Generalized functional differential equations: existence and uniqueness of solutions

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Abstract. We study generalized nonlinear functional differential equations arising in various applications, for instance in the control theory, or if there is a need to incorporate impulsive and/or delay effects into the underlying system. The main result of the paper provides a general existence and uniqueness theorem for such equations, and we also give many illustrative examples. The proofs are based on the theory of generalized Volterra operators in the spaces of continuous and discontinuous functions.

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1   Introduction

We introduce a broad class of functional differential equations driven by a general measure (in the paper we call these equations *generalized* for brevity). The class includes ordinary, delay, impulsive, difference equations and their combinations as well as important types of equations with distributed control and equations with discontinuous noise (e.g. of Poisson type). We illustrate the general theory with several examples. However, we do not intend to present an exhaustive theory of the equations included in the examples treating them rather as auxiliary to the main framework. That is why the list of references related to the particular classes of equations considered below [1–3, 10, 12, 13] is by far not complete. We cite therefore only very few papers and refer the reader to the references in these and other papers for more information.

The analysis framework is organized in a way that has become customary in the contemporary theory of functional differential equations (see e.g. [4] and the references therein). An

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essential feature of this construction is to consider the initial (prehistory) function as a part of the equation itself, which in particular, gives an opportunity to include equations with unbounded delays and avoid "nasty" functional spaces.

To be able to establish the well-posedness of the initial value problem, we formulate and prove a fixed point theorem for generalized Volterra operators in L^p-spaces with respect to an arbitrary measure, thus extending similar results proved in the series of papers [5,15,16]. The fixed point theorem of the present paper can also be used in many other applications.

We start with a brief description of the functional spaces which we need to define a generalized functional differential equation.

Let $\tilde{\mu}$ be a σ-additive, finite measure defined on the family $\mathfrak{B}$ of all Borel subsets of the interval $[0, T]$, and let μ be its standard extension, i.e. a σ-additive, finite and complete measure which is defined on the minimal σ-algebra $\mathfrak{L}$ containing $\mathfrak{B}$ and all subsets of measure zero and which satisfies $\mu(E) = \tilde{\mu}(E)$ for any $E \in \mathfrak{B}$. Any set from $\mathfrak{L}$ will alternatively be called μ-measurable.

As usual, we say that a function $y : [0, T] \to \mathbf{R}$ is *μ-measurable* if it satisfies the following condition: for any Borel subset $B \subset \mathbf{R}$ the set $y^{-1}(B) \in \mathfrak{L}$. The restriction of μ to the familiy of all μ-measurable subsets of an arbitrary set $S \subset [0, T]$, $S \in \mathfrak{L}$, will again be denoted by μ. The Lebesgue integral of a μ-measurable function y defined on S will be denoted by $\int_S y(s)\, \mu(ds)$. If $\mu = \text{mes}$ is the Lebesgue measure, then we will write $\int_S y(s)\, ds$. The measure $\mu \otimes \nu$ stands below for the product of two measures μ and ν. The indicator (the characteristic function) $\mathbf{1}_S$ of a set S is given by

$$\mathbf{1}_S(t) \equiv \begin{cases} 1 & \text{if } t \in S, \\ 0 & \text{if } t \notin S. \end{cases}$$

The space $L^p(S, \mathbf{R}^n, \mu)$, $1 \le p < \infty$ consists of all functions $y : S \to \mathbf{R}^n$ (more exactly, of equivalence classes), which are p-integrable with respect to (w.r.t.) the measure μ; the standard norm in this space is given by $\|y\|_{L^p} = \left(\int_S |y(s)|^p\, \mu(ds) \right)^{1/p}$. The space $L^\infty(S, \mathbf{R}^n, \mu)$ contains all μ-bounded (i.e. bounded up to a set of measure zero), μ-measurable functions $y : S \to \mathbf{R}^n$, the norm being defined $\|y\|_{L^\infty} = \text{ess sup}_{t \in S} |y(t)|$. In the case $S = [0, T]$ we will use the shorter notation $L^p \equiv L^p([0, T], \mathbf{R}^n, \mu)$ for any $1 \le p \le \infty$.

Let us now define the space $W_1^p \equiv W_1^p([-0, T], \mathbf{R}^n, \mu)$, $1 \le p \le \infty$. It contains all μ-measurable functions $y : [0, T] \to \mathbf{R}^n$ which are absolutely continuous w.r.t. the measure μ and whose "derivative", w.r.t. μ belongs to L^p:

$$x \in W_1^p \iff \exists y \in L^p \ \exists \alpha \in \mathbf{R}^n \text{ so that } \forall\, t \in [0, T] \quad x(t) = \alpha + \int_{[0,t]} y(s)\mu(ds). \tag{1.1}$$

For notational convenience, we will assume that the functions from W_1^p have an auxiliary value at -0, which we will treat as the left-hand limit at 0. That is why we introduced the "interval" $[-0, T]$ in the notation of W_1^p. From (1.1) we conclude that the functions $x \in W_1^p$ are *cadlag*, (see e.g. [7]) i.e. they are right-continuous and have left-hand limits at any point $t \in [0, T]$ including $t = 0$. The definition (1.1) also implies that the value of the jump of a function $x \in W_1^p$ is equal to

$$x(t) - x(t - 0) = y(t)\mu\{t\} \tag{1.2}$$

for any $t \in [0, T]$. In particular, $x \in W_1^p$ is continuous at $t \in [0, T]$ if $\mu(\{t\}) = 0$ and for continuity of x at a point t of positive measure we have to require that $y(t) = 0$.

The definition (1.1) determines a one-to-one mapping $x \mapsto (\alpha, y)$ between the spaces W_1^p and $\mathbf{R}^n \times L^p$, and in our notation we may also write $\alpha = x(-0)$. The mapping $x \mapsto y$ from (1.1) produces "differentiation" operator δ_μ which can be used to indroduce a norm in W_1^p:

$$\|x\|_{W_1^p} = |x(-0)|_{\mathbf{R}^n} + \|\delta_\mu x\|_{L^p}.$$

With this definition, the spaces W_1^p and $\mathbf{R}^n \times L^p$ become isometric. In the particular case of the Lebesgue measure $\mu = \text{mes}$, we obtain the usual differentiation of an absolutely continuous function x: $\delta_{\text{mes}} x = \dot{x}$. In this case we also may write $x(-0) = x(0)$ arriving at the standard space $W_1^p([0, T], \mathbf{R}^n, \text{mes})$ of absolutely continuous functions [4].

The main target of the paper is the following generalized nonlinear differential equation:

$$dx(t) = (Fx)(t) \, \mu(dt), \qquad t \in [0, T], \tag{1.3}$$

where $F : W_1^p \to L^p$ is a given (nonlinear) operator and $x \in W_1^p$ is an unknown function (solution) that should satisfy the initial condition

$$x(-0) = \alpha. \tag{1.4}$$

The central result of the paper describes the conditions providing existence and uniqueness of solutions of the initial value problem (1.3)–(1.4).

Using the introduced notation of the "derivative" of a function w.r.t. the measure μ we can rewrite the equation (1.3) as

$$(\delta_\mu x)(t) = (Fx)(t), \qquad t \in [0, T]. \tag{1.5}$$

Applying the isomorphism between the spaces W_1^p and $\mathbf{R}^n \times L^p$ described in (1.1) yields the following integral equation in the space W_1^p:

$$x(t) - x(-0) = \int_{[0,t]} (Fx)(s) \, \mu(ds), \qquad t \in [0, T]. \tag{1.6}$$

Equivalently, we can rewrite (1.5) in the form of an integral equation w.r.t. $y = \delta_\mu x$ in the space L^p:

$$y(t) = \left(F\left(x(-0) + \int_{[0,\cdot]} y(s) \, \mu(ds) \right) \right)(t), \qquad t \in [0, T]. \tag{1.7}$$

Both representations of the main equation (1.5) will be used below.

Normally, the continuity assumption is required in existence and uniqueness theorems:

(1.1) The operator $F : W_1^p \to L^p$ is continuous.

However, we will in many cases only assume that the operator F has the following Volterra-type property adjusted to arbitrary measures: for any $t \in [0, T]$ such that $\mu([0, t]) > 0$, the equality $x(s) = \hat{x}(s)$, $s \in [-0, t)$ implies the equality $(Fx)(s) = (F\hat{x})(s)$ $s \in [0, t]$. In particular, if $\mu(\{0\}) > 0$, then the Volterra operator F produces the same value $(Fx)(0)$ for any $x \in W_1^p([-0, T], \mathbf{R}^n, \mu)$ with the same auxiliary value (1.4).

Remark 1.1. At the points, where $\mu(\{t\}) = 0$, we can assume, without loss of generality, that in the definition of the Volterra property the intervals are equal, i.e. both are either $[0, t]$ or $[0, t)$. However, in the case $\mu(\{t\}) > 0$, it is essential that the intervals differ, i.e. that the image of a function completely depends on the values of the function at strictly preceding times.

A method of studying existence and uniqueness we propose in this paper goes back to the theory of generalized Volterra operators originally suggested by the second author, see e.g. [15]). We will apply this theory either to equation (1.6) or to equation (1.7). **We stress that these results do not require continuity of the operator** F. We also remark that some examples described in Section 2 are non-Volterra. These examples are only meant to illustrate the general algorithm of how to represent various equations with deviated argument in the standard form (1.3) (or (1.5)). This algorithm is an essential part of the theory of functional differential equations known as *Azbelev's theory*, see e.g. [4] and the references therein.

We also note that the existence and uniqueness in the case when F in (1.3) is an affine operator (more precisely, $(Fx)(t) = \int_{[0,t)} Q(t,s)dx(s) + f(t))$ was studied in [10].

To be able to proceed with further analysis, we need some auxiliary results about the introduced functional spaces and mappings in these spaces.

First of all, we will often use the following "integration by parts formula":

$$\int_{[t_1,t_2)} u(s-0)\,dv(s) = u(t_2-0)v(t_2-0) - u(t_1-0)v(t_1-0) - \int_{[t_1,t_2)} v(s+0)\,du(s), \quad (1.8)$$

which holds for arbitrary functions $u, v : [0, T] \to \mathbf{R}^n$ of bounded variation and any points $0 \le t_1 < t_2 \le T$.

Without loss of generality, we may assume that all functions of finite variation (in particular, functions belonging to W_1^p) are cadlag. Therefore, we can always replace $v(s+0)$ with $v(s)$ in formula (1.8).

The following result is well-known (see e.g. [6]).

Proposition 1.2. *Let S be a μ-measurable subset of the interval $[0, T]$. The linear integral operator $(Qy)(t) \equiv \int_S Q(t,s)\,y(s)\,\mu(ds)$ is bounded as an operator from $L^p(S, \mathbb{R}^n, \mu)$ to $L^q(S, \mathbb{R}^n, \mu)$ ($1 \le p, q < \infty$) if the kernel $Q : S \times S \to \mathbb{R}^{n \times n}$ is a $\mu \otimes \mu$-measurable and satisfies the following condition:*

(1.2) *For μ-almost all $t \in S$ it is required that $Q(t, \cdot) \in L^{p'}(S, \mathbf{R}^{n \times n}, \mu)$, where*

$$p' = \begin{cases} p/(p-1) & \text{if } p > 1, \\ \infty & \text{if } p = 1, \end{cases}$$

and the function ϑ, given as $\vartheta(t) \equiv \|Q(t, \cdot)\|_{L^{p'}}$, belongs to the space $L^q(S, \mathbf{R}, \mu)$.

Note that condition **(1.2)** is fulfilled if for almost all $(t, s) \in S \times S$ the kernel Q satisfies the inequality $|Q(t,s)|_{\mathbf{R}^{n \times n}} \le \varphi(t)$ for some $\varphi \in L^q(S, \mathbf{R}, \mu)$.

Proposition 1.3. *The superposition operator $(Nu)(t) \equiv f(t, u(t))$ is continuous as an operator from $L^q(S, \mathbb{R}^l, \mu)$ to $L^p(S, \mathbb{R}^n, \mu)$ ($1 \le p, q < \infty$) if $f : S \times \mathbf{R}^l \to \mathbf{R}^n$ is a Carathéodory function satisfying $|f(t,x)|_{\mathbf{R}^n}^p \le a(t) + b|x|_{\mathbf{R}^l}^q$ for almost all $t \in S$ and all $u \in \mathbb{R}^l$, where $b \ge 0$ and $a \in L^p(S, \mathbf{R}, \mu)$.*

2   Some examples of the equation (1.3)

In this section we review the notions of a difference equation and its solutions as finite collections of vectors and describe the concept of a functional differential equation and its absolutely continuous solutions, which was suggested and developed by the participants of the Perm Seminar in Russia led by Prof. N. V. Azbelev [4]. Let us also remark that a constantly growing interest to hybrid systems has initiated analysis of objects combining functional differential and difference equations [11].

2.1 Functional differential equations

The example below is a functional differential equation (see e.g. [4])

$$\dot{x}(t) = (\widetilde{F}x)(t), \qquad t \in [0, T], \tag{2.1}$$

where $\widetilde{F} : W_1^p([0, T], \mathbf{R}^n, \text{mes}) \to L^p([0, T], \mathbf{R}^n, \text{mes})$ is a (nonlinear) operator, mes is the Lebesgue measure, $1 \le p < \infty$. In the results presented in the monograph [4] equation (2.1) is assumed to satisfy the following condition.

(2.1) The operator $\widetilde{F} : W_1^p([0, T], \mathbf{R}^n, \text{mes}) \to L^p([0, T], \mathbf{R}^n, \text{mes})$ is continuous.

In Section 3 we describe more specific examples of equation (2.1). All of them include the Volterra property on $\widetilde{F}$, which is not necessarily fulfilled in (2.1).

2.2 Nonlinear difference equations

By this we mean the following system of equations:

$$\Delta x_i \equiv x_i - x_{i-1} = f_i(x_{-1}, x_0, x_1, \ldots, x_m), \qquad i = 0, 1, \ldots, m, \tag{2.2}$$

where $x_0, \ldots, x_m \in \mathbf{R}^n$ are unknown vectors and $x_{-1} = \alpha$ is the initial condition. It is assumed that the functions $f_i : \mathbf{R}^{(m+2)n} \to \mathbf{R}^n$ are continuous.

In this case, the measure μ of a set $S \subset [0, m]$ is equal to the number of integers contained in S. Now we put

$$x(-0) = x_{-1}; \quad x(t) = x_{i-1} \text{ for } t \in [i-1, i), \quad i = 1, \ldots, m; \quad x(m) = x_m.$$

Then "the derivative" of x at $t = 0, 1, \ldots, m$ is given as

$$(\delta_\mu x)(i) = x_i - x_{i-1} = \Delta x_i;$$

while its values $(\delta_\mu x)(t)$ where $t \in (i-1, i)$, $i = 1, \ldots, m$ may be defined arbitrarily or may remain undefined, as $\mu((i-1, i)) = 0$. Indeed, for any $t \in [i-1, i)$ we have

$$x(t) = x(-0) + \sum_{j=0}^{i-1} \Delta x_i = x(-0) + \int_{[0,t]} (\delta_\mu x)(s)\, \mu(ds),$$

and similarly for $t = m$:

$$x(m) = x(-0) + \sum_{j=0}^{m} \Delta x_i = x(-0) + \int_{[0,m]} (\delta_\mu x)(s)\, \mu(ds).$$

Let also $(Fx)(i) \equiv f_i(x_{-1}, x_0, x_1, \ldots, x_m)$, $i = 0, 1, \ldots, m$, again defining the values $(Fx)(t)$ on the set $(i-1, i)$ arbitrarily. By this definition, the operator F acts from $W_1^p([-0, m], \mathbf{R}^n, \mu)$ to $L^p([0, m], \mathbf{R}^n, \mu)$ for any $1 \le p \le \infty$, and equation (2.2) becomes the functional differential equation (1.5).

Note that for the measure just defined we have $L^p([0, m], \mathbf{R}^n, \mu) \simeq \mathbf{R}^{(m+1)n}$ ($m+1$ jumps at the points $t = 0, 1, \ldots, m$) and $W_1^p([-0, m], \mathbf{R}^n, \mu) \simeq \mathbf{R}^{(m+2)n}$ (m constants on the sets $[i-1, i)$, $i = 1, \ldots, m$, plus the values at end points $x(-0) = x_{-1}$, $x(m) = x_m$).

3   Examples with Volterra operators

In this section we assume that the operator F in (1.3) is Volterra.

3.1   Linear nonhomogeneous equation with the unknown function in the differential

This equation, which was studied in [10], is given by

$$dx(t) = \left(\int_{[0,t)} \mathcal{Q}(t,s) dx(s) + B(t)x(-0) + g(t) \right) \mu(dt), \qquad t \in [0,T], \tag{3.1}$$

or, equivalently, by

$$(\delta_\mu x)(t) = \int_{[0,t)} \mathcal{Q}(t,s)(\delta_\mu x)(s)\mu(ds) + B(t)x(-0) + g(t), \qquad t \in [0,T]. \tag{3.2}$$

The assumptions we put on the equation (1.3) are as follows.

(3.1a) $g : [0,T] \to \mathbf{R}^n$ is a μ-measurable function belonging to the space $L^p([0,T],\mathbf{R}^n,\mu)$;

(3.1b) $\mathcal{Q}(t,s)$ is a $n \times n$-matrix with the entries that are $\mu \otimes \mu-$measurable functions defined for $t \in [0,T]$, $s \in [0,t)$. In some cases we find it convenient to extend the function $\mathcal{Q}(t,s)$ to the set $[0,T] \times [0,T]$ assuming that $\mathcal{Q}(t,s) = 0$ for the corresponding (t,s).

(3.1c) For μ-almost all $t \in [0,T]$ the function $\mathcal{Q}(t,\cdot)$ belongs to the space $L^{p'}(S(t),\mathbf{R}^{n \times n},\mu)$, where $S(t) = [0,t)$,

$$p' = \begin{cases} p/(p-1) & \text{if } p > 1, \\ \infty & \text{if } p = 1, \end{cases}$$

and the function ϑ, defined by $\vartheta(t) \equiv \|\mathcal{Q}(t,\cdot)\|_{L^{p'}(S(t),\mathbf{R}^{n \times n},\mu)}$, belongs to the space $L^q([0,T],\mathbf{R},\mu)$.

(3.1d) The function $B : [0,T] \to \mathbf{R}^{n \times n}$ is μ-measurable and belongs to $L^p([0,T],\mathbf{R}^{n \times n},\mu)$.

In [10] it is shown that under the assumptions **(3.1a)**–**(3.1d)** equation (3.1) with the initial condition (1.4) has a unique solution $x \in W_1^p([-0,T],\mathbf{R}^n,\mu)$ for any $\alpha \in \mathbf{R}^n$. The proof suggested in [10] is based on the standard iteration procedure.

Specific examples of the equation (3.1) can be found in [10]. Below we generalize these examples to the nonlinear case.

3.2   Nonlinear differential equations with delay

In this subsection we demonstrate how delay equations can be written in the standard form (2.1). Note that we consider only the case of distributed delays. Some more involved examples can be found in [4].

Let

$$\dot{x}(t) = f\left(t, \int_{(-\infty,t)} d_s R(t,s) x(s) \right), \qquad t \in [0,T]. \tag{3.3}$$

It is assumed that this equation is supplied with the "prehistory" condition:

$$x(s) = \psi(s), \qquad s < 0. \tag{3.4}$$

Following [4] we will now include this condition into the equation (3.3) in such a way that the initial condition (1.4) remains unchanged.

We separate conditions for $s < 0$ and $s = 0$, in particular, for the following reason: since ψ is often assumed to belong to a space consisting of measurable functions, the functional $\psi(\cdot) \longmapsto \psi(0)$ may have no sense. On the other hand, if ψ is continuous and the solutions of (3.3) are supposed to be continuous for all $t \in (-\infty, T]$, as well, then we can assume that $\psi(0) = \alpha$. Let us however stress that even in this continuous case separating the conditions for $s < 0$ and $s = 0$ may be technically useful (see e.g. [4]).

We now list the assumptions on $R(t,s)$ and $f(t,u)$, which we need to be able to rewrite (3.3) in the form (2.1). Let us choose two real numbers $p, q \in [1, \infty)$ and a natural number m.

(3.2a) The entries of $mn \times n$-matrix function $R(\cdot, \cdot)$ are Lebesgue measurable on $[0, T] \times (-\infty, T]$.

(3.2b) For any $t \in [0, T]$ the function $R(t, \cdot)$ is of bounded variation.

(3.2c) $\mathrm{Var}_{s \in [0,T]} R(\cdot, s) \in L^q([0, T], \mathbf{R}, \mathrm{mes})$.

(3.2d) $\int_{(-\infty,0)} d_s R(\cdot, s) \psi(s) \in L^q([0, T], \mathbf{R}, \mathrm{mes})$.

(3.2e) The function $f : [0, T] \times \mathbf{R}^{mn} \to \mathbf{R}^n$ is Carathéodory (i.e. $f(\cdot, u)$ is Lebesgue measurable for each $u \in \mathbf{R}^{mn}$ and $f(t, \cdot)$ is continuous for mes-almost all $t \in [0, T]$) and for some $a \in L^1([0, T], \mathbf{R}, \mathrm{mes})$ and $b \geq 0$ satisfies $|f(t, u)|^p_{\mathbf{R}^n} \leq a(t) + b|u|^q_{\mathbf{R}^{mn}}$ ($t \in [0, T]$ and $u \in \mathbf{R}^{mn}$).

For instance, the equation

$$\dot{x}(t) = f\big(t, x(h_1(t)), \ldots, x(h_m(t))\big), \quad t \in [0, T]; \qquad x(s) = \psi(s), \quad s < 0, \qquad (3.5)$$

with the delay condition $h(t) \leq t, t \in [0, T]$, can be rewritten in the form (3.3) if we put

$$R(t,s) = \Big(\mathbf{1}_{(-\infty,h_1(t)]}(s) \cdot I, \ldots, \mathbf{1}_{(-\infty,h_m(t)]}(s) \cdot I\Big)^T, \qquad I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & 1 \end{pmatrix}_{n \times n}.$$

Evidently, $R(\cdot, \cdot)$ satisfies the assumptions **(3.2a)**–**(3.2d)** for any $q \geq 1$ if $h_i(\cdot)$ is Lebesgue measurable.

To represent the system (3.3)–(3.4) in the form (1.3) we put

$$Q(t,s) = -R(t,s) + \mathbf{1}_{[0,t)}(s) \cdot R(t, t-0), \qquad t \in [0, T], \ s \in (-\infty, T]$$

$$\widetilde{f}(t,u) = f\left(t, u + \int_{(-\infty,0)} d_s R(t,s)\, \psi(s)\right), \qquad t \in [0, T].$$

Then $Q(t, t-0) = 0$, $Q(t, -0) = -R(t, -0)$, and using the integration by parts formula (1.8) we obtain

$$\int_{[0,t)} Q(t,s) dx(s) = \int_{[0,t)} Q(t, s+0) dx(s)$$

$$= Q(t, t-0)x(t-0) - Q(t, -0)x(-0) - \int_{[0,t)} d_s Q(t,s) x(s-0)$$

$$= R(t, -0)x(0) - \int_{[0,t)} d_s Q(t,s) x(s-0)$$

$$= R(t, -0)x(0) + \int_{[0,t)} d_s R(t,s) x(s), \qquad t \in [0, T].$$

Minding

$$\int_{[0,t)} \mathcal{Q}(t,s)dx(s) = \int_{[0,t)} \mathcal{Q}(t,s)\dot{x}(s)ds, \qquad t \in (0,T],$$

we see that (3.3)–(3.4) become

$$\dot{x} = (N \circ Q)\,x,$$

where $(Q\,x)(t) \equiv -R(t,-0)\,x(0) + \int_{[0,t)} \mathcal{Q}(t,s)\,\dot{x}(s)ds$ and $(Nu)(t) \equiv \widetilde{f}(t,u(t))$, $t \in (0,T]$. Propositions 1.2–1.3 and the assumptions **(3.2a)–(3.2e)** ensure that the operator $\widetilde{F} \equiv N \circ Q$ continuously acts from $W_1^p([0,T],\mathbf{R}^n,\text{mes})$ to $L^p([0,T],\mathbf{R}^n,\text{mes})$. This operator is Volterra.

3.3 Linear difference equations with delay

We describe a particular case of the difference equation (2.2) which can also be represented in the form (3.1) or (3.2).

Let

$$\Delta x_0 = g_0, \qquad \Delta x_i = \sum_{j=0}^{i-1} A_{ij}x_j + g_i, \qquad i = 1,\ldots,m, \tag{3.6}$$

where we assume that A_{ij} are $n \times n-$matrices and g_0, g_i are $n-$vectors, $i = 1,\ldots,m$, $j = 0,\ldots,m-1$. Using the equality $x_j = x_{-1} + \sum_{p=0}^{j} \Delta x_p$ we rewrite equation (3.6) as follows:

$$\sum_{j=0}^{i-1} A_{ij}x_j = \sum_{j=0}^{i-1} A_{ij}x_{-1} + \sum_{j=0}^{i-1} A_{ij} \sum_{p=0}^{j} \Delta x_p = \sum_{j=0}^{i-1} A_{ij}x_{-1} + \sum_{j=0}^{i-1}\sum_{p=j}^{i-1} A_{ip}\Delta x_j.$$

Then we define $\mathcal{Q}_{ij} = \sum_{p=j}^{i-1} A_{ip}$, and represent equation (3.6) as

$$\Delta x_0 = g_0, \qquad \Delta x_i = \sum_{j=0}^{i-1} \mathcal{Q}_{ij}\Delta x_j + \mathcal{Q}_{i0}x_{-1} + g_i, \qquad i = 1,\ldots,m.$$

As in Subsection 2.2, the measure μ of a set $S \subset [0,m]$ is now equal to the number of integers contained in S. The $\mu \otimes \mu$-measurable function is defined as $\mathcal{Q}: [0,m] \times [0,m] \to \mathbf{R}^n$, $\mathcal{Q}(i,j) = \mathcal{Q}_{ij}$ for integers, while the values of $\mathcal{Q}(t,s)$ at the points (t,s), where at least one component is not an integer, are not needed. Then we define the μ-measurable function $g: [0,m] \to \mathbf{R}^n$ by setting $g(i) = g_i$ and observing that for $t \in (i-1,i)$, $i = 1,\ldots,m$ the values $g(t)$ may be disregarded. Finally, we choose an arbitrary $1 \le p \le \infty$ and define the function $x \in W_1^p([-0,m],\mathbf{R}^n,\mu)$ to equal

$$x(-0) = x_{-1}; \quad x(t) = x_{i-1} \quad \text{for all } t \in [i-1,i), \quad i = 1,\ldots,m; \quad x(m) = x_m.$$

"The derivative" $(\delta_\mu x)(i) = x_i - x_{i-1} = \Delta x_i$, of this function can be defined arbitrarily (or remain undefined) for any $t \in (i-1,i)$, $i = 1,\ldots,m$.

Thus, equation (3.6) becomes

$$(\delta_\mu x)(t) = \int_{[0,t)} \mathcal{Q}(t,s)(\delta_\mu x)(s)\mu(dt) + \mathcal{Q}(t,0)x(-0) + g(t), \qquad t \in [0,m],$$

and we obtain the representation (3.2).

3.4 Impulsive differential equations with delay

We return to the functional differential equation (2.1), but in this subsection we assume that a countable (in particularly, finite) set $\mathcal{T} \subset (0, T]$ is given and at any time $\tau \in \mathcal{T}$ the solution can make a jump $\Delta x(\tau) \equiv x(\tau) - x(\tau - 0)$.

To formalize the notion of such an impulsive functional differential equation we suppose that to any $\tau \in \mathcal{T}$ a positive number $\mathcal{M}(\tau)$ is assigned in such a way that the series $\sum_{\tau \in \mathcal{T}} \mathcal{M}(\tau)$ converges. Then we are able to define a finite measure μ on $[0, T]$ by putting

$$\mu \equiv \mathrm{mes} + \mu_{\mathcal{T}}, \qquad \mu_{\mathcal{T}} \equiv \sum_{\tau \in \mathcal{T}} \nu_\tau \mathcal{M}(\tau), \tag{3.7}$$

where ν_τ is the Dirac measure at τ. In other words, the measure $\mu(S)$ of a set $S \in \mathfrak{L}$ is equal to the sum of its Lebesgue measure $\mathrm{mes}(S)$ and $\sum_{\tau \in \mathcal{T} \cap S} \mathcal{M}(\tau)$.

Below we consider an impulsive functional differential equation under the following assumptions.

The behavior of the solution $x(\cdot)$ outside $\mathcal{T}$ is governed by equation (2.1) with the nonlinear operator $\widetilde{F} : W_1^p([-0, T], \mathbf{R}^n, \mu) \to L^p([0, T], \mathbf{R}^n, \mathrm{mes})$, $p \in [1, \infty)$, satisfying the following condition.

(3.4a) The operator $\widetilde{F}$ is Volterra, i.e. for any $t \in (0, T]$ and any $x, \widehat{x} \in W_1^p([-0, T], \mathbf{R}^n, \mu)$, for which $x(s) = \widehat{x}(s)$ ($s \in [-0, t)$), the equality $(\widetilde{F}x)(s) = (\widetilde{F}\widehat{x})(s)$ is satisfied almost everywhere on $[0, t]$ w.r.t. the Lebesgue measure mes.

Further, we assume that the value of the jump $\Delta x(\tau)$ at time $\tau \in \mathcal{T}$ may only depend on the values of the solution $x(t)$ for $t \in [0, \tau)$. More precisely, we impose the following requirement on the jumps:

$$\Delta x(\tau) = Y(\tau, x), \qquad \tau \in \mathcal{T}, \tag{3.8}$$

where the vector functional (possibly nonlinear) $Y : \mathcal{T} \times W_1^p([-0, T], \mathbf{R}^n, \mu) \to \mathbf{R}^n$ satisfies the following assumptions.

(3.4b) For any $\tau \in \mathcal{T}$ and arbitrary $x, \widehat{x} \in W_1^p([-0, T], \mathbf{R}^n, \mu)$, satisfying $x(s) = \widehat{x}(s)$ for all $s \in [-0, \tau)$, one has $Y(\tau, x) = Y(\tau, \widehat{x})$.

(3.4c) For any $x \in W_1^p([-0, T], \mathbf{R}^n, \mu)$ one has $Y(\cdot, x)/\mathcal{M}(\cdot) \in L^p([0, T], \mathbf{R}^n, \mu_{\mathcal{T}})$, or equivalently, $\int_{[0,T]} \left| Y(\tau, x)/\mathcal{M}(\tau) \right|_{\mathbf{R}^n}^p \mu_{\mathcal{T}}(d\tau) = \sum_{\tau \in \mathcal{T}} \left| Y(\tau, x) \right|_{\mathbf{R}^n}^p \mathcal{M}(\tau)^{1-p} < \infty$.

Let us verify that under the assumptions **(3.4a)–(3.4c)** the system (2.1),(3.8) can be represented in the general form (1.3).

To see it, we put

$$(Fx)(t) = \begin{cases} (\widetilde{F}x)(t) & \text{if } t \in [0, T] - \mathcal{T}, \\ Y(t, x)/\mathcal{M}(t) & \text{if } t \in \mathcal{T}. \end{cases} \tag{3.9}$$

From **(3.4c)** it follows that the operator F acts from $W_1^p([-0, T], \mathbf{R}^n, \mu)$ to $L^p([0, T], \mathbf{R}^n, \mu)$, and it is Volterra due to the assumptions **(3.4a)**, **(3.4b)**.

We claim further that the operator $F : W_1^p \to L^p$ (defined by (3.9)) becomes continuous if the following conditions are fulfilled.

(3.4d) The operator $\widetilde{F} : W_1^p([-0, T], \mathbf{R}^n, \mu) \to L^p([0, T], \mathbf{R}^n, \mathrm{mes})$ is continuous.

(3.4e) The mapping $x \in W_1^p([-0,T],\mathbf{R}^n,\mu) \mapsto \mathrm{Y}(\cdot,x)/\mathcal{M}(\cdot) \in L^p([0,T],\mathbf{R}^n,\mu_{\mathcal{T}})$ is continuous.

To prove it, we choose any convergent sequence $x_n \to x$ in the space $W_1^p([-0,T],\mathbf{R}^n,\mu)$. Then

$$\|Fx_n - Fx\|_{L^p}^p = \int_{[0,T]} |(Fx_n)(t) - (Fx)(t)|_{\mathbf{R}^n}^p \, \mu(dt)$$

$$= \int_{[0,T]} |(\widetilde{F}x_n)(t) - (\widetilde{F}x)(t)|_{\mathbf{R}^n}^p \, dt + \sum_{\tau \in \mathcal{T}} |\mathrm{Y}(\tau,x_n)/\mathcal{M}(\tau) - \mathrm{Y}(\tau,x)/\mathcal{M}(\tau)|_{\mathbf{R}^n}^p \mathcal{M}(\tau).$$

It remains to observe that $\int_{[0,T]} |(\widetilde{F}x_n)(t) - (\widetilde{F}x)(t)|_{\mathbf{R}^n}^p \, dt \to 0$ due to **(3.4d)**, while **(3.4e)** implies

$$\sum_{\tau \in \mathcal{T}} |\mathrm{Y}(\tau,x_n)/\mathcal{M}(\tau) - \mathrm{Y}(\tau,x)/\mathcal{M}(\tau)|_{\mathbf{R}^n}^p \mathcal{M}(\tau) \to 0.$$

Let us now look closer at the affine case. In this case, the equation (2.1) converts into

$$\dot{x}(t) = \int_{(-\infty,t)} d_s R(t,s)x(s) + \widetilde{g}(t), \quad t \in [0,T] - \mathcal{T}, \qquad x(s) = \psi(s), \quad s < 0. \tag{3.10}$$

We observe as well that defining the measure μ by the formula (3.7) yields the following "derivative" of a function x which is absolutely continuous w.r.t. this measure:

$$(\delta_\mu x)(t) = \begin{cases} \dot{x}(t) & \text{if } t \in [0,T] - \mathcal{T}, \\ \Delta x(t)/\mathcal{M}(t) & \text{if } t \in \mathcal{T}. \end{cases}$$

Therefore, an arbitrary affine and bounded vector functional $\mathrm{Y} : \mathcal{T} \times W_1^p([-0,T],\mathbf{R}^n,\mu) \to \mathbf{R}^n$, which is affine and bounded w.r.t. the second variable, becomes

$$\mathrm{Y}(\tau,x) = \int_{[0,\tau]} \mathcal{W}(\tau,s)\dot{x}(s)ds + \sum_{\sigma \in \mathcal{T}, \sigma < \tau} \mathfrak{W}(\tau,\sigma)\Delta x(\sigma) + \omega(\tau)x(-0) + \omega_0(\tau) \tag{3.11}$$

if the following three assumptions are fulfilled.

(3.4f) For any $\tau \in \mathcal{T}$ the $n \times n$-matrix function $\mathcal{W}(\tau,\cdot)$ belongs to $L^{p'}(S(\tau),\mathbf{R}^{n \times n},\mathrm{mes})$, where $S(\tau) = [0,\tau]$,

$$p' = \begin{cases} p/(p-1) & \text{if } p > 1, \\ \infty & \text{if } p = 1, \end{cases}$$

and the function ϑ, defined by

$$\vartheta(\tau) = \|\mathcal{W}(\tau,\cdot)\|_{L^{p'}(S(\tau),\mathbf{R}^{n \times n},\mathrm{mes})} / \mathcal{M}(\tau),$$

belongs to $L^p([0,T],\mathbf{R}^n,\mu_{\mathcal{T}})$, i.e.

$$\sum_{\tau \in \mathcal{T}} \left(\mathcal{M}(\tau)^{(1-p)/p} \|\mathcal{W}(\tau,\cdot)\|_{L^{p'}(S(\tau),\mathbf{R}^{n \times n},\mathrm{mes})}\right)^p < \infty.$$

(3.4g) For any $\tau \in \mathcal{T}$ the $n \times n$-matrix function $\mathfrak{W}(\tau,\cdot) \in L^{p'}(S(\tau),\mathbf{R}^{n \times n},\mu_{\mathcal{T}})$, where p' and $S(\tau)$ are defined above, and $\mathfrak{v} \in L^p([0,T],\mathbf{R}^n,\mu_{\mathcal{T}})$, where the function $\mathfrak{v}$ is defined by

$$\mathfrak{v}(\tau) = \|\mathfrak{W}(\tau,\cdot)\|_{L^{p'}(S(\tau),\mathbf{R}^{n \times n},\mu_{\mathcal{T}})} / \mathcal{M}(\tau);$$

in other words,

$$\sum_{\tau \in \mathcal{T}} \left(\mathcal{M}(\tau)^{(1-p)/p} \|\mathfrak{W}(\tau,\cdot)\|_{L^{p'}(S(\tau),\mathbf{R}^{n \times n},\mu_{\mathcal{T}})}\right)^p < \infty.$$

(3.4h) For the $n \times n$-matrix $\omega(\tau)$ and the n-dimensional vector $\omega_0(\tau)$ the following holds
true:

$$\sum_{\tau \in \mathcal{T}} \mathcal{M}(\tau)|\omega(\tau)|_{\mathbf{R}^{n \times n}}^p < \infty, \qquad \sum_{\tau \in \mathcal{T}} \mathcal{M}(\tau)|\omega_0(\tau)|_{\mathbf{R}^n}^p < \infty.$$

Thanks to the assumptions **(3.1a)** (put on the function $\widetilde{g}$), **(3.2a)–(3.2d)** (where $m = 1$,
$q = p$), **(3.4f)–(3.4h)** the system (3.10)–(3.11) is equivalent to equation (3.2). In order to prove
this fact, we put

$$\mathcal{Q}(t,s) = \begin{cases} -R(t,s) + 1_{[0,t)}(s) \cdot R(t,t-0) & \text{if } t,s \in (0,T] - \mathcal{T}, \\ \mathcal{W}(t,s) & \text{if } t \in \mathcal{T},\ s \in (0,T] - \mathcal{T}, \\ \mathfrak{W}(t,s) & \text{if } t,s \in \mathcal{T}; \end{cases}$$

$$B(t) = \begin{cases} R(t,-0) & \text{if } t \in (0,T] - \mathcal{T}, \\ \omega(t) & \text{if } t \in \mathcal{T}; \end{cases}$$

$$g(t) = \begin{cases} \widetilde{g}(t) + \int_{(-\infty,0)} d_s R(t,s)\psi(s) & \text{if } t \in (0,T] - \mathcal{T}, \\ \omega_0(t) & \text{if } t \in \mathcal{T}, \end{cases}$$

and substitute these functions to the right-hand side $(Fx)(t) \equiv \int_{[0,t)} \mathcal{Q}(t,s)(\delta_\mu x)(s)\mu(ds) + B(t)x(-0) + g(t)$ of equation (3.2).

For an arbitrary $t \in (0,T] - \mathcal{T}$ the integration by parts formula (1.8) and the observations
$\mathcal{Q}(t,t-0) = 0$ and $x(s-0) = x(s)$ for almost all $s \in [0,T]$ yield

$$\begin{aligned}
(Fx)(t) &= \int_{[0,t)} \mathcal{Q}(t,s)dx(s) + R(t,-0)x(-0) + \widetilde{g}(t) + \int_{(-\infty,0)} d_s R(t,s)\psi(s) \\
&= \mathcal{Q}(t,t-0)x(t-0) - \mathcal{Q}(t,-0)x(-0) \\
&\quad - \int_{[0,t)} d_s \mathcal{Q}(t,s)x(s-0) + R(t,-0)x(-0) + \widetilde{g}(t) + \int_{(-\infty,0)} d_s R(t,s)\psi(s) \\
&= \int_{[0,t)} d_s R(t,s)x(s) + \int_{(-\infty,0)} d_s R(t,s)\psi(s) + \widetilde{g}(t) = \int_{(-\infty,t)} d_s R(t,s)x(s) + \widetilde{g}(t).
\end{aligned}$$

Thus, for $t \in (0,T] - \mathcal{T}$ equation (3.2) with the functions $\mathcal{Q}(t,s)$, $B(t)$, $g(t)$ coincide with
equation (3.10).

For $\tau \in \mathcal{T}$ we have

$$\begin{aligned}
(Fx)(\tau) &= \frac{1}{\mathcal{M}(\tau)}\left(\int_{[0,\tau)} \mathcal{Q}(\tau,s)dx(s) + \omega(\tau)x(-0) + \omega_0(\tau)\right) \\
&= \frac{1}{\mathcal{M}(\tau)}\left(\int_{[0,\tau)-\mathcal{T}} \mathcal{W}(\tau,s)dx(s) + \int_{[0,\tau)\cap\mathcal{T}} \mathfrak{W}(\tau,s)dx(s) + \omega(\tau)x(-0) + \omega_0(\tau)\right) \\
&= \frac{1}{\mathcal{M}(\tau)}\left(\int_{[0,\tau]} \mathcal{W}(\tau,s)\dot{x}(s)ds + \sum_{\sigma\in\mathcal{T},\sigma<\tau} \mathfrak{W}(\tau,\sigma)\Delta x(\sigma) + \omega(\tau)x(-0) + \omega_0(\tau)\right),
\end{aligned}$$

and we arrive at (3.11).

4 Existence and uniqueness of solutions

In this section we consider the general equation (1.3) with the initial condition (1.4).

4.1   Volterra operators in the space $L^p\left([0,T],\mathbf{R}^n,\mu\right)$

In the sequel we will always assume that the (nonlinear) operator $F : W_1^p([-0,T],\mathbf{R}^n,\mu) \to L^p([0,T],\mathbf{R}^n,\mu)$ is Volterra. The operators considered in Section 3 have this property, including the operator defined by (3.9) if the assumptions **(3.4a)**, **(3.4b)** are fulfilled. We just remark that, unlike the usual derivative, "the differentiation" $\delta_\mu : W_1^p([-0,T],\mathbf{R}^n,\mu) \to L^p([0,T],\mathbf{R}^n,\mu)$ is not Volterra if the interval $[0,T]$ has points of positive measure.

Definition 4.1. If there exists a number $\xi \in (0,T)$ and a function $u_\xi : [-0,\xi) \to \mathbf{R}^n$ satisfying the initial condition (1.4), the extension $u : [-0,T] \to \mathbf{R}^n$,

$$u(t) = \begin{cases} u_\xi(t) & \text{if } t \in [-0,\xi), \\ u_\xi(\xi - 0) & \text{if } t \in [\xi,T] \end{cases}$$

of which belongs to the space $W_1^p\left([-0,T],\mathbf{R}^n,\mu\right)$ and which μ-almost everywhere on $[0,\xi)$ satisfies equation (1.3), then the initial value problem (1.3), (1.4) is called locally solvable, and the function u_ξ is called its local solution defined on $[-0,\xi)$. The function $u \in W_1^p\left([-0,T],\mathbf{R}^n,\mu\right)$, satisfying the condition (1.4) and equation (1.3) on the entire $[0,T]$, is called a global solution. The function $u_\eta : [-0,\eta) \to \mathbf{R}^n$, whose restriction u_ξ to any subinterval $[-0,\xi) \subset [-0,\eta)$, $0 < \xi < \eta$, is a local solution, and $\lim_{\xi \to \eta - 0} \int_{[0,\xi)} |(\delta_\mu u_\xi)(s)|\, ds = \infty$, is called an unextendable solution. A solution (local, global and unextendable) u_η is called an extension of a local solution u_ξ if $\eta > \xi$ and $u_\eta(t) = u_\xi(t)$ for $t \in [-0,\xi)$.

Let us make use of the representation (1.7) of the functional differential equation (1.3) and rewrite the initial value problem for this equation with the initial condition (1.4) as an equation in the space $L^p\left([0,t],\mathbf{R}^n,\mu\right)$

$$y(t) = \left(F\left(\alpha + \int_{[0,\cdot]} y(s)\,\mu(ds) \right) \right)(t), \qquad t \in [0,T]. \tag{4.1}$$

This is an equation w.r.t. $y = \delta_\mu x$. Given $x \in W_1^p\left([-0,T],\mathbf{R}^n,\mu\right)$, the norm of the restriction of the image $y = Fx : [0,T] \to \mathbf{R}^n$ to the subinterval $[0,t)$, calculated in the space $L^p\left([0,t),\mathbf{R}^n,\mu\right)$, is, in general, a discontinuous function of t. This is due to the fact that the measure μ is not assumed to be absolutely continuous w.r.t. the Lebesgue measure. This fact explains why a straightforward application of the classical Volterra theory and its known generalizations to equation (1.7) is impossible. Below we apply an idea of a generalized Volterra property which was suggested in the paper [15].

Let B be a normed space. Suppose that to any $\gamma \in [0,1]$ we assign an equivalence relation $v(\gamma)$ for the elements of the space B. Assume further that the family $\mathfrak{V} = \left\{ v(\gamma) \mid \gamma \in [0,1] \right\}$ satisfies the following conditions:

$$v(0) = B^2; \quad v(1) = \left\{ (x,x) \ x \in B \right\}; \quad \gamma > \eta \Rightarrow v(\gamma) \subset v(\eta).$$

Finally, we assume that the relations $v(\gamma) \in \mathfrak{V}$ are closed under addition and multiplication by scalars, i.e. that for every $\gamma \in (0,1)$ and any $x,\widehat{x},y,\widehat{y} \in B$, λ we have

$$(x,\widehat{x}) \in v(\gamma), \quad (y,\widehat{y}) \in v(\gamma) \ \Rightarrow \ (x+y,\widehat{x}+\widehat{y}) \in v(\gamma), \quad (\lambda x, \lambda \widehat{x}) \in v(\gamma).$$

Definition 4.2. We say that an operator $\Phi : B \to B$ is Volterra w.r.t. the family $\mathfrak{V}$ of equivalence relations (satisfying the above conditions) if for every $\gamma \in (0,1)$ and any $x,y \in B$ the equality $(x,y) \in v(\gamma)$ implies the equality $(\Phi x, \Phi y) \in v(\gamma)$.

As an example, let us consider the following family $\mathfrak{V}$ of equivalence relations $v(\gamma)$ in the space $L^p([0,T],\mathbf{R}^n,\mu)$: we write $(y,\widehat{y}) \in v(\gamma), 0 < \gamma < 1$ if $y(s) = \widehat{y}(s)$ for almost all (w.r.t. μ) $s \in [0,\xi(\gamma))$, where

$$\xi(\gamma) = \sup\{\, t \mid \mu([0,t)) \le \gamma \cdot M \,\}, \qquad M = \mu([0,T))$$

(the set $[0,0)$ is considered to be empty, so that $\mu[0,0) = 0$). Evidently, any Volterra operator in the space $L^p([0,T],\mathbf{R}^n,\mu)$ is also Volterra w.r.t. the just defined family $\mathfrak{V}$. In particular, the following operator generated by the equation (1.7) is Volterra w.r.t. $\mathfrak{V}$:

$$\Phi : L^p([0,T],\mathbf{R}^n,\mu) \to L^p([0,T],\mathbf{R}^n,\mu), \qquad \Phi y = F\left(\alpha + \int_{[0,(\cdot))} y(s)\,\mu(ds)\right). \tag{4.2}$$

Lemma 4.3. *For every $\gamma \in (0,1)$ one has the estimates $\mu([0,\xi(\gamma))) \le \gamma \cdot M$, $\mu([0,\xi(\gamma)]) \ge \gamma \cdot M$.*

Proof. Let $\mu([0,\xi(\gamma))) = \gamma \cdot M + \varepsilon$, $\varepsilon > 0$. Due to [14, pp. 86–87] there exists a positive δ such that $\mu([\xi(\gamma) - \delta, \xi(\gamma))) < \varepsilon$. Thus, $\mu([0,\xi(\gamma) - \delta)) > \gamma \cdot M$, which contradicts the definitions of $\xi(\gamma)$. The second estimate can be proved similarly. $\qquad\square$

4.2 The initial value problem for the general equation

Theorem 4.4. *Let the Volterra operator $F : W_1^p([-0,T],\mathbf{R}^n,\mu) \to L^p([0,T],\mathbf{R}^n,\mu)$ satisfy the condition*

(4.2a) *there are $\Delta > 0$, $q < 1$ such that for any ξ_1, ξ_2 satisfying $0 \le \xi_1 < \xi_2 \le T$, $\mu((\xi_1,\xi_2)) < \Delta$ and arbitrary $x,\widehat{x} \in W_1^p([-0,T],\mathbf{R}^n,\mu)$, satisfying the initial condition (1.4) the following holds true: if for all $t \in [-0,\xi_1]$ one has $x(t) = \widehat{x}(t)$, then*

$$\left(\int_{(\xi_1,\xi_2)} \left| (Fx)(s) - (F\widehat{x})(s) \right|_{\mathbf{R}^n}^p \mu(ds)\right)^{1/p} \le q\|x - \widehat{x}\|_{W_1^p}.$$

In this case the initial value problem (1.3), (1.4) has a unique global solution, and any of its local solutions is the restriction of the global one.

Proof. The proof consists of verifying the conditions of Corollary from Theorem 4 proved in the paper [15, p. 448] for the operator (4.2). These conditions describe the property which in this paper is called "local contraction" and which guarantee unique solvability of the equation (4.1) and hence of the initial value problem (1.3), (1.4).

Denote $\gamma_0 = \frac{\mu(\{0\})}{M}$. If $\gamma_0 = 0$, then for any δ, $0 < \delta < \frac{\Delta}{M}$ the interval $[0, \xi(\delta))$ is not empty, as $\mu([0, \xi(\delta)]) \ge \delta M > 0 = \mu(\{0\})$, see Lemma 4.3. The isomorphism of the spaces W_1^p and $L^p \times \mathbf{R}^n$ allows for using **(4.2a)** for $y, \widehat{y} \in L^p([0,T],\mathbf{R}^n,\mu)$. Thus,

$$\left(\int_{[0,\xi(\delta))} \left| (\Phi y)(s) - (\Phi\widehat{y})(s) \right|^p \mu(ds)\right)^{1/p} = \left(\int_{(0,\xi(\delta))} \left| (\Phi y)(s) - (\Phi\widehat{y})(s) \right|^p \mu(ds)\right)^{1/p}$$
$$\le q\|y - \widehat{y}\|_{L^p}.$$

If $\gamma_0 > 0$, then for any δ, $0 < \delta \le \gamma_0$ the interval $[0, \xi(\delta))$ is empty and hence

$$\left(\int_{[0,\xi(\delta))} \left| (\Phi y)(s) - (\Phi\widehat{y})(s) \right|^p \mu(ds)\right)^{1/p} = 0 \le q\|y - \widehat{y}\|_{L^p}$$

for all $y, \widehat{y} \in L^p([0,T], \mathbf{R}^n, \mu)$.

Let us choose an arbitrary $\gamma \in [0,1)$ and any $v(\gamma)$-equivalent elements $y, \widehat{y} \in L^p([0,T], \mathbf{R}^n, \mu)$, which means that $y(t) = \widehat{y}(t)$ for almost all (w.r.t. μ) $t \in [0, \xi(\gamma))$. Due to the Volterra property of the operator Φ we obtain $(\Phi x)(t) = (\Phi \widehat{x})(t)$ for almost all (w.r.t. μ) $t \in [0, \xi(\gamma)]$. According to Lemma 4.3, $\mu((\xi(\gamma), \xi(\gamma + \delta))) \leq \delta \cdot M$, so that for any $\delta < \frac{\Delta}{M}$ and all $y, \widehat{y} \in L^p([0,T], \mathbf{R}^n, \mu)$ we have

$$\left(\int_{[0,\xi(\gamma+\delta))} | (\Phi y)(s) - (\Phi \widehat{y})(s)|^p \mu(ds) \right)^{1/p} = \left(\int_{(\xi(\gamma),\xi(\gamma+\delta))} | (\Phi y)(s) - (\Phi \widehat{y})(s)|^p \mu(ds) \right)^{1/p}$$
$$\leq q \|y - \widehat{y}\|_{L^p}.$$

We have verified the properties of the operator Φ which guarantee the unique solvability of the equation (4.1). $\qquad \square$

Remark 4.5. The operators F and (4.2) in the above theorem do not need to be continuous. An example of a local contraction in the space $L^p([0,T], \mathbb{R}^n, \text{mes})$ which is nowhere continuous, can be found in [16].

4.3 The initial value problem for impulsive equations

In this subsection we apply Theorem 4.4 to the impulsive system (2.1), (3.8). Assuming that the conditions **(3.4a)–(3.4c)** are fulfilled and defining the measure μ on $[0,T]$ by (3.7) we can reduce the system (2.1),(3.8) to equation (1.3) with the operator $F : W_1^p([-0,T], \mathbf{R}^n, \mu) \to L^p([0,T], \mathbf{R}^n, \mu)$ given by (3.9). Under these assumptions the operator is Volterra (see Subsection 3.4).

Theorem 4.6. *Let the Volterra operator* $\widetilde{F} : W_1^p([-0,T], \mathbf{R}^n, \mu) \to L^p([0,T], \mathbf{R}^n, \text{mes})$ *satisfy the condition*

(4.3a) *there are* $\Delta_{\widetilde{F}} > 0$, $q_{\widetilde{F}}$ *such that for any* ξ_1, ξ_2 *satisfying* $0 \leq \xi_1 < \xi_2 \leq T$, $\mu((\xi_1, \xi_2)) < \Delta_{\widetilde{F}}$ *and arbitrary* $x, \widehat{x} \in W_1^p([-0,T], \mathbf{R}^n, \mu)$ *satisfying the initial condition (1.4), the following holds true: if for all* $t \in [-0, \xi_1]$ *one has* $x(t) = \widehat{x}(t)$, *then*

$$\left(\int_{(\xi_1,\xi_2)} | (\widetilde{F}x)(s) - (\widetilde{F}\widehat{x})(s)|_{\mathbf{R}^n}^p \, ds \right)^{1/p} \leq q_{\widetilde{F}} \|x - \widehat{x}\|_{W_1^p}.$$

Let, in addition, the vector functional $Y : \mathcal{T} \times W_1^p([-0,T], \mathbf{R}^n, \mu) \to \mathbf{R}^n$ *have the property*

(4.3b) *there exist a subset* $\mathfrak{T} \subset \mathcal{T}$, *for which* $\mathcal{T} - \mathfrak{T}$ *is finite, and the numbers* $\Delta_Y > 0$, q_Y *such that for any* ξ_1, ξ_2 *satisfying* $0 \leq \xi_1 < \xi_2 \leq T$, $\mu((\xi_1, \xi_2)) < \Delta_Y$ *and arbitrary* $x, \widehat{x} \in W_1^p([-0,T], \mathbf{R}^n, \mu)$, *satisfying the condition (1.4), the following holds true: if for all* $t \in [-0, \xi_1]$ *one has* $x(t) = \widehat{x}(t)$, *then*

$$\left(\sum_{\tau \in \mathfrak{T} \cap (\xi_1,\xi_2)} |Y(\tau,x) - Y(\tau,\widehat{x})|^p \mathcal{M}(\tau)^{1-p} \right)^{1/p} \leq q_Y \|x - \widehat{x}\|_{W_1^p}.$$

If now $q_{\widetilde{F}} + q_Y < 1$, *then the initial value problem (2.1), (3.8), (1.4) has a unique global solution, and any of its local solutions is a restriction of the global one.*

Proof. Letting $\tau_1, \tau_2, \ldots, \tau_m$ be all the elements of $\mathcal{T} - \mathfrak{T}$, we define

$$\Delta = \min\left\{ \Delta_{\widetilde{F}}, \Delta_Y, \mathcal{M}(\tau_1), \ldots, \mathcal{M}(\tau_m) \right\}.$$

For any ξ_1, ξ_2 such that $\mu((\xi_1, \xi_2)) < \Delta$ and for all $x, \widehat{x} \in W_1^p([-0, T], \mathbf{R}^n, \mu)$ satisfying (1.4) and the equality $x(t) = \widehat{x}(t)$, $t \in [0, \xi_1]$ the assumptions **(4.3a)**, **(4.3b)** and the formula (3.9) for the operator $F : W_1^p([-0, T], \mathbf{R}^n, \mu) \to L^p([0, T], \mathbf{R}^n, \mu)$ yield

$$\left(\int_{(\xi_1, \xi_2)} \left| (Fx)(s) - (F\widehat{x})(s) \right|^p \mu(ds) \right)^{1/p}$$

$$\leq \left(\int_{(\xi_1, \xi_2)} \left| (\widetilde{F}x)(s) - (\widetilde{F}\widehat{x})(s) \right|^p ds \right)^{1/p} + \left(\sum_{\tau \in \mathcal{T} \cap (\xi_1, \xi_2)} |Y(\tau, x) - Y(\tau, \widehat{x})|^p \mathcal{M}(\tau)^{1-p} \right)^{1/p}$$

$$\leq (q_{\widetilde{F}} + q_Y) \|x - \widehat{x}\|_{W_1^p}.$$

In the last inequality we utilized the fact that the interval (ξ_1, ξ_2) does not contain points τ_i, $1 \leq i \leq m$, because $\mu(\{\tau_i\}) > \mu((\xi_1, \xi_2))$. Thus, we have proved that the operator $F : W_1^p([-0, T], \mathbf{R}^n, \mu) \to L^p([0, T], \mathbf{R}^n, \mu)$ satisfies the assumption **(4.2a)**. Now, due to Theorem 4.4 the initial value problem (1.4), (2.1), (3.8) has a unique global solution. $\square$

4.4 Local solutions of the initial value problem for the generalized functional differential equation

Solvability of the initial value problem (1.3), (1.4) can also be obtained when the operator F does not satisfy the assumptions of Theorem 4.4, but is completely continuous (i.e. continuous and compact) instead.

Theorem 4.7. *Let the Volterra operator $F : W_1^p([-0, T], \mathbf{R}^n, \mu) \to L^p([0, T], \mathbf{R}^n, \mu)$ be completely continuous. Then the initial value problem (1.3), (1.4) has a local solution and any local solution is part of either some global solution or some unextendable solution.*

Proof. We will prove solvability of the equation (4.1), which is equivalent to the initial value problem (1.3), (1.4). We will use the assumptions that $\Phi : L^p([0, T], \mathbf{R}^n, \mu) \to L^p([0, T], \mathbf{R}^n, \mu)$ defined by (4.2) is Volterra and completely continuous. Due to the latter property, for any $\varepsilon > 0$, $r > 0$ there exists a positive $\Delta = \Delta(\varepsilon, r)$ such that the following condition holds true:

(4.4a) for every $y \in L^p([0, T], \mathbf{R}^n, \mu)$, $\|y\|_{L^p} \leq r$ and each μ-measurable subset $S \subset [0, T]$, $\mu(S) < \Delta(\varepsilon, r)$ we have $\int_S |(\Phi y)(s)|^p \mu(ds) < \varepsilon^p$.

Let $\widetilde{\Delta}$ be the least upper bound for all possible numbers Δ satisfying the above property. Observe that the condition **(4.4a)** is also fulfilled for any subset $S \subset [0, T]$ satisfying $\mu(S) < \widetilde{\Delta}(\varepsilon, r)$.

The solution will be constructed successively extending its domain step by step.

Step 1. If $\mu(\{0\}) > 0$, then the Volterra property of the operator Φ implies that the value $(\Phi y)(0)$ will be the same for all $y \in L^p([0, T], \mathbf{R}^n, \mu)$. Put

$$H_0 = \begin{cases} (\Phi 0)(0) & \text{if } \mu(\{0\}) > 0, \\ 0 & \text{if } \mu(\{0\}) = 0. \end{cases}$$

This vector coincides with the value of the solution $z(t)$ at $t = 0$ in the case when $\mu(\{0\}) > 0$. Let $r_1 = |H_0| (\mu(\{0\}))^{1/p} + 1$, $\varepsilon = 1$, define $\Delta_1 = \widetilde{\Delta}(r_1, \varepsilon)$ to be the least upper bound of all numbers $\Delta(r_1, \varepsilon)$ satisfying **(4.4a)** and put $\gamma_0 = \frac{\mu(\{0\})}{M}$, $\gamma_1 = \gamma_0 + \frac{\Delta_1}{2M}$. Minding $\xi(\gamma_0) = 0$ and setting $\xi(\gamma_1) = \xi_1$ we obtain, due to Lemma 4.3, two estimates $\mu((0, \xi_1)) \leq \frac{\Delta_1}{2}$, $\mu((0, \xi_1]) \geq \frac{\Delta_1}{2}$.

By $\mathfrak{B}_1$ we denote the subset of the space $L^p([0, \xi_1), \mathbf{R}^n, \mu)$ coinciding with the entire ball

$$\{y : \|y\|_{L^p([0, \xi_1), \mathbf{R}^n, \mu)} \leq r_1\}$$

if $\mu(\{0\}) = 0$, and with the part of this ball containing all functions with the property $y(0) = H_0$ if $\mu(\{0\}) > 0$. This set is closed and convex. Define now the following operators:

$$\Pi_{\xi_1} : L^p([0, T], \mathbf{R}^n, \mu) \to L^p([0, \xi_1), \mathbf{R}^n, \mu), \qquad (\Pi_{\xi_1} y)(t) = y(t), \qquad t \in [0, \xi_1);$$

$$P_{\xi_1} : L^p([0, \xi_1), \mathbf{R}^n, \mu) \to L^p([0, T], \mathbf{R}^n, \mu), \qquad (P_{\xi_1} y)(t) = \begin{cases} y(t) & \text{if } t \in [0, \xi_1), \\ 0 & \text{if } t \in [\xi_1, T]. \end{cases}$$

Then the operator $\Pi_{\xi_1} \Phi P_{\xi_1}$ acting in the space $L^p([0, \xi_1), \mathbf{R}^n, \mu)$ is completely continuous with $\mathfrak{B}_1 \subset L^p([0, \xi_1), \mathbf{R}^n, \mu)$ being its invariant subset. Therefore, the equation $\Pi_{\xi_1} \Phi P_{\xi_1} y_{\xi_1} = y_{\xi_1}$ has a solution in $\mathfrak{B}_1$, which we denote by z_{ξ_1}. This function is a local solution of equation (4.1).

Step 2. The function $\Phi P_{\xi_1} z_{\xi_1} : [0, T] \to \mathbf{R}^n$ is an extension of the local solution $z_{\xi_1} : [0, \xi_1) \to \mathbf{R}^n$ and due to the Volterra property of the operator Φ satisfies equation (4.1) on the interval $[0, \xi_1]$. Choosing $r_2 = \left(\int_{[0, \xi_1]} |(\Phi P_{\xi_1} z_{\xi_1})(s)|^p ds \right)^{1/p} + 1$, $\varepsilon = 1$ we denote by $\Delta_2 = \widetilde{\Delta}(r_2, \varepsilon)$ the least upper bound of the values $\Delta(r_2, \varepsilon)$, for which **(4.4a)** holds true. Let $\gamma_2 = \gamma_1 + \frac{\Delta_2}{2M}$, $\xi_2 = \xi(\gamma_2)$. From Lemma 4.3 we then obtain $\mu((\xi_1, \xi_2)) \leq \frac{\Delta_2}{2}$, $\mu((\xi_1, \xi_2]) \geq \frac{\Delta_2}{2}$.

Now we define a subset $\mathfrak{B}_2$ of the space $L^p([0, \xi_2), \mathbf{R}^n, \mu)$ containing all functions $y : [0, \xi_2) \to \mathbf{R}^n$ such that $\|y\|_{L^p([0, \xi_2), \mathbf{R}^n, \mu)} \leq r_2$ and $y(t) = (\Phi P_{\xi_1} z_{\xi_1})(t)$ on $[0, \xi_1]$. This subset is closed and convex. Define also

$$\Pi_{\xi_2} : L^p([0, T], \mathbf{R}^n, \mu) \to L^p([0, \xi_2), \mathbf{R}^n, \mu), \qquad (\Pi_{\xi_2} y)(t) = y(t), \qquad t \in [0, \xi_2);$$

$$P_{\xi_2} : L^p([0, \xi_2), \mathbf{R}^n, \mu) \to L^p([0, T], \mathbf{R}^n, \mu), \qquad (P_{\xi_2} y)(t) = \begin{cases} y(t) & \text{if } t \in [0, \xi_2), \\ 0 & \text{if } t \in [\xi_2, T]. \end{cases}$$

The operator $\Pi_{\xi_2} \Phi P_{\xi_2} : L^p([0, \xi_2), \mathbf{R}^n, \mu) \to L^p([0, \xi_2), \mathbf{R}^n, \mu)$ is completely continuous and $\mathfrak{B}_2$ is invariant w.r.t. this operator. Therefore, the equation $\Pi_{\xi_2} \Phi P_{\xi_2} y_{\xi_2} = y_{\xi_2}$ has a solution in the space $L^p([0, \xi_2), \mathbf{R}^n, \mu)$, which we denote by z_{ξ_2}. This is a local solution of equation (4.1), with z_{ξ_1} being its restriction.

Further steps are performed similarly. Let us show that the final result of the described procedure consisting of countably or finitely many (if $\xi_n = T$ for some n) steps yields a global or an unextendable solution. Indeed, we would otherwise obtain numbers ϱ, $\widehat{T}$, for which $\xi_i \leq \widehat{T} < T$ and $\|z_{\xi_i}\|_{L^p([0, \xi_2), \mathbf{R}^n, \mu)} \leq \varrho$. after any step. But in this case we would obtain the estimate $\widetilde{\Delta}(r_i, \varepsilon) \geq \widetilde{\Delta}(\varrho, \varepsilon)$ for any i. Therefore $\mu((\xi_{i-1}, \xi_i]) \geq \widetilde{\Delta}(\varrho, \varepsilon)$, which contradicts the assumption that μ is a finite measure on $[0, T]$. $\qquad \square$

4.5 Local solutions of the initial value problem for impulsive equations

Theorem 4.7 gives us an opportunity to prove one more result on solvability of the impulsive system (2.1), (3.8).

Theorem 4.8. *Let the Volterra operator $\widetilde{F} : W_1^p([-0, T], \mathbf{R}^n, \mu) \to L^p([0, T], \mathbf{R}^n, mes)$ be completely continuous and let the functional $Y : \mathcal{T} \times W_1^p([-0, T], \mathbf{R}^n, \mu) \to \mathbf{R}^n$ satisfy the assumptions (3.4b), (3.4e) and the condition*

(4.4b) *there is a function $\varphi : \mathcal{T} \times \mathbf{R} \to \mathbf{R}$, such that for any $r > 0$ and all $x \in W_1^p([-0, T], \mathbf{R}^n, \mu)$, $\|x\|_{W_1^p} \leq r$ one has*

$$|Y(\tau, x)|_{\mathbf{R}^n} \leq \varphi(\tau, r),$$

and, in addition, for any $r > 0$ the following estimate holds true:

$$\sum_{\tau \in \mathcal{T}} (\mathcal{M}(\tau))^{1-p} (\varphi(\tau, r))^p < \infty.$$

Then the initial value problem (1.4), (2.1), (3.8) has a local solution, and any local solution of this problem is part of either some global or some unextendable solution.

Proof. From **(4.4b)** we immediately obtain **(3.4c)**. As we demonstrated in Subsection 3.4, the operator $F : L^p([0, T], \mathbf{R}^n, \mu) \to L^p([0, T], \mathbf{R}^n, \mu)$ defined by (3.9) is continuous and Volterra.

We show now that for any $r > 0$ the image FU_r of the ball $U_r \subset W_1^p([-0, T], \mathbf{R}^n, \mu)$ of radius r is contained in a compact subset of the space $L^p([0, T], \mathbf{R}^n, \mu)$, i.e. it is relatively compact. To do this, we make use of the following simple statement: Let S be an arbitrary μ-measurable subset of the interval $[0, T]$ and $G \subset L^p([0, T], \mathbf{R}^n, \mu)$. Then G is relatively compact if and only if the subsets $G|_S \subset L^p(S, \mathbf{R}^n, \mu)$, $G|_{\overline{S}} \subset L^p(\overline{S}, \mathbf{R}^n, \mu)$ of the restrictions of G to S and $\overline{S} = [0, T] - S$, respectively, are both relatively compact. Indeed, if $\mathcal{G}$ is a finite ε-net of G, then the restrictions $\mathcal{G}|_S, \mathcal{G}|_{\overline{S}}$ of functions from $\mathcal{G}$ constitute finite ε-nets for the sets $G|_S, G|_{\overline{S}}$, respectively. Conversely, the set of functions obtained by "gluing" together each of the functions from a finite $\varepsilon/2$-net for $G|_S$ with each of the functions from a finite $\varepsilon/2$-net for $G|_{\overline{S}}$ produces a finite ε-net for the set G.

We apply now this observation to $G = FU_r$ assuming that $S = \mathcal{T}$.

From the assumptions of the theorem we deduce that $\widetilde{F}U_r$ is relatively compact in the space $L^p([0, T], \mathbf{R}^n, mes)$. By the definition of the operator F, the set $FU_r|_{\overline{\mathcal{T}}} \subset L^p(\overline{\mathcal{T}}, \mathbf{R}^n, \mu)$, is relatively compact, too, where $\overline{\mathcal{T}} = [0, T] - \mathcal{T}$.

We prove now that the set $FU_r|_{\mathcal{T}}$ is, in fact, relatively compact in the space $L^p(\mathcal{T}, \mathbf{R}^n, \mu) = L^p(\mathcal{T}, \mathbf{R}^n, \mu_{\mathcal{T}})$. To do this, we observe that the space $L^p(\mathcal{T}, \mathbf{R}^n, \mu_{\mathcal{T}})$ is isometric to the space $l^p(\mathbf{R}^n)$ of countable sequences $\zeta = \{\zeta_i\}$, $\zeta_i \in \mathbf{R}^n$, $i = 1, 2, \ldots$, satisfying $\sum_{i=1}^{\infty} |\zeta_i|_{\mathbf{R}^n}^p < \infty$, the norm being given by the formula $\|\zeta\|_{l^p} = \left(\sum_{i=1}^{\infty} |\zeta_i|_{\mathbf{R}^n}^p\right)^{1/p}$. More precisely, this isometry is defined as the mapping which assigns the sequence $\{(\mathcal{M}(\tau_i))^{1/p} y(\tau_i)\}$ belonging to the space $l^p(\mathbf{R}^n)$ to an arbitrary $y \in L^p(\mathcal{T}, \mathbf{R}^n, \mu_{\mathcal{T}})$. Any $y \in FU_r|_{\mathcal{T}}$ can be represented as $y(\tau) = Y(\tau, x)/\mathcal{M}(\tau)$, $\tau \in \mathcal{T}$, where $\|x\|_{W_1^p} \leq r$. Hence for any natural number i_0 we obtain

$$\sum_{i=i_0}^{\infty} |(\mathcal{M}(\tau_i))^{1/p} y(\tau_i)|_{\mathbf{R}^n}^p = \sum_{i=i_0}^{\infty} (\mathcal{M}(\tau_i))^{1-p} |Y(\tau_i, x)|_{\mathbf{R}^n}^p \leq \sum_{i=i_0}^{\infty} (\mathcal{M}(\tau_i))^{1-p} (\varphi(\tau_i, r))^p.$$

From this, the assumption **(4.4b)** and the compactness criterion in the space $l^p(\mathbf{R}^n)$ (see e.g. [9, p. 32]) we conclude that the subset $FU_r|_{\mathcal{T}}$ is relatively compact in $L^p(\mathcal{T}, \mathbf{R}^n, \mu)$. □

Example 4.9. Let $\mathcal{T} = \{\tau_i\}$, where the sequence $\{\tau_i\} \subset (0, 1)$ is strictly decreasing and $\tau_i \to 0$. Consider the impulsive system

$$\dot{x}(t) = 0, \qquad t \in [0, 1]; \qquad x(\tau_i + 0) = x(\tau_i) + \sqrt{\frac{|x(\tau_i)|}{2^i}}. \tag{4.3}$$

Let us fix an arbitrary convergent series $\sum_i \mathcal{M}(\tau_i)$ with positive terms and define the measure μ by (3.7). As $\mu(\{0\}) = 0$, then μ-absolutely continuous function $t \mapsto x(t)$ is also continuous at $t = 0$, i.e. $x(-0) = x(0)$. Therefore we may disregard the auxiliary point -0 which results in the domain $[0,1]$.

The operator $\widetilde{F} : W_1^p([0,1], \mathbf{R}, \mu) \to L^p([0,1], \mathbf{R}, \text{mes})$ is simply zero in this case, so that it trivially satisfies the assumptions of Theorem 4.6 and Theorem 4.8 for any p.

The functional

$$Y(\tau_i, x) = \sqrt{\frac{|x(\tau_i)|}{2^i}},$$

which determines the size of the jumps, can be considered acting from $\mathcal{T} \times W_1^1([0,1], \mathbf{R}, \mu)$ to $\mathbf{R}$, and it also satisfies the assumptions **(3.4b)**, **(3.4c)**, **(3.4e)** for $p = 1$. This gives us opportunity to search for solutions in the space $W_1^1([0,1], \mathbf{R}, \mu)$. Moreover, the functional Y satisfies the estimate

$$Y(\tau_i, x) \leq \sqrt{\frac{r}{2^i}},$$

$\|x\|_{W_1^1} \leq r$. Thus, all the assumptions of Theorem 4.8 are fulfilled, so that any initial value $x(0) = \alpha$ gives rise to a solution for any real α.

However, the functional Y does not satisfy the requirement **(4.3b)** of Theorem 4.6 ensuring uniqueness of the solutions. In the example the initial condition $\alpha = 0$ does produce two solutions: the zero solution and this one: $x(t) = \frac{1}{2^i}$, $t \in (\tau_{i+1}, \tau_i]$, $i = 1, 2, \dots$

Unlike (4.3), the initial value problem

$$\dot{x}(t) = 0, \quad t \in [0,1]; \quad x(\tau_i + 0) = x(\tau_i) + \frac{|x(\tau_i)|}{2^i}; \quad x(0) = \alpha \tag{4.4}$$

satisfies all the assumptions of Theorem 4.6. Therefore the problem (4.4) has a unique solution in the space $W_1^1([0,1], \mathbf{R}, \mu)$. For instance, the only solution for $\alpha = 0$ is the zero function.

5   Outlook

The central results of the present paper can be used to a further development of the theory of functional differential equations with an arbitrary driven measure, in the spirit of the monograph [4]. This development might include boundary value problems, stability analysis, control theory, difference equations, stochastic functional differential and difference equations driven by Poisson-type noises etc. Some preliminary results can e.g. be found in the papers [2, 3, 8, 10, 11, 13].

Acknowledgements

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