathbb{R}. \tag{5.1}$$

By Lemma 5.3 $\mathrm{O}(1)$ is (f,σ)-ordinary. Hence the assertion is a consequence of Theorem 3.1 (A1). □

Corollary 5.5. *Assume (A,Z) is an evanescent m-pair, $K \in \mathrm{K}(A)$, $w \in \mathrm{O}(1)$, $b = \Delta^m w$, and f is continuous and locally equibounded. Then*

$$\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) + Z = w + \mathbb{R} + Z. \tag{5.2}$$

Proof. If f is continuous and locally equibounded, then $\mathrm{O}(1)$ is f-regular. Hence, using (5.1), and Theorem 3.1 (A4) we obtain (5.2). □

Corollary 5.6. *Assume (A,Z) is an evanescent m-pair, $K \in \mathrm{K}(A)$, $w \in \mathrm{O}(1)$, $b = \Delta^m w$, and f is continuous and bounded. Then*

$$\mathrm{O}(1) \cap \mathrm{Sol}_1(\mathrm{E}) + Z = \mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) + Z = w + \mathbb{R} + Z. \tag{5.3}$$

Proof. Since the set $\mathrm{O}(1)$ is f-regular, the assertion is a consequence of Corollary 5.5 and Theorem 3.1 (A5). □

Let $k \in \mathbb{N}$ and $Z \subset \mathrm{SQ}$. We define

$$\mathrm{Per}(k) = \{x \in \mathrm{SQ} : \ x \text{ is } k\text{-periodic}\}, \qquad \mathrm{Val}(k) = \{x \in \mathrm{SQ} : \ \mathrm{card}(x(\mathbb{N})) \le k\}.$$
$$\mathrm{Per}(k,Z) = \mathrm{Per}(k) + Z, \qquad \mathrm{Val}(k,Z) = \mathrm{Val}(k) + Z,$$

Corollary 5.7. *Assume (A,Z) is an evanescent m-pair, $K \in \mathrm{K}(A)$, $k \in \mathbb{N}$, and f is locally equibounded. Then*

(1) *if $\Delta^{-m} b \cap \mathrm{Per}(k,Z) \neq \emptyset$, then* $\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) \subset \mathrm{Per}(k,Z)$,

(2) *if $\Delta^{-m} b \cap \mathrm{Val}(k,Z) \neq \emptyset$, then* $\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) \subset \mathrm{Val}(k,Z)$.

Proof. If $w \in \Delta^{-m} b \cap \mathrm{Per}(k,Z)$, then by Corollary 5.4

$$\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) \subset w + \mathbb{R} + Z \subset \mathrm{Per}(k) + Z = \mathrm{Per}(k,Z),$$

and we obtain (1). Analogously we obtain (2). □

Corollary 5.8. *Assume f is continuous and locally equibounded, (A,Z) is an evanescent m-pair, $K \in \mathrm{K}(A)$, and $w \in \Delta^{-m} b$. Then*

(1) *if $w \in \mathrm{Per}(k,Z)$, then* $\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) + Z = \mathrm{Per}(k,Z) \cap \mathrm{Sol}(\mathrm{E}) + Z = w + \mathbb{R} + Z$,

(2) *if $w \in \mathrm{Val}(k,Z)$, then* $\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) + Z = \mathrm{Val}(k,Z) \cap \mathrm{Sol}(\mathrm{E}) + Z = w + \mathbb{R} + Z$.

Proof. Since f is continuous and locally equibounded, the set $\mathrm{O}(1)$ is f-regular. Moreover, since the pair (A, Z) is evanescent, we have $Z + \mathrm{O}(1) \subset \mathrm{O}(1)$. Using Theorem 3.1 (A4) and (5.1) we have

$$\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) + Z = \mathrm{O}(1) \cap \Delta^{-m}b + Z = w + \mathbb{R} + Z.$$

By Corollary 5.7, $\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) \subset \mathrm{Per}(k, Z)$. Hence

$$\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) \subset \mathrm{Per}(k, Z) \cap \mathrm{Sol}(\mathrm{E}).$$

Since $\mathrm{Per}(k, Z) \subset \mathrm{O}(1)$, we get $\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) = \mathrm{Per}(k, Z) \cap \mathrm{Sol}(\mathrm{E})$ and we obtain (1). Similarly we obtain (2). □

Corollary 5.9. *Assume f is continuous and bounded, (A, Z) is an evanescent m-pair, $K \in \mathrm{K}(A)$, and $w \in \Delta^{-m}b$. Then*

(1) *if $w \in \mathrm{Per}(k, Z)$, then*

$$\begin{aligned}\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) + Z &= \mathrm{O}(1) \cap \mathrm{Sol}_1(\mathrm{E}) + Z = \mathrm{Per}(k, Z) \cap \mathrm{Sol}(\mathrm{E}) + Z \\ &= \mathrm{Per}(k, Z) \cap \mathrm{Sol}_1(\mathrm{E}) + Z = w + \mathbb{R} + Z,\end{aligned}$$

(2) *if $w \in \mathrm{Val}(k, Z)$, then*

$$\begin{aligned}\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) + Z &= \mathrm{O}(1) \cap \mathrm{Sol}_1(\mathrm{E}) + Z = \mathrm{Val}(k, Z) \cap \mathrm{Sol}(\mathrm{E}) + Z \\ &= \mathrm{Val}(k, Z) \cap \mathrm{Sol}_1(\mathrm{E}) + Z = w + \mathbb{R} + Z.\end{aligned}$$

Proof. By Theorem 3.1 (A5) we have

$$\mathrm{O}(1) \cap \mathrm{Sol}(\mathrm{E}) + Z = \mathrm{O}(1) \cap \mathrm{Sol}_1(\mathrm{E}) + Z$$

and

$$\mathrm{Per}(k, Z) \cap \mathrm{Sol}(\mathrm{E}) + Z = \mathrm{Per}(k, Z) \cap \mathrm{Sol}_1(\mathrm{E}) + Z.$$

Hence (1) is a consequence of Corollary 5.8 (1). Analogously we obtain (2). □

6 Remarks

In this section, we present some examples of f-regular sets. These sets are used in Theorem 3.1. Next, we discuss the condition $w \circ \sigma \in \mathrm{O}(w)$ which is important in Theorem 3.2. Finally, we present some tests that are helpful in verifying whether a given kernel K fulfills the assumptions of Theorems 3.1 and 3.2.

Remark 6.1. If $K \in \mathrm{K}(m)$, then, by (2.3), $r^m K' \in \mathrm{o}(1)$. Hence for any f-regular sequence y there exists an index p such that y is (K, f, p)-regular.

We say that a subset W of SQ is $\mathrm{o}(1)$-*invariant* if

$$\mathrm{o}(1) + W \subset W.$$

Note that if W is $\mathrm{o}(1)$-invariant and (A, Z) is an evanescent m-pair, then $Z + W \subset W$.

Example 6.2. If f is continuous and bounded, then SQ is f-regular and $\mathrm{o}(1)$-invariant. If f is continuous and locally equibounded, then $\mathrm{O}(1)$ is f-regular and $\mathrm{o}(1)$-invariant.

Example 6.3. If f is continuous and locally equibounded, then the set of all convergent sequences $x \in \mathrm{SQ}$ is f-regular and $\mathrm{o}(1)$-invariant. More generally, the set

$$\{x \in \mathrm{SQ} : \mathrm{L}(x) \text{ is finite}\}$$

is f-regular and $\mathrm{o}(1)$-invariant.

Example 6.4. Assume U is a uniform neighborhood of a set $Y \subset \mathbb{R}$ and f is continuous and bounded on $\mathbb{N} \times U$. Then the sets

$$W_L = \{y \in \mathrm{SQ} : L(y) \subset Y\}, \quad W_\infty = \{y \in \mathrm{SQ} : \lim y \in Y\}$$

are f-regular and $\mathrm{o}(1)$-invariant.

By Lemma 3.6 (3) the condition $w \circ \sigma \in \mathrm{O}(w)$ implies $\mathrm{O}(w) \subset \mathrm{O}(w, \sigma)$. Moreover, subsets of $\mathrm{O}(w, \sigma)$ play an important role in Theorem 3.2. Below, we discuss the condition $w \circ \sigma \in \mathrm{O}(w)$.

Example 6.5. If $s \in (0, \infty)$, $w_n = n^s$, and $\sigma(n) = \mathrm{O}(n)$, then $w \gg 0$ and $w \circ \sigma \in \mathrm{O}(w)$.

Justification. Obviously, $w \gg 0$. If M is a positive constant such that $\sigma(n) \leq Mn$ for any n. then $w(\sigma(n)) = (\sigma(n))^s \leq (Mn)^s = M^s w_n$. Hence $w \circ \sigma \in \mathrm{O}(w)$. □

Example 6.6. If $\mathrm{O}(w_{n+1}) = \mathrm{O}(w_n)$, and the sequence $\sigma(n) - n$ is bounded, then $w \circ \sigma \in \mathrm{O}(w)$.

Justification. Choose $k \in \mathbb{N}$ such that $|\sigma(n) - n| \leq k$ for any n. Since $w_{n+1} = \mathrm{O}(w_n)$, there exists a constant $M > 1$ such that $|w_{n+1}| \leq M|w_n|$ for large n. Then

$$|w_{n+2}| \leq M|w_{n+1}| \leq M^2|w_n|, \ \ldots, \ |w_{n+k}| \leq M^k|w_n|.$$

Hence, for any $p \in \mathbb{N}(0, k)$, we have

$$|w_{n+p}| \leq M^k|w_n|$$

for large n. Analogously, since $w_n = \mathrm{O}(w_{n+1})$, there exists a constant $Q > 1$ such that for any $p \in \mathbb{N}(0, k)$, we have

$$|w_{n-p}| \leq Q^k|w_n|$$

for large n. Now, if $L = \max(M^k, Q^k)$, then $|w(\sigma(n))| \leq L|w_n|$ for large n. □

Remark 6.7. If $s \in \mathbb{R}$ and $w_n = n^s$, then $\mathrm{O}(w_{n+1}) = \mathrm{O}(w_n)$. Similarly, if $\lambda \in (0, \infty)$ and $w_n = \lambda^n$, then $\mathrm{O}(w_{n+1}) = \mathrm{O}(w_n)$. On the other hand, if $w_n = n^n$, then $(w_{n+1}) \notin \mathrm{O}(w_n)$.

In our main theorems we assume that (A, Z) is an m-pair and $K \in \mathrm{K}(A)$. The basic example of an m-pair is $(\mathrm{A}(t), \mathrm{o}(n^{m-t}))$. Hence in our theory, the answer to the following question is very important: whether for a given kernel $K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ the relation $K \in \mathrm{K}(\mathrm{A}(t)) = \mathrm{K}(t)$ is fulfilled? Below we present some lemmas concerning this problem. These lemmas are analogous to the classical tests for absolute convergence of series.

For $n \in \mathbb{N}$ let

$$K^*(n) = n \max\{|K(n,1)|, |K(n,2)|, \ldots, |K(n,n)|\},$$

$$K_*(n) = n \min\{|K(n,1)|, |K(n,2)|, \ldots, |K(n,n)|\}.$$

Note that

$$K_* \leq K' \leq K^*. \tag{6.1}$$

Moreover if $|K|$ is nondecreasing with respect to second variable, then

$$K_*(n) = n|K(n,1)|, \quad K^*(n) = n|K(n,n)|$$

for any n, if $|K|$ is nonincreasing with respect to second variable, then

$$K_*(n) = n|K(n,n)|, \quad K^*(n) = n|K(n,1)|$$

for any n.

Lemma 6.8 (Comparison test 1)**.** *Assume $a, b, c \in \mathrm{SQ}$, and A is a linear subspace of SQ, such that $\mathrm{O}(1)A \subset A$, and $\mathrm{Fin} + A \subset A$. Then*

(1) *if $|b_n| \le |c_n|$ for large n and $c \in A$, then $b \in A$,*

(2) *if $|b_n| \ge |a_n|$ for large n and $a \notin A$, then $b \notin A$.*

Proof. For the proof of (1) see [33, Lemma 3.8]. (2) is a consequence of (1). □

Lemma 6.9 (Comparison test 2)**.** *Assume A is a linear subspace of SQ, such that*

$$\mathrm{O}(1)A \subset A, \quad \textit{and} \quad \mathrm{Fin} + A \subset A.$$

Moreover, let $L \in \mathrm{K}(A)$, $c \in \mathrm{A}$, and

$$K' \le L' \quad \textit{or} \quad |K| \le |L| \quad \textit{or} \quad K^* \le |c|.$$

Then $K \in \mathrm{K}(A)$.

Proof. The assertion is an easy consequence of Lemma 6.8. □

Lemma 6.10 (Logarithmic test)**.** *Assume $t \in [1, \infty)$,*

$$u_*(n) = -\frac{\ln K_*(n)}{\ln n}, \qquad u^*(n) = -\frac{\ln K^*(n)}{\ln n}.$$

Then

(1) *if* $\liminf u^*(n) > t$, *then* $K \in \mathrm{K}(t)$,

(2) *if* $\lim u^*(n) = \infty$, *then* $K \in \mathrm{K}(\infty)$,

(3) *if* $u_*(n) \le t$ *for large* n, *then* $K \notin \mathrm{K}(t)$,

(4) *if* $\limsup u_*(n) < t$, *then* $K \notin \mathrm{K}(t)$.

Proof. The assertion is a consequence of (6.1), Lemma 6.8 and [33, Lemma 6.2]. □

Lemma 6.11 (Raabe's type test)**.** *Assume $t \in [1, \infty)$,*

$$u_*(n) = n\left(\frac{K_*(n)}{K_*(n+1)} - 1\right), \qquad u^*(n) = n\left(\frac{K^*(n)}{K^*(n+1)} - 1\right).$$

Then

(1) *if* $\liminf u^*(n) > t$, *then* $K \in \mathrm{K}(t)$,

(2) *if* $\lim u^*(n) = \infty$, *then* $K \in \mathrm{K}(\infty)$,

(3) *if* $u_*(n) \leq t$ *for large* n, *then* $K \notin \mathrm{K}(t)$,

(4) *if* $\limsup u_*(n) < t$, *then* $K \notin \mathrm{K}(t)$.

Proof. The assertion is a consequence of (6.1), Lemma 6.8 and [33, Lemma 6.3]. □

Lemma 6.12 (Schlömilch's type test)**.** *Assume* $t \in [1, \infty)$,

$$u_*(n) = n \ln \frac{K_*(n)}{K_*(n+1)}, \qquad u^*(n) = n \ln \frac{K^*(n)}{K^*(n+1)}$$

Then

(1) *if* $\liminf u^*(n) > t$, *then* $K \in \mathrm{K}(t)$,

(2) *if* $\lim u^*(n) = \infty$, *then* $K \in \mathrm{K}(\infty)$,

(3) *if* $u_*(n) \leq t$ *for large* n, *then* $K \notin \mathrm{K}(t)$,

(4) *if* $\limsup u_*(n) < t$, *then* $K \notin \mathrm{K}(t)$.

Proof. The assertion is a consequence of (6.1), Lemma 6.8 and [33, Lemma 6.4]. □

Lemma 6.13 (Gauss's type test)**.** *Let* $t \in [1, \infty)$, $\lambda, \mu \in \mathbb{R}$, $s \in (-\infty, -1)$, *and*

$$\frac{K_*(n)}{K_*(n+1)} = 1 + \frac{\lambda}{n} + \mathrm{O}(n^s), \qquad \frac{K^*(n)}{K^*(n+1)} = 1 + \frac{\mu}{n} + \mathrm{O}(n^s).$$

Then

(a) *if* $\mu > t$, *then* $K \in \mathrm{K}(t)$,

(b) *if* $\lambda \leq t$, *then* $K \notin \mathrm{K}(t)$.

Proof. The assertion is a consequence of (6.1), Lemma 6.8 and [33, Lemma 6.5]. □

Lemma 6.14 (Bertrand's type test)**.** *Assume* $t \in [1, \infty)$ *and*

$$\frac{K_*(n)}{K_*(n+1)} = 1 + \frac{t}{n} + \frac{\lambda_n}{n \ln n}, \qquad \frac{K^*(n)}{K^*(n+1)} = 1 + \frac{t}{n} + \frac{\mu_n}{n \ln n}.$$

Then

(1) *if* $\liminf \mu_n > 1$, *then* $K \in \mathrm{K}(t)$,

(2) *if* $\lambda_n \leq 1$ *for large* n, *then* $K \notin \mathrm{K}(t)$,

(3) *if* $\limsup \lambda_n < 1$, *then* $K \notin \mathrm{K}(t)$.

Proof. The assertion is a consequence of (6.1), Lemma 6.8 and [33, Lemma 6.7]. □

7 Examples of applications

If the kernel K of equation (E) satisfies some additional conditions, then from Theorem 3.1 we can obtain many interesting results. Some of them are presented below.

Corollary 7.1. *Assume* $x \in \mathrm{Sol(E)}$ *is* (f,σ)*-ordinary,* $y \in \Delta^{-m}b$ *is* f*-regular,*

$$s \in (-\infty, -m), \quad \text{and} \quad \limsup_{n\to\infty} n^{-s} \sum_{k=1}^{n} |K(n,k)| < \infty. \tag{7.1}$$

Then

$$x \in \Delta^{-m}b + \mathrm{O}(n^{m+s}) \quad \text{and} \quad y \in \mathrm{Sol(E)} + \mathrm{O}(n^{m+s}). \tag{7.2}$$

Proof. By (7.1), $K' = \mathrm{O}(n^s)$. Using Example 2.6 and Theorem 3.1 (A2) and (A3) we obtain (7.2). □

Corollary 7.2. *Assume* $x \in \mathrm{Sol(E)}$ *is* (f,σ)*-ordinary,* $y \in \Delta^{-m}b$ *is* f*-regular,*

$$\limsup_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} |K(n,k)|} < \lambda < 1. \tag{7.3}$$

Then

$$x \in \Delta^{-m}b + \mathrm{o}(\lambda^n) \quad \text{and} \quad y \in \mathrm{Sol(E)} + \mathrm{o}(\lambda^n). \tag{7.4}$$

Proof. By (7.3), $K' = \mathrm{o}(\lambda^n)$. Using Example 2.7 and Theorem 3.1 (A2) and (A3) we obtain (7.4). □

Corollary 7.3. *Assume* $x \in \mathrm{Sol(E)}$ *is* (f,σ)*-ordinary,* $y \in \Delta^{-m}b$ *is* f*-regular,*

$$s \in (-\infty, 0] \quad \text{and} \quad \liminf_{n\to\infty} n\left(\frac{K'_n}{K'_{n+1}} - 1\right) > m - s. \tag{7.5}$$

Then

$$x \in \Delta^{-m}b + \mathrm{o}(n^s) \quad \text{and} \quad y \in \mathrm{Sol(E)} + \mathrm{o}(n^s). \tag{7.6}$$

Proof. Using (7.5) and [33, Lemma 6.3], we get $K' \in \mathrm{A}(m-s)$. Using Example 2.6 and Theorem 3.1 (A2) and (A3) we obtain (7.6). □

Corollary 7.4. *Assume* $x \in \mathrm{Sol(E)}$ *is* (f,σ)*-ordinary,* $y \in \Delta^{-m}b$ *is* f*-regular,*

$$s \in (-\infty, 0], \quad t > m - s \quad \text{and} \quad K(n,k) = \frac{(n-1)!}{k(t+1)(t+2)\cdots(t+n)}.$$

Then

$$x \in \Delta^{-m}b + \mathrm{o}(n^s) \quad \text{and} \quad y \in \mathrm{Sol(E)} + \mathrm{o}(n^s). \tag{7.7}$$

Proof. For any n we have

$$K^*(n) = \frac{n!}{(t+1)(t+2)\cdots(t+n)}.$$

Hence

$$n\left(\frac{K^*(n)}{K^*(n+1)} - 1\right) = n\left(\frac{t+n+1}{n+1} - 1\right) = \frac{nt}{n+1} \to t > m - s.$$

By Lemma 6.11 we have $K \in \mathrm{K}(m-s)$. Using Example 2.6 and Theorem 3.1 (A2) and (A3) we obtain (7.7). □

Corollary 7.5. *Assume* $f(n,t) = e^t$, $s \in [1, \infty)$,

$$W = \{y \in \mathrm{SQ} : y(\mathbb{N}) \subset (-\infty, 1)\}, \quad U = \{y \in \mathrm{SQ} : \limsup_{n\to\infty} y_n < \infty\},$$

and

$$\sum_{n=3}^{\infty} n^{m+s-1} \sum_{k=1}^{n} |K(n,k)| \le \frac{-1+\ln 7}{7}.$$

Then

$$W \cap \Delta^{-m} b \subset \mathrm{Sol}_3(\mathrm{E}) + \mathrm{A}(s) \quad \textit{and} \quad U \cap \Delta^{-m} b \subset \mathrm{Sol}(\mathrm{E}) + \mathrm{A}(s).$$

Proof. By assumption $K' \in \mathrm{A}(m+s)$. Obviously, the set U is f-regular. Using Example 2.6 and Theorem 3.1 (A4) we obtain

$$U \cap \Delta^{-m} b \subset \mathrm{Sol}(\mathrm{E}) + \mathrm{A}(s).$$

Note that

$$r_3^m K' \le \sum_{n=3}^{\infty} n^{m-1} K'_n \le \sum_{n=3}^{\infty} n^{m+s-1} \sum_{k=1}^{n} |K(n,k)| \le \frac{-1+\ln 7}{7}.$$

Assume $y \in W$ and $n \in \mathbb{N}$. Then

$$f(n, y_n + 7r_3^m K') = \exp(y_n + 7r_3^m K') \le \exp(1 - 1 + \ln 7) = 7.$$

Hence any $y \in W$ is $(K, f, 3)$-regular and, by Theorem 3.1 (A1), we have

$$W \cap \Delta^{-m} b \subset \mathrm{Sol}_3(\mathrm{E}) + \mathrm{A}(s). \qquad \square$$

Corollary 7.6. *Assume* $x \in \mathrm{Sol}(\mathrm{E})$ *is* (f, σ)*-ordinary,* $y \in \Delta^{-m} b$ *is* f*-regular, and*

$$K(n,k) = k 2^{-\sqrt{n}}$$

for any $n \in \mathbb{N}$ *and* $k \in \mathbb{N}(1, n)$*. Then*

$$x \in \Delta^{-m} b + \mathrm{A}(\infty) \quad \textit{and} \quad y \in \mathrm{Sol}(\mathrm{E}) + \mathrm{A}(\infty). \tag{7.8}$$

Proof. For any n we have $K^*(n) = n^2 2^{-\sqrt{n}}$. Hence

$$n \ln \frac{K^*(n)}{K^*(n+1)} = n \ln \left(\left(\frac{n}{n+1} \right)^2 2^{\left(\sqrt{n+1} - \sqrt{n}\right)} \right) = 2 \ln \left(\frac{n}{n+1} \right)^n + n \left(\sqrt{n+1} - \sqrt{n} \right) \ln 2 \to \infty.$$

By Lemma 6.12 we have $K \in \mathrm{K}(\infty)$. Using Example 2.7 and Theorem 3.1 (A2) and (A3) we obtain (7.8). $\square$

Corollary 7.7. *Assume* $u \in \mathrm{O}(1)$, $f(n,t) = e^{-t} + u_n$, $\lambda \in (e^{-1}, 1)$,

$$b_n = \frac{n!}{n^n}, \quad \textit{and} \quad K(n,k) = \left(\frac{k}{n+1} \right)^{kn}$$

for any $n \in \mathbb{N}$ *and* $k \in \mathbb{N}(1, n)$*. Then for any* $\varphi \in \mathrm{Pol}(m-1)$ *such that* $\lim_{n\to\infty} \varphi(n) = \infty$ *there exists a solution* x *of* (E) *such that*

$$x_n = \varphi(n) + \mathrm{o}(\lambda^n).$$

Proof. Note that

$$K^*(n) = n\left(\frac{n}{n+1}\right)^{n^2}, \qquad \sqrt[n]{K^*(n)} = \sqrt[n]{n}\left(\frac{n}{n+1}\right)^{n} \to \frac{1}{e} < \lambda,$$

$$\frac{b_{n+1}}{b_n} = \left(\frac{n}{n+1}\right)^{n} \to \frac{1}{e} < \lambda.$$

Hence $K' \in \mathrm{o}(\lambda^n)$ and $b \in \mathrm{o}(\lambda^n)$. Moreover, φ is f-regular and $\mathrm{o}(\lambda^n)$ is an evanescent m-space. Therefore, the assertion follows from Corollary 4.11. □

References

[1] C. T. H. Baker, Y. Song, Periodic solutions of non-linear discrete Volterra equations with finite memory, *J. Comput. Appl. Math.* **234**(2010), No. 9, 2683–2698. MR2652117; https://doi.org/10.1016/j.cam.2010.01.019

[2] L. Berezansky, M. Migda, E. Schmeidel, Some stability conditions for scalar Volterra difference equations, *Opuscula Math.* **36**(2016), No. 4, 459–470. MR3488501; https://doi.org/10.7494/OpMath.2016.36.4.45

[3] M. Bohner, S. Stević, Asymptotic behavior of second-order dynamic equations, *Appl. Math. Comput.* **188**(2007), No. 2, 1503–1512. MR1202198; https://doi.org/10.1016/j.amc.2006.11.016

[4] M. Bohner, S. Stević, Trench's perturbation theorem for dynamics equations, *Discrete Dyn. Nat. Soc. Vol.* **2007**, Art. ID 75672, 11 pp. MR2375475; https://doi.org/10.1155/2007/75672

[5] M. Bohner, N. Sultana, Subexponential solutions of linear Volterra difference equations, *Nonauton. Dyn. Syst.* **2**(2015) 63–76. MR3412180; https://doi.org/10.1515/msds-2015-0005

[6] M. R. Crisci, V. B. Kolmanovskii, E. Russo, A. Vecchio, Boundedness of discrete Volterra equations, *J. Math. Anal. Appl.* **211**(1997), 106–130. MR1813211; https://doi.org/10.1080/10236190008808251

[7] M. R. Crisci, V. B. Kolmanovskii, E. Russo, A. Vecchio, Stability of difference Volterra equations: direct Liapunov method and numerical procedure, *Advances in difference equations, II. Comput. Math. Appl.* **36**(1998), No. 10–12, 77–97. MR1666128; https://doi.org/10.1016/S0898-1221(98)80011-4

[8] M. R. Crisci, V. B. Kolmanovskii, E. Russo, A. Vecchio, On the exponential stability of discrete Volterra systems, *J. Differ. Equations Appl.* **6**(2000), No. 6, 667–680. MR1813211; https://doi.org/10.1080/10236190008808251

[9] J. Diblík, M. Růžičková, E. Schmeidel, Asymptotically periodic solutions of Volterra difference equations, *Tatra Mt. Math. Publ.* **43**(2009), 43–61. MR3146295; https://doi.org/10.1016/j.amc.2013.10.055

[10] J. Diblík, E. Schmeidel, M. Růžičková, Asymptotically periodic solutions of Volterra system of difference equations, *Comput. Math. Appl.* **59**(2010), 2854–2867. MR2607992; https://doi.org/10.1016/j.camwa.2010.01.055

[11] J. Diblík, M. Růžičková, L. E. Schmeidel, M. Zbaszyniak, Weighted asymptotically periodic solutions of linear Volterra difference equations, *Abstr. Appl. Anal.* **2011**, Art. ID 370982, 14 pp. MR2795073; https://doi.org/370982

[12] J. Diblík, E. Schmeidel, On the existence of solutions of linear Volterra difference equations asymptotically equivalent to a given sequence, *Appl. Math. Comput.* **218**(2012), No. 18, 9310–9320. MR2923029; https://doi.org/10.1016/j.amc.2012.03.010

[13] A. Drozdowicz, J. Popenda, Asymptotic behavior of the solutions of the second order difference equations, *Proc. Amer. Math. Soc.* **99**(1987), No. 1, 135–140. MR0866443; https://doi.org/10.2307/2046284

[14] A. Drozdowicz, J. Popenda, Asymptotic behavior of the solutions of an *n*th order difference equations, *Comment. Math. Prace Mat.* **29**(1990), 161–168. MR1059121

[15] S. Elaydi, Stability and asymptoticity of Volterra difference equations: a progress report, *J. Comput. Appl. Math.* **228**(2009), No. 2, 504–513. MR2523167; https://doi.org/10.1016/j.cam.2008.03.023

[16] K. Gajda, T. Gronek, E. Schmeidel, On the existence of a weighted asymptotically constant solutions of Volterra difference equations of nonconvolution type, *Discrete Contin. Dyn. Syst. Ser. B* **19**(2014), No. 8, 2681–2690. MR3275021; https://doi.org/10.3934/dcdsb.2014.19.2681

[17] T. Gronek, E. Schmeidel, Existence of bounded solution of Volterra difference equations via Darbo's fixed-point theorem, *J. Difference Equ. Appl.* **19**(2013), No. 10, 1645–1653. MR3173509; https://doi.org/10.1080/10236198.2013.769974

[18] I. Győri, E. Awwad, On the boundedness of the solutions in nonlinear discrete Volterra difference equations, *Adv. Difference Equ.* **2012**, 2012:2, 20 pp. MR2916343; https://doi.org/10.1186/1687-1847-2012-2

[19] I. Győri, F. Hartung, Asymptotic behavior of nonlinear difference equations, *J. Difference Equ. Appl.* **18**(2012), No. 9, 1485–1509. MR2974133; https://doi.org/10.1080/10236198.2011.574619

[20] I. Győri, L. Horváth, Asymptotic representation of the solutions of linear Volterra difference equations, *Adv. Difference Equ.* **2008**, Art. ID 932831, 22 pp. MR2407437

[21] I. Győri, D. W. Reynolds, Sharp conditions for boundedness in linear discrete Volterra equations, *J. Difference Equ. Appl.* **15**(2009), No. 11–12, 1151–1164. MR2569138; https://doi.org/10.1080/10236190902932726

[22] I. Győri, D. W. Reynolds, On asymptotically periodic solutions of linear discrete Volterra equations, *Fasc. Math.* **44**(2010), 53–67. MR2722631

[23] M. N. Islam, Y. N. Raffoul, Uniform asymptotic stability in linear Volterra difference equations, *PanAmer. Math. J.* **11**(2001), No. 1, 61–73. MR1820712

[24] T. M. Khandaker, Y. N. Raffoul, Stability properties of linear Volterra discrete systems with nonlinear perturbation, *J. Difference Equ. Appl.* **8**(2002), No. 10, 857–874. MR1918529

[25] V. Kolmanovskii, L. Shaikhet, Some conditions for boundedness of solutions of difference Volterra equations, *Appl. Math. Lett.* **16**(2003), 857–862. MR2005259; https://doi.org/10.1016/S0893-9659(03)90008-5

[26] R. Medina, Asymptotic behavior of Volterra difference equations, *Comput. Math. Appl.* **41**(2001), No. 5–6, 679–687. MR1822595; https://doi.org/10.1016/S0898-1221(00)00312-6

[27] J. Migda, Asymptotic properties of solutions of nonautonomous difference equations, *Arch. Math. (Brno)* **46**(2010), No. 1, 1–11. MR2644450

[28] J. Migda, Asymptotic properties of solutions of higher order difference equations, *Math. Bohem.* **135**(2010), No. 1, 29–39. MR2643353

[29] J. Migda, Asymptotically polynomial solutions of difference equations, *Adv. Difference Equ.* **2013**, 2013:92, 1–16. MR3053806; https://doi.org/10.1186/1687-1847-2013-92

[30] J. Migda, Approximative solutions of difference equations, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 13, 1–26. MR3183611; https://doi.org/10.14232/ejqtde.2014.1.13

[31] J. Migda, Iterated remainder operator, tests for multiple convergence of series and solutions of difference equations, *Adv. Difference Equ.* **2014**, 2014:189, 1–18. MR3357342; https://doi.org/10.1186/1687-1847-2014-189

[32] J. Migda, Regional topology and approximative solutions of difference and differential equations, *Tatra Mountains Math. Publ.* **63**(2015), 183–203. MR3411445; https://doi.org/10.1515/tmmp-2015-0031

[33] J. Migda, Qualitative approximation of solutions to difference equations, *Electron. J. Qual. Theory Differ. Equ.* **2015**, No. 32, 1–26. MR3353203; https://doi.org/10.14232/ejqtde.2015.1.32

[34] J. Migda, Approximative solutions to difference equations of neutral type, *Appl. Math. Comput.* **268**(2015), 763–774. MR3399460; https://doi.org/10.1016/j.amc.2015.06.097

[35] J. Migda, Asymptotically polynomial solutions to difference equations of neutral type, *Appl. Math. Comput.* **279**(2016), 16–27. MR345800; https://doi.org/10.1016/j.amc.2016.01.001

[36] J. Migda, Mezocontinuous operators and solutions of difference equations, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 11, 1–16. MR3475095; https://doi.org/10.14232/ejqtde.2016.1.11

[37] J. Migda, M. Migda, Asymptotic behavior of solutions of discrete Volterra equations, *Opuscula Math.* **36**(2016), No. 2, 265–278. MR3437219; https://doi.org/10.7494/OpMath.2016.36.2.265

[38] M. Migda, J. Migda, Bounded solutions of nonlinear discrete Volterra equations, *Math. Slovaca* **66**(2016), No. 5, 1169–1178. MR3602612; https://doi.org/10.1515/ms-2016-0212

[39] M. Migda, J. Morchało, Asymptotic properties of solutions of difference equations with several delays and Volterra summation equations, *Appl. Math. Comput.* **220**(2013), 365–373. MR3091861; https://doi.org/10.1016/j.amc.2013.06.032

[40] M. Migda, M. Růžičková, E. Schmeidel, Boundedness and stability of discrete Volterra equations, *Adv. Difference Equ.* **2015**, 2015:47, 11 pp. MR3314614; https://doi.org/10.1186/s13662-015-0361-6

[41] J. Morchało, Volterra summation equations and second order difference equations, *Math. Bohem.* **135**(2010), No. 1, 41–56. MR2643354

[42] J. Popenda, Asymptotic properties of solutions of difference equations, *Proc. Indian Acad. Sci. Math. Sci.* **95**(1986), No. 2, 141–153. MR0913886; https://doi.org/10.1007/BF02881078

[43] Y. N. Raffoul, Boundedness and periodicity of Volterra systems of difference equations, *J. Differ. Equations Appl.* **4**(1998), No. 4, 381–393. MR1657161; https://doi.org/10.1080/10236199808808150

[44] D. W. Reynolds, On asymptotic constancy for linear discrete summation equations, *Comput. Math. Appl.* **64**(2012), No. 10–12, 2335–2344. MR2966869; https://doi.org/10.1016/j.camwa.2012.05.013

[45] S. Stević, Growth theorems for homogeneous second-order difference equations, *ANZIAM J.* **43**(2002), No. 4, 559–566. MR1903917; https://doi.org/10.1017/S1446181100012141

[46] S. Stević, Asymptotic behaviour of second-order difference equation, *ANZIAM J.* **46**(2004), No. 1, 157–170. MR2075520; https://doi.org/10.1017/S1446181100013742

[47] S. Stević, Growth estimates for solutions of nonlinear second-order difference equations, *ANZIAM J.* **46**(2005), No. 3, 439–448. MR2124934; https://doi.org/10.1017/S1446181100008361

[48] E. Yankson, Stability of Volterra difference delay equations, *Electron. J. Qual. Theory Differ. Equ.* **2006**, No. 20, 1–14. MR2263079; https://doi.org/10.14232/ejqtde.2006.1.20

[49] A. Zafer, Oscillatory and asymptotic behavior of higher order difference equations, *Math. Comput. Modelling* **21**(1995), No. 4, 43–50. MR1317929; https://doi.org/10.1016/0895-7177(95)00005-M