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Andronov–Hopf and Bautin bifurcation in a tritrophic food chain model with Holling functional response types IV and II

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Abstract. The existence of an Andronov–Hopf and Bautin bifurcation of a given system of differential equations is shown. The system corresponds to a tritrophic food chain model with Holling functional responses type *IV* and *II* for the predator and superpredator, respectively. The linear and logistic growth is considered for the prey. In the linear case, the existence of an equilibrium point in the positive octant is shown and this equilibrium exhibits a limit cycle. For the logistic case, the existence of three equilibrium points in the positive octant is proved and two of them exhibit a simultaneous Hopf bifurcation. Moreover the Bautin bifurcation on these points are shown.

Keywords: Andronov–Hopf bifurcation, Bautin bifurcation, limit cycle, food chain model.

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1 Introduction

In the task of understanding the complexity presented by the interactions among the different populations living in a habitat, the mathematical modeling has been a very important rule in ecology in the last decades. Some of the models which have been studied are the tritrophic systems (see Ref. [3] and references therein). In particular, in this work we analyzed a tritrophic model given by the following differential equation system,

$$\begin{aligned}\frac{dx}{dt} &= h(x) - f(x)y,\\ \frac{dy}{dt} &= c_1 y f(x) - g(y)z - c_2 y,\\ \frac{dz}{dt} &= c_3 g(y) z - d_2 z,\end{aligned} \tag{1.1}$$

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where x represents the density of a prey that gets eaten by a predator of density y (mesopredator), and the species y feeds the top predator z (superpredator). The function $h(x)$ represents the growth rate of the prey population in absence of the predators, and the functions $f(x)$ and $g(y)$ are the functional responses for the mesopredator and the superpredator, respectively. The parameters c_1, c_2, c_3 and d_2 are positive and we are interested to find the stable solutions in the positive octant $\Omega = \{x > 0,\, y > 0,\, z > 0\}$. There are different proposals of functional responses in literature, among which are the Holling type (see Refs. [5, 10, 11]). In Ref. [3], is considered the case when $h(x)$ is a logistic map and the functional responses f and g are Holling type *II*. Using the averaging theory, they proved that the system (1.1) has an equilibrium point which exhibits a triple Andronov–Hopf bifurcation. It implies the existence of a stable periodic orbit contained in the domain of interest.

The local dynamics of the differential system (1.1) has been analyzed in Ref. [2], when $h(x)$ is a linear map, and the functional responses f and g are Holling type *III*. They proved the existence of two equilibrium points which exhibit simultaneously a zero-Hopf bifurcation in Ω. In Ref. [1], the authors analyzed the case when $h(x)$ is a linear map, and the functional responses f and g are Holling type *III* and Holling type *II*, respectively. They proved that there is a domain in the parameter space where the system (1.1) has a stable periodic orbit which results from an Andronov–Hopf bifurcation.

In this paper we are interested in analyzed the dynamics of the differential system (1.1) when the functional responses $f(x)$ and $g(y)$ are Holling type *IV* and *II*, respectively. In particular, we are interested in stable equilibrium points or stable limit cycles inside the positive octant Ω. We consider two cases, the linear case, taking $h(x) = \rho x$, and the logistic case taking $h(x) = \rho x(1 - \frac{x}{R})$. The functions f and g will be

$$f(x) = \frac{a_1 x}{b_1 + x^2}, \qquad g(y) = \frac{a_2 y}{b_2 + y},$$

where a_1, b_1, a_2, b_2 are positive constants. Explicitly, we will study the differential system

$$\begin{aligned} \frac{dx}{dt} &= h(x) - \frac{a_1 xy}{x^2 + b_1}, \\ \frac{dy}{dt} &= \frac{c_1 a_1 yx}{x^2 + b_1} - \frac{a_2 yz}{b_2 + y} - c_2 y, \\ \frac{dz}{dt} &= \frac{c_3 a_2 yz}{b_2 + y} - d_2 z. \end{aligned} \tag{1.2}$$

Along this manuscript the terms **linear** or **logistic case** will be used to refer cases when the prey has either linear or logistic growth rate, respectively.

The main results in this paper are contained in Sections 2 and 3.

2 Linear case

In this section we consider the differential system (1.2) with a linear growth for the prey, this means that the function $h(x) = \rho x$ and then the differential system becomes

$$\begin{aligned} \dot{x} &= -\frac{a_1 xy}{b_1 + x^2} + \rho x, \\ \dot{y} &= -c_2 y + \frac{a_1 c_1 xy}{b_1 + x^2} - \frac{a_2 yz}{b_2 + y}, \\ \dot{z} &= \left(-d_2 + \frac{a_2 c_3 y}{b_2 + y}\right) z. \end{aligned} \tag{2.1}$$

In next lemma we show the existence of an equilibrium point in the positive octant Ω under certain conditions on the parameters involved in the system of differential equations.

Lemma 2.1. *The differential system (2.1) has only one equilibrium point $p_0 = (x_0, y_0, z_0) \in \Omega$ if*

(a) $a_2c_3 - d_2 > 0$,

(b) $a_1y_0 - \rho b_1 > 0$,

(c) $c_2y_0 - c_1x_0\rho < 0$.

Moreover, if ones of above condition does not hold, then the differential system (2.1) does not have any equilibrium point in Ω.

Proof. The equilibrium points of the differential system (2.1) are the solutions of

$$\begin{aligned} -\frac{a_1xy}{b_1+x^2} + \rho x, &= 0,\\ -c_2y + \frac{a_1c_1xy}{b_1+x^2} - \frac{a_2yz}{b_2+y} &= 0,\\ \left(-d_2 + \frac{a_2c_3y}{b_2+y}\right) z &= 0. \end{aligned}$$

By multiplying the above equations by the term $\left(b_1+x^2\right)(b_2+y)$, (which is always positive in Ω), we obtain that an equilibrium point in Ω must satisfy the system

$$\begin{aligned} \rho\left(b_1+x^2\right) - a_1y &= 0,\\ (b_2+y)\left(c_2\left(b_1+x^2\right) - a_1c_1x\right) + a_2z\left(b_1+x^2\right) &= 0,\\ d_2(b_2+y) - a_2c_3y &= 0. \end{aligned} \tag{2.2}$$

From the third equation in system (2.2), $y_0 = \frac{d_2b_2}{a_2c_3-d_2}$ and it is positive by hypothesis (a).

Substituting $y = y_0$ in the first equation of (2.2), we obtain a unique positive solution $x = x_0$ by hypothesis (b). Now, substituting $x = x_0$ and $y = y_0$ in the second equation of system (2.2), we have that the unique solution $z = z_0$ of this equation is positive, if and only if, $\left(c_2\left(b_1+x_0^2\right) - a_1c_1x_0\right) < 0$, but, from the first equation in system (2.2), we have that $b_1 + x_0^2 = a_1y_0/\rho$, then $\left(c_2\left(b_1+x_0^2\right) - a_1c_1x_0\right) = \frac{a_1}{\rho}\left(c_2y_0 - c_1x_0\rho\right)$, and $z_0 > 0$ by hypothesis (c).

Clearly, if ones of the conditions $a_2c_3 - d_2 > 0$, $a_1y_0 - \rho b_1 > 0$ or $c_2y_0 - c_1x_0\rho < 0$ does not hold then the differential system (2.1) has no equilibrium points in Ω. □

In order to simplify the expression of the equilibrium point p_0 we introduce a new parameters given by the next result.

Lemma 2.2. *If the parameters of the system (2.1) satisfy the conditions (a), (b) and (c) in Lemma 2.1, then there exist $k_1 > 0$, $k_2 > 0$ and $k_3 > 0$, such that the parameters a_1, a_2 and b_2 involved in the differential system (2.1) can be written as*

$$a_2 = \frac{d_2\rho + k_1^2}{c_3\rho}, \qquad b_2 = \frac{b_1k_1^2 + k_2^2}{a_1d_2}, \qquad a_1 = \frac{b_1c_2k_1^2 + c_2k_2^2 + k_3}{c_1k_1k_2}, \tag{2.3}$$

and the unique equilibrium point of the system (2.1) in Ω, is

$$p_0 = \left(\frac{k_2}{k_1}, \frac{c_1k_2\rho\left(b_1{k_1}^2 + {k_2}^2\right)}{k_1\left(b_1c_2{k_1}^2 + c_2{k_2}^2 + k_3\right)}, \frac{c_1c_3k_2k_3\rho}{b_1c_2d_2{k_1}^3 + c_2d_2k_1{k_2}^2 + d_2k_1k_3}\right).$$

Proof. The solutions of system (2.2) are

$$p_0 = \left(\frac{\sqrt{a_1b_2d_2 + b_1\rho(d_2 - a_2c_3)}}{\sqrt{\rho(a_2c_3 - d_2)}}, \frac{b_2d_2}{a_2c_3 - d_2}, \frac{c_1c_3\sqrt{\rho(a_2c_3 - d_2)}\sqrt{\Delta_1} - b_2c_2c_3d_2}{d_2(a_2c_3 - d_2)} \right),$$

$$p_1 = \left(-\frac{\sqrt{a_1b_2d_2 + b_1\rho(d_2 - a_2c_3)}}{\sqrt{\rho(a_2c_3 - d_2)}}, \frac{b_2d_2}{a_2c_3 - d_2}, -\frac{c_3\left(c_1\sqrt{\rho(a_2c_3 - d_2)}\sqrt{\Delta_1} + b_2c_2d_2\right)}{d_2(a_2c_3 - d_2)} \right),$$

$$\Delta_1 = a_1b_2d_2 + b_1\rho(d_2 - a_2c_3).$$

Since $p_1 \notin \Omega$, by Lemma 2.1 $p_0 \in \Omega$, and $\rho(a_2c_3 - d_2) > 0$, then there exists $k_1 > 0$ such that $a_2 = \frac{d_2\rho + k_1^2}{c_3\rho}$. Hence

$$p_0 = \left(\frac{\sqrt{a_1b_2d_2 - b_1{k_1}^2}}{k_1}, \frac{b_2d_2\rho}{{k_1}^2}, \frac{c_3\rho\left(c_1k_1\sqrt{a_1b_2d_2 - b_1{k_1}^2} - b_2c_2d_2\right)}{d_2{k_1}^2} \right).$$

Moreover, $a_1b_2d_2 - b_1{k_1}^2 > 0$, then there exists $k_2 > 0$ such that $b_2 = \frac{b_1k_1^2 + k_2^2}{a_1d_2}$, then

$$p_0 = \left(\frac{k_2}{k_1}, \frac{\rho\left(b_1{k_1}^2 + {k_2}^2\right)}{a_1{k_1}^2}, \frac{c_3\rho\left(k_2(a_1c_1k_1 - c_2k_2) - b_1c_2{k_1}^2\right)}{a_1d_2{k_1}^2} \right).$$

Since $k_2(a_1c_1k_1 - c_2k_2) - b_1c_2{k_1}^2 > 0$, then there exists $k_3 > 0$ such that $a_1 = \frac{b_1c_2{k_1}^2 + c_2{k_2}^2 + k_3}{c_1k_1k_2}$, and

$$p_0 = \left(\frac{k_2}{k_1}, \frac{c_1k_2\rho\left(b_1{k_1}^2 + {k_2}^2\right)}{k_1\left(b_1c_2{k_1}^2 + c_2{k_2}^2 + k_3\right)}, \frac{c_1c_3k_2k_3\rho}{b_1c_2d_2{k_1}^3 + c_2d_2k_1{k_2}^2 + d_2k_1k_3} \right).$$

□

Lemma 2.3. *Under the hypothesis of Lemma 2.2 and considering that the parameters a_1, a_2 and b_1 satisfy the conditions (2.3) and*

$$k_2 = \sqrt{2}\sqrt{b_1}k_1, \quad d_2 = \frac{12b_1{k_1}^4}{5k_3}, \quad k_3 = \frac{3}{2}b_1{k_1}^2\rho, \quad \text{and} \quad c_2 = c_{20}(\rho) := \frac{9{k_1}^2 + 38\rho^2}{52\rho}, \tag{2.4}$$

then the equilibrium point p_0 is given by

$$p_0 = \left(\sqrt{2}\sqrt{b_1}, \frac{52\sqrt{2}\sqrt{b_1}c_1\rho^2}{9{k_1}^2 + 64\rho^2}, \frac{65\sqrt{b_1}c_1c_3\rho^4}{\sqrt{2}\left(18{k_1}^4 + 128{k_1}^2\rho^2\right)} \right)$$

and the eigenvalues of the linear approximation of system (2.1) at p_0 are

$$\alpha = \frac{64\rho}{39} \quad \textit{and} \quad \pm i\omega,$$

where

$$\omega^2 = \frac{{k_1}^2}{4} > 0.$$

Proof. Taking into account the assignations of the parameters a_1, a_2 and b_1 given by (2.3), the characteristic polynomial of the linear approximation M_{p_0} of differential system (2.1) at the equilibrium point p_0 is $P(\lambda) = det(\lambda I - M_{p_0}) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$, where,

$$\begin{aligned}
A_1 &= -\frac{\rho\left(2{k_2}^2\left(d_2\rho + {k_1}^2\right) + d_2k_3\right)}{\left(b_1{k_1}^2 + {k_2}^2\right)\left(d_2\rho + {k_1}^2\right)},\\
A_2 &= \frac{d_2\rho^2\left(b_1{k_1}^2 + {k_2}^2\right)h_1 + {k_1}^2\rho\left(b_1{k_1}^2 - {k_2}^2\right)h_2 + d_2{k_1}^2k_3\left(b_1{k_1}^2 + {k_2}^2\right)}{\left(b_1{k_1}^2 + {k_2}^2\right)^2\left(d_2\rho + {k_1}^2\right)},\\
A_3 &= -\frac{2d_2{k_1}^2{k_2}^2k_3\rho}{\left(b_1{k_1}^2 + {k_2}^2\right)^2\left(d_2\rho + {k_1}^2\right)},\\
h_1 &= \left(b_1c_2{k_1}^2 - c_2{k_2}^2 + k_3\right),\\
h_2 &= \left(b_1c_2{k_1}^2 + c_2{k_2}^2 + k_3\right).
\end{aligned}$$

If we consider the assignments for k_2, k_3 and d_2 given by (2.4), then A_1, A_2 and A_3 reduce to

$$A_1 = -\frac{64\rho}{39}, \qquad A_2 = \frac{1}{78}\left(\rho(-26c_2 + 19\rho) + 24{k_1}^2\right) \quad \text{and} \quad A_3 = -\frac{16{k_1}^2\rho}{39}.$$

The characteristic polynomial $P(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$ has a pair of purely imaginary roots $\pm i\omega$ and a real root α if and only if $P(\lambda) = (\lambda - \alpha)(\lambda^2 + \omega^2) = \lambda^3 - \alpha\lambda^2 + \omega^2\lambda - \alpha\omega^2$. Thus comparing coefficients, $P(\lambda)$ has a pair of purely imaginary roots $\pm i\omega$ and a real root α if and only if $A_2 > 0$ and

$$A_1A_2 - A_3 = 0, \tag{2.5}$$

where $\omega = \sqrt{A_2}$ and $\alpha = -A_1$. Since $A_1A_2 - A_3 = -\frac{16\rho(-52c_2\rho + 9{k_1}^2 + 38\rho^2)}{1521}$, solving equation (2.5) for the parameter c_2, we have that if

$$c_2 = \frac{9{k_1}^2 + 38\rho^2}{52\rho},$$

then $A_2 = \frac{{k_1}^2}{4} > 0$. Thus, we conclude that the characteristic polynomial $P(\lambda)$ has a pair of purely imaginary roots $\pm i\omega$ and a real root α, where $\alpha = \frac{64\rho}{39}$ and $\omega = \frac{k_1}{2}$. The equilibrium point p_0 becomes

$$p_0 = \left(\sqrt{2}\sqrt{b_1}, \frac{52\sqrt{2}\sqrt{b_1}c_1\rho^2}{9{k_1}^2 + 64\rho^2}, \frac{65\sqrt{b_1}c_1c_3\rho^4}{\sqrt{2}\left(18{k_1}^4 + 128{k_1}^2\rho^2\right)}\right).$$

□

In order to compute the Lyapunov coefficients and a regularity condition, from now in this section

$$b_1 = 1, \qquad k_1 = 1, \qquad c_1 = 1 \quad \text{and} \quad c_3 = 1.$$

Remark 2.4. If the assumptions of Lemma 2.3 are satisfied, then the linear approximation of the differential system (2.1) at p_0 has the eigenvalues $\alpha = \frac{64\rho}{39}$ and $\pm\frac{i}{2}$, when $c_2 = c_{20}(\rho)$, hence, by continuity on the eigenvalues, the linear approximation of the differential system at p_0 has a pair of complex eigenvalues,

$$\lambda(p_0, c_2, \rho) = \xi(p_0, c_2, \rho) \pm i\omega(p_0, c_2, \rho),$$

when c_2 is in a neighborhood of $c_{20}(\rho)$.

In order to compute the first Lyapunov coefficient ℓ_1, we apply the Kuznetsov formula, (see Ref. [7]). Taking into account the assumptions of Lemma 2.3 and using the Mathematica software, we obtain the first Lyapunov coefficient of the differential system (2.1) at the equilibrium point p_0.

Lemma 2.5. *If the hypotheses of Lemma 2.3 hold, then the first Lyapunov coefficient of the differential system (2.1) at the equilibrium point p_0 is*

$$\ell_1(p_0, c_{20}(\rho), \rho) = \frac{(64\rho^2+9)(30074175488\rho^8+9866010240\rho^6-2504091294\rho^4-1131103197\rho^2-80677701)}{169\rho^3(4096\rho^2+1521)(16384\rho^2+1521)(100\rho^4+1252\rho^2+81)}.$$

Corollary 2.6. *There exists a unique real number $\rho_0 > 0$ such that $\ell_1(p_0, c_{20}(\rho_0), \rho_0) = 0$.*

Proof. By Lemma 2.5, $\ell_1(p_0, c_{20}(\rho), \rho) = 0$ if and only if

$$30074175488\rho^8 + 9866010240\rho^6 - 2504091294\rho^4 - 1131103197\rho^2 - 80677701 = 0. \tag{2.6}$$

According to the Descartes rule, there is a unique real number $\rho_0 > 0$ such that $\ell_1(p_0, c_{20}(\rho_0), \rho_0) = 0$. Indeed, solving numerically equation (2.6) for the parameter ρ, we have that ρ_0 (≈ 0.57721). □

Since $\ell_1(p_0, c_{20}(\rho), \rho)$ takes positive and negative values, we will verify the transversality conditions to have Andronov–Hopf or Bautin bifurcation. At first we state the following result proposed as an exercise in Ref. [6], whose proof is straight forward and we omit the details.

Lemma 2.7. *Let $M(\tau)$ be a parameter-dependent real $(n \times n)$-matrix which has a simple pair of complex eigenvalues $\xi(\tau) \pm i\omega(\tau)$ such that $\xi(\tau_0) = 0$ and $\omega(\tau_0) := \omega_0 > 0$. Then, the derivative of the real part of the complex eigenvalues at τ_0 is given by*

$$\frac{d\xi}{d\tau}(\tau_0) = \mathrm{Re}\left(\overline{\mathbf{p}}^{tr} \cdot \left(\frac{dM}{d\tau}(\tau_0) \cdot \mathbf{q}\right)\right),$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{C}^n$ are eigenvectors satisfying the normalization conditions

$$M(\tau_0)\mathbf{q} = i\omega_0, \qquad M^{tr}(\tau_0)\mathbf{p} = -i\omega_0, \qquad \overline{\mathbf{q}}^{tr} \cdot \mathbf{q} = 1 \quad \text{and} \quad \overline{\mathbf{p}}^{tr} \cdot \mathbf{q} = 1.$$

We know proceed to show the regularity condition in order to obtain a Bautin bifurcation.

Lemma 2.8 (Bautin regularity condition)**.** *If the parameters a_1, a_2, b_2, k_2, k_3 and d_2 satisfy the hypothesis of Lemma 2.3, then the map $(c_2, \rho) \mapsto (\xi(p_0, c_2, \rho), \ell_1(p_0, c_2, \rho))$ is regular at (c_0, ρ_0), where $\xi(p_0, c_2, \rho)$ is given in Remark 2.4 and $c_0 := c_{20}(\rho_0)$.*

Proof. By hypothesis, the linear approximation of the differential system (2.1) at p_0 depends only on the parameters c_2 and ρ, let $M_{p_0}(c_2, \rho)$ be this linear approximation. By Lemma 2.5, the complex numbers $\pm\frac{i}{2}$ are eigenvalues of $M_{p_0}(c_0, \rho_0)$, hence, the real part of the complex eigenvalues of $M_{p_0}(c_0, \rho_0)$, are

$$\xi(p_0, c_0, \rho_0) = 0, \tag{2.7}$$

let $\mathbf{p}$ and $\mathbf{q}$ be eigenvectors of $M_{p_0}(c_0,\rho_0)$ for the corresponding eigenvalues $-\frac{i}{2}$ and $\frac{i}{2}$, respectively, such that

$$\overline{\mathbf{q}}^{tr}\cdot\mathbf{q}=1 \quad\text{and}\quad \overline{\mathbf{p}}^{tr}\cdot\mathbf{q}=1. \tag{2.8}$$

By (2.7) and (2.8), we can apply Lemma 2.7 to the linear approximation $M_{p_0}(c_2,\rho)$, then taking into account the values of $\overline{\mathbf{p}}$, $\mathbf{q}$, and $\frac{\partial M_{p_0}(c_2,\rho)}{\partial c_2}$, which we compute with the Mathematica software, we have that the partial derivative of the real part of the eigenvalues $\xi(c_2,\rho)\pm i\omega(c_2,\rho)$ of $M_{p_0}(c_2,\rho)$, with respect to the parameter c_2, at the point (c_0,ρ_0), takes the value

$$\frac{\partial\xi}{\partial c_2}(c_0,\rho_0)=\overline{\mathbf{p}}^{tr}\left(\frac{\partial M_{p_0}(c_0,\rho_0)}{\partial c_2}\cdot\mathbf{q}\right)=-\frac{1664\rho_0^2}{16384\rho_0^2+1521}. \tag{2.9}$$

Applying Lemma 2.7 one more time, and taking into account the values of $\overline{\mathbf{p}}$, $\mathbf{q}$ and $\frac{\partial M(c_2,\rho)}{\partial\rho}$ it follows that

$$\frac{\partial\xi}{\partial\rho}(c_0,\rho_0)=\overline{\mathbf{p}}^{tr}\left(\frac{\partial M_{p_0}(c_0,\rho_0)}{\partial\rho}\cdot\mathbf{q}\right)=\frac{32\left(38\rho_0^2-9\right)}{16384\rho_0^2+1521}. \tag{2.10}$$

From the Kuznetsov formula (see Ref. [4]), the first Lyapunov coefficient at the equilibrium point p_0 is given by

$$\ell_1(p_0,c_2,\rho)=\frac{\operatorname{Re}\mathbf{C}_1(c_2,\rho)}{\omega(c_2,\rho)}-\xi(c_2,\rho)\frac{\operatorname{Im}\mathbf{C}_1(c_2,\rho)}{\omega^2(c_2,\rho)}, \tag{2.11}$$

where $\mathbf{C}_1(c_2,\rho)$ is a function that takes complex values as a differentiable function in the variables (c_2,ρ). Notice that, from Corollary 2.6, (2.7), (2.11) and since $\omega(c_0,\rho_0)=1/2$,

$$\operatorname{Re}\mathbf{C}_1(c_0,\rho_0)=0. \tag{2.12}$$

Hence, from (2.11), (2.7) and (2.12), the partial derivative of $\ell_1(c_2,\rho)$ with respect to c_2 at the point (c_0,ρ_0) is given by

$$\frac{\partial\ell_1}{\partial c_2}(c_0,\rho_0)=\frac{1}{\omega^2(c_0,\rho_0)}\left(\omega(c_0,\rho_0)\operatorname{Re}\left(\frac{\partial\mathbf{C}_1}{\partial c_2}(c_0,\rho_0)\right)-\operatorname{Im}\mathbf{C}_1(c_0,\rho_0)\frac{\partial\xi}{\partial c_2}(c_0,\rho_0)\right)$$

and the partial derivative of $\ell_1(c_2,\rho)$ with respect to ρ at the point (c_0,ρ_0) is given by

$$\frac{\partial\ell_1}{\partial\rho}(c_0,\rho_0)=\frac{1}{\omega^2(c_0,\rho_0)}\left(\omega(c_0,\rho_0)\operatorname{Re}\left(\frac{\partial\mathbf{C}_1}{\partial\rho}(c_0,\rho_0)\right)-\operatorname{Im}\mathbf{C}_1(c_0,\rho_0)\frac{\partial\xi}{\partial\rho}(c_0,\rho_0)\right),$$

thus, the determinant of interest reduces to

$$\det\begin{pmatrix}\frac{\partial\xi}{\partial c_2}(c_0,\rho_0) & \frac{\partial\xi}{\partial\rho}(c_0,\rho_0)\\ \frac{\partial\ell_1}{\partial c_2}(c_0,\rho_0) & \frac{\partial\ell_1}{\partial\rho}(c_0,\rho_0)\end{pmatrix}$$
$$=\frac{\frac{\partial\xi}{\partial c_2}(c_0,\rho_0)\operatorname{Re}\left(\frac{\partial\mathbf{C}_1}{\partial\rho}(c_0,\rho_0)\right)-\frac{\partial\xi}{\partial\rho}(c_0,\rho_0)\operatorname{Re}\left(\frac{\partial\mathbf{C}_1}{\partial c_2}(c_0,\rho_0)\right)}{\omega(c_0,\rho_0)}. \tag{2.13}$$

Numerically, one has that $\operatorname{Re}\left(\frac{\partial\mathbf{C}_1}{\partial c_2}(c_0,\rho_0)\right)\approx-0.9053$ and $\operatorname{Re}\left(\frac{\partial\mathbf{C}_1}{\partial\rho}(c_0,\rho_0)\right)\approx 2.48325$, and by Corollary 2.6, $\rho_0\approx 0.57721$. Then by (2.9), (2.10) and (2.13)

$$\det\begin{pmatrix}\frac{\partial\xi}{\partial c_2}(c_0,\rho_0) & \frac{\partial\xi}{\partial\rho}(c_0,\rho_0)\\ \frac{\partial\ell_1}{\partial c_2}(c_0,\rho_0) & \frac{\partial\ell_1}{\partial\rho}(c_0,\rho_0)\end{pmatrix}\approx-0.18205.$$

Hence, the map $(c_2,\rho)\mapsto(\xi(p_0,c_2,\rho),\ell_1(p_0,c_2,\rho))$ is regular at (c_0,ρ_0). □

Theorem 2.9. *If the parameters a_1, a_2, b_2, k_2, k_3 and d_2 satisfy the hypothesis given in Lemma 2.3, then the differential system (2.1) exhibits an Andronov–Hopf bifurcation at $p_0 = \left(\sqrt{2}, \frac{52\sqrt{2}\rho^2}{64\rho^2+9}, \frac{65\rho^4}{\sqrt{2}(128\rho^2+18)}\right)$, with respect to the parameter c_2 and its bifurcation value is $c_{20}(\rho)$, where $\rho > 0$ and $\rho \neq \rho_0$. Moreover, if $\rho > \rho_0$ the bifurcation is subcritical and if $\rho < \rho_0$ the bifurcation is supercritical.*

Proof. From Lemma 2.3, the linearization $M_{p_0}(c_2, \rho)$ of differential system (2.1) at p_0 has a positive real eigenvalue and a conjugate pair of pure imaginary eigenvalues if $c_2 = c_{20}(\rho)$. From Lemma 2.8, the derivative of the real part of the complex eigenvalues is

$$\frac{\partial \xi}{\partial c_2}(c_{20}(\rho), \rho) = -\frac{1664\rho^2}{16384\rho^2 + 1521},$$

which is negative for $\rho \neq 0$, and hence the transversality condition holds. The Lemma 2.5 and Corollary 2.6 give a negative first Lyapunov coefficient if $\rho < \rho_0$, and a positive first Lyapunov coefficient if $\rho > \rho_0$. Then the hypotheses of Andronov–Hopf bifurcation Theorem (see Refs. [7–9]) hold and we conclude the proof. □

Lemma 2.10 (Second Lyapunov coefficient)**.** *If we have the assumptions given in Lemma 2.3, then the second Lyapunov coefficient of the differential system (2.1) at the equilibrium point p_0 is given by*

$$\begin{aligned} &\ell_2(p_0, c_{20}(\rho), \rho) \\ &\quad = -\frac{\left(64\rho^2+9\right)^2 s_1(\rho)}{65804544\rho^9\left(4096\rho^2+1521\right)^3\left(16384\rho^2+1521\right)^3\left(16384\rho^2+13689\right)s_2(\rho)^2}, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} s_1(\rho) = {} & 16840883183715777818700442083618979 84\rho^{26}- \\ & 1215935242531623572771231495897900 6464\rho^{24}+ \\ & 8445100013575163080629632314879089 0496\rho^{22}+ \\ & 8837077023725222111636106636084589 3632\rho^{20}- \\ & 1096187767147018347479406414333015 49056\rho^{18}- \\ & 1943701583272810733849076799850623 79520\rho^{16}- \\ & 1131112389947859122362340150200812 175360\rho^{14}- \\ & 3318961131049573767114954168264784 2816\rho^{12}- \\ & 5137221528028189621494819571679640 576\rho^{10}- \\ & 3059987578855189075459643887660630 32\rho^{8}+ \\ & 3044031039595973584672804736756489 7\rho^{6}+ \\ & 7031366298566120280440132136776088\rho^{4}+ \\ & 4929327082244952423283726256955 84\rho^{2}+ \\ & 12343578321586192504727388915456, \\ s_2(\rho) = {} & 100\rho^4 + 1252\rho^2 + 81. \end{aligned}$$

Moreover, if $\rho = \rho_0$, then $\ell_2(p_0, c_{20}(\rho), \rho) \neq 0$, where ρ_0 is given in the Corollary 2.6.

Proof. In order to compute the second Lyapunov coefficient ℓ_2, we apply the Kuznetsov formula, (see Ref. [4]). Taking into account the assumptions of this Lemma and using the Mathematica software, we obtain that the second Lyapunov coefficient $\ell_2(p_0, c_{20}(\rho), \rho)$, of the differential system (2.1) at the equilibrium point p_0 is given by (2.14) and $\ell_2(p_0, c_{20}(\rho_0), \rho_0) \approx 7.40065$. □

Corollary 2.6, Lemma 2.8 and Lemma 2.10 provide the validity of the necessary and sufficient conditions to apply the Bautin bifurcation theorem (see Ref. [4]). In summary we have the following result.

Theorem 2.11 (Bautin bifurcation in linear growth)**.** *If the parameters a_1, a_2, b_2, k_2, k_3 and d_2 satisfy the hypothesis given in Lemma 2.3, then the differential system (2.1) exhibits a Bautin bifurcation at p_0, with respect to the parameters c_2 and ρ and its critical bifurcation value is $(c_{20}(\rho_0), \rho_0)$.*

3 Logistic case

In this section we consider the differential system (1.2) with a logistic growth for the prey, this means that the function $h(x) = \rho x \left(1 - \frac{x}{R}\right)$ and we will analyze the differential system

$$
\begin{aligned}
\dot{x} &= \rho x \left(1 - \frac{x}{R}\right) - \frac{a_1 x y}{b_1 + x^2}, \\
\dot{y} &= \frac{a_1 c_1 x y}{b_1 + x^2} - \frac{a_2 y z}{b_2 + y} - c_2 y, \\
\dot{z} &= z \left(\frac{a_2 c_3 y}{b_2 + y} - d_2\right).
\end{aligned}
\tag{3.1}
$$

In order to make ecological sense we assume that all parameters of the system (3.1) are positive.

Lemma 3.1. *If the parameters a_1, a_2, b_1, c_1 and R, satisfy*

$$
\begin{aligned}
a_1 &= \frac{\rho\left(b_1 + x_0^2\right)(R - x_0)}{R y_0}, \qquad a_2 = \frac{d_2(b_2 + y_0)}{c_3 y_0}, \qquad R = k_2 + x_0, \\
c_1 &= \frac{c_2 c_3 y_0 (k_2 + x_0) + k_3}{c_3 k_2 \rho x_0}, \qquad b_1 = k_2 x_0 + k_4,
\end{aligned}
\tag{3.2}
$$

then the unique equilibrium points of the differential system (3.1) in the region Ω are

$$
\begin{aligned}
p_1 &= \left(x_0, y_0, \frac{2k_3}{d_2(k_7 + k_8 + 6x_0)}\right), \\
p_2 &= \left(\frac{k_7}{2} + x_0, y_0, \frac{c_2 c_3 k_7 y_0 (k_8 + 2x_0)(k_7 + k_8 + 6x_0) + 2k_3(k_7 + 2x_0)(k_8 + 4x_0)}{2d_2 x_0 (k_7 + k_8 + 4x_0)(k_7 + k_8 + 6x_0)}\right), \\
p_3 &= \left(\frac{k_8}{2} + x_0, y_0, \frac{c_2 c_3 k_8 y_0 (k_7 + 2x_0)(k_7 + k_8 + 6x_0) + 2k_3(k_7 + 4x_0)(k_8 + 2x_0)}{2d_2 x_0 (k_7 + k_8 + 4x_0)(k_7 + k_8 + 6x_0)}\right).
\end{aligned}
$$

Where, $x_0 > 0$, $y_0 > 0$, $k_3 > 0$, $k_7 \geq 0$, $k_8 \geq 0$ and

$$
k_2 = \frac{4x_0 + k_7 + k_8}{2}, \qquad k_4 = \frac{1}{4} k_5 k_6, \qquad k_5 = 2x_0 + k_7, \qquad k_6 = 2x_0 + k_8. \tag{3.3}
$$

Proof. The equilibrium points of the differential system (3.1) are the solutions of the system,

$$\begin{aligned} \rho x\left(1-\frac{x}{R}\right)-\frac{a_1 x y}{b_1+x^2} &= 0, \\ \frac{a_1 c_1 x y}{b_1+x^2}-\frac{a_2 y z}{b_2+y}-c_2 y &= 0, \\ z\left(\frac{a_2 c_3 y}{b_2+y}-d_2\right) &= 0. \end{aligned}$$

Multiplying the above equations by $\left(b_1+x^2\right)\left(b_2+y\right)$, (which is always positive in the region Ω), we obtain that the equilibrium point must satisfy (3.4). Correspondingly each solution of (3.4) must be an equilibrium point of the differential system (3.1).

$$\begin{aligned} a_1 R y-\rho\left(b_1+x^2\right)(R-x) &= 0, \\ \left(b_2+y\right)\left(c_2\left(b_1+x^2\right)-a_1 c_1 x\right)+a_2 z\left(b_1+x^2\right) &= 0, \\ d_2\left(b_2+y\right)-a_2 c_3 y &= 0. \end{aligned} \tag{3.4}$$

A point $\left(x_0, y_0, z_0\right) \in \Omega$ is an equilibrium point of the differential system (3.1) if

$$\begin{aligned} a_1 R y_0-\rho\left(b_1+x_0^2\right)\left(R-x_0\right) &= 0, \\ \left(b_2+y_0\right)\left(c_2\left(b_1+x_0^2\right)-a_1 c_1 x_0\right)+a_2 z_0\left(b_1+x_0^2\right) &= 0, \\ d_2\left(b_2+y_0\right)-a_2 c_3 y_0 &= 0. \end{aligned} \tag{3.5}$$

We suppose $x_0>0$, $y_0>0$ and $z_0>0$. Note that the first equation of the system (3.5) is a linear equation in terms of a_1, and it has the unique solution,

$$a_1=\frac{\rho\left(b_1+x_0{}^2\right)\left(R-x_0\right)}{R y_0}.$$

Since $a_1>0$, $R-x_0$ must be positive, so there exists $k_2>0$ such that $R=x_0+k_2$. A similar argument using the third equation of system (3.5), we obtain that:

$$a_2=\frac{d_2\left(b_2+y_0\right)}{c_3 y_0}.$$

Using the values of a_1, a_2 and R, and solving the second equation of system (3.5) for z_0, we have that

$$z_0=\frac{c_1 c_3 k_2 \rho x_0-c_2 c_3 y_0\left(k_2+x_0\right)}{d_2\left(k_2+x_0\right)}.$$

Since $z_0>0$, there must exists $k_3>0$, such that $c_1 c_3 k_2 \rho x_0-c_2 c_3 y_0\left(k_2+x_0\right)=k_3$. Then

$$c_1=\frac{c_2 c_3 y_0\left(k_2+x_0\right)+k_3}{c_3 k_2 \rho x_0}.$$

Therefore, if a_1, a_2, R and c_1 satisfy (3.2), then $\left(x_0, y_0, z_0\right)$ is a solution of system (3.4) in Ω, where $z_0=\frac{k_3}{d_2\left(k_2+x_0\right)}$. Moreover, the system (3.4) takes the form

$$\begin{aligned} \frac{k_2 \rho y\left(b_1+x_0{}^2\right)}{y_0}-\rho\left(b_1+x^2\right)\left(k_2-x+x_0\right) &= 0, \\ \left(b_2+y\right)\left(c_2\left(b_1+x^2\right)-Q\right)+\frac{d_2 z\left(b_1+x^2\right)\left(b_2+y_0\right)}{c_3 y_0} &= 0, \\ \frac{b_2 d_2\left(y_0-y\right)}{y_0} &= 0, \end{aligned} \tag{3.6}$$

where $Q = \frac{x(b_1+x_0{}^2)(c_2c_3y_0(k_2+x_0)+k_3)}{c_3x_0y_0(k_2+x_0)}$. Solving the third equation of system (3.6) for y, we have that $y = y_0$. Moreover, the first equation of system (3.6) reduce to:

$$\rho(x-x_0)\left(x^2-k_2x+b_1-k_2x_0\right)=0.$$

Hence, the solutions of this equation are

$$x_0, \qquad x_1 := \frac{1}{2}\left(k_2-\sqrt{k_2(k_2+4x_0)-4b_1}\right), \qquad x_2 := \frac{1}{2}\left(k_2+\sqrt{k_2(k_2+4x_0)-4b_1}\right).$$

Thus, a necessary condition to have at least two solutions of system (3.6) in Ω is that $k_2(k_2+4x_0)-4b_1 \geq 0$. On the other hand, $x_1 > 0$ if and only if $0 < k_2^2-(k_2(k_2+4x_0)-4b_1) = 4(b_1-k_2x_0)$, then, $x_1 > 0$ if and only if there exists $k_4 > 0$ such that $b_1 = k_2x_0+k_4$, which is a hypothesis in (3.2). Let $k_5 = k_2-\sqrt{k_2{}^2-4k_4} > 0$, then $k_4 = \frac{1}{4}k_5k_6$, where $k_6 = 2k_2-k_5 > 0$. Hence, $x_1 = \frac{k_5}{2}$ and $x_2 = \frac{k_6}{2}$.

Substituting b_1, k_4, k_5, k_2, $y = y_0$ and $x = x_1$ in the second equation of system (3.6) and solving this equation for z, we have that

$$z_1 = \frac{c_2c_3k_6y_0(k_5-2x_0)(k_5+k_6+2x_0)+2k_3k_5(k_6+2x_0)}{2d_2x_0(k_5+k_6)(k_5+k_6+2x_0)}.$$

Moreover, if $k_5-2x_0 \geq 0$, then $z_1 > 0$. In the same way, replacing $y = y_0$ and $x = x_2$ in (3.6), we obtain

$$z_2 = \frac{c_2c_3k_5y_0(k_6-2x_0)(k_5+k_6+2x_0)+2k_3k_6(k_5+2x_0)}{2d_2x_0(k_5+k_6)(k_5+k_6+2x_0)}.$$

Also, if $k_6-2x_0 \geq 0$, then $z_2 > 0$.

Let $k_7 = k_5-2x_0$ and $k_8 = k_6-2x_0$, then $x_1 = \frac{k_7}{2}+x_0$, $x_2 = \frac{k_8}{2}+x_0$, and z_1, z_2 becomes

$$z_1 = \frac{c_2c_3k_7y_0(k_8+2x_0)(k_7+k_8+6x_0)+2k_3(k_7+2x_0)(k_8+4x_0)}{2d_2x_0(k_7+k_8+4x_0)(k_7+k_8+6x_0)},$$

$$z_2 = \frac{c_2c_3k_8y_0(k_7+2x_0)(k_7+k_8+6x_0)+2k_3(k_7+4x_0)(k_8+2x_0)}{2d_2x_0(k_7+k_8+4x_0)(k_7+k_8+6x_0)}.$$

Therefore, the unique equilibrium points of the differential system (3.1) in the region Ω, are:

$$p_1 = (x_0, y_0, z_0), \quad p_2 = (x_1, y_0, z_1) \quad \text{and} \quad p_3 = (x_2, y_0, z_2),$$

which completes the proof. □

Remark 3.2. Choosing the values k_7, k_8 adequately, we obtain one, two or three equillibria.

1. If $k_7 = k_8 = 0$, then $p_1 = p_2 = p_3 = \left(x_0, y_0, \frac{k_3}{3d_2x_0}\right)$, is the unique equilibrium point of the differential system (3.1) in Ω.

2. If $k_7 = 0$, and $k_8 > 0$ then $p_1 = p_2 = \left(x_0, y_0, \frac{2k_3}{d_2k_8+6d_2x_0}\right)$, and

$$p_3 = \left(\frac{k_8}{2}+x_0, y_0, \frac{c_2c_3k_8y_0(k_8+6x_0)+4k_3(k_8+2x_0)}{d_2(k_8+4x_0)(k_8+6x_0)}\right),$$

hence, there are two equilibrium points of the differential system (3.1) in Ω.

3. If $k_7 > 0,\ k_8 > 0$ and $k_7 \neq k_8$ then

$$p_1 = \left(x_0, y_0, \frac{2k_3}{d_2(k_7+k_8+6x_0)}\right),$$

$$p_2 = \left(\frac{k_7}{2}+x_0, y_0, \frac{c_2c_3k_7y_0(k_8+2x_0)(k_7+k_8+6x_0)+2k_3(k_7+2x_0)(k_8+4x_0)}{2d_2x_0(k_7+k_8+4x_0)(k_7+k_8+6x_0)}\right),$$

$$p_3 = \left(\frac{k_8}{2}+x_0, y_0, \frac{c_2c_3k_8y_0(k_7+2x_0)(k_7+k_8+6x_0)+2k_3(k_7+4x_0)(k_8+2x_0)}{2d_2x_0(k_7+k_8+4x_0)(k_7+k_8+6x_0)}\right)$$

are three different equilibrium points of system (3.1) in Ω.

3.1 One equilibrium point of the differential system

In this subsection, we assume that the parameters $a_1,\ a_2,\ b_1,\ c_1,\ R,\ k_2,\ k_4,\ k_5$ and k_6 satisfy the conditions (3.2) and (3.3) of Lemma 3.1, and $k_7 = k_8 = 0$. Then according to Remark 3.2, $p_1 = \left(x_0, y_0, \frac{k_3}{3d_2x_0}\right)$ is the unique equilibrium point of the differential system (3.1) in Ω.

Proposition 3.3. *The equilibrium point p_1 is not hyperbolic and it has a local unstable manifold of dimension 2.*

Proof. Under the hypothesis of this subsection, the characteristic polynomial of the linear approximation M_{p_1} of differential system (3.1) at p_1 is

$$P(\lambda) = -\lambda^3 + \frac{k_3}{3b_2c_3x_0+3c_3x_0y_0}\lambda^2 - \frac{(\rho(b_2+y_0)(3c_2c_3x_0y_0+k_3)+3b_2d_2k_3)}{9c_3x_0y_0(b_2+y_0)}\lambda.$$

The eigenvalues of M_{p_1} are

$$\lambda_1 = 0,$$

$$\lambda_2 = \frac{k_3y_0 - \sqrt{y_0\left(4c_3x_0(b_2+y_0)(-\rho(b_2+y_0)(3c_2c_3x_0y_0+k_3)-3b_2d_2k_3)+{k_3}^2y_0\right)}}{6c_3x_0y_0(b_2+y_0)},$$

and

$$\lambda_3 = \frac{k_3y_0 + \sqrt{y_0\left(4c_3x_0(b_2+y_0)(-\rho(b_2+y_0)(3c_2c_3x_0y_0+k_3)-3b_2d_2k_3)+{k_3}^2y_0\right)}}{6c_3x_0y_0(b_2+y_0)}.$$

Since $\lambda_1 = 0$, the equilibrium point p_1 of differential system (3.1) is not hyperbolic. Moreover, if λ_2 and λ_3 are complex then

$$\mathrm{Re}(\lambda_2) = \mathrm{Re}(\lambda_3) = \frac{k_3y_0}{6c_3x_0y_0(b_2+y_0)} > 0.$$

It is not difficult to see that if λ_2 and λ_3 are real, then $\lambda_2 > 0$ and $\lambda_3 > 0$. Therefore the equilibrium point p_1 of differential system (3.1) has a local unstable manifold of dimension 2. □

Corollary 3.4. *The differential system (3.1) does not exhibit an Andronov–Hopf bifurcation at the equilibrium point $p_1 = \left(x_0, y_0, \frac{k_3}{3d_2x_0}\right)$.*

3.2 Two equilibrium points of the differential system

From now on this subsection, we assume that the parameters a_1, a_2, b_1, c_1, R, k_2, k_4, k_5 and k_6 satisfy the conditions (3.2) and (3.3) of Lemma 3.1, $k_7 = 0$ and $k_8 > 0$. By Remark 3.2, $p_1 = \left(x_0, y_0, \frac{2k_3}{d_2k_8+6d_2x_0}\right)$, and $p_2 = \left(\frac{k_8}{2} + x_0, y_0, \frac{c_2c_3k_8y_0(k_8+6x_0)+4k_3(k_8+2x_0)}{d_2(k_8+4x_0)(k_8+6x_0)}\right)$ are the unique two equilibrium points of differential system (3.1) in Ω.

Proposition 3.5. *The equilibrium point p_1 is not hyperbolic and it has a local unstable manifold of dimension 2.*

Proof. Considering the assignations for the parameters a_1, a_2, b_1, c_1, R, k_2, k_4, k_5, k_6 and k_7 given in this subsection, the characteristic polynomial of the linear approximation M_{p_1} of differential system (3.1) at p_1 is

$$\begin{aligned} P(\lambda) = & -\lambda^3 + \frac{2k_3}{c_3(b_2+y_0)(k_8+6x_0)}\lambda^2 \\ & - \frac{(\rho(b_2+y_0)(k_8+2x_0)(c_2c_3y_0(k_8+6x_0)+2k_3)+2b_2d_2k_3(k_8+6x_0))}{c_3y_0(b_2+y_0)(k_8+6x_0)^2}\lambda, \end{aligned}$$

and the eigenvalues of M_{p_1} are

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= \frac{k_3}{c_3(b_2+y_0)(k_8+6x_0)} - \sqrt{\frac{Q1}{{c_3}^2y_0(b_2+y_0)^2(k_8+6x_0)^2}}, \\ \lambda_3 &= \frac{k_3}{c_3(b_2+y_0)(k_8+6x_0)} + \sqrt{\frac{Q1}{{c_3}^2y_0(b_2+y_0)^2(k_8+6x_0)^2}}, \\ Q_1 = & -2{b_2}^2c_3d_2k_3(k_8+6x_0) - c_3\rho(b_2+y_0)^2(k_8+2x_0)(c_2c_3y_0(k_8+6x_0)+2k_3) \\ & + k_3y_0(k_3 - 2b_2c_3d_2(k_8+6x_0)). \end{aligned}$$

Then the equilibrium point p_1 of differential system (3.1) is not hyperbolic. Moreover, if λ_2 and λ_3 are complex then

$$\mathrm{Re}(\lambda_2) = \mathrm{Re}(\lambda_3) = \frac{k_3}{c_3(b_2+y_0)(k_8+6x_0)} > 0.$$

And it can be verify that if λ_2 and λ_3 are real then $\lambda_2 > 0$ and $\lambda_3 > 0$. Therefore the equilibrium point p_1 of differential system (3.1) is not hyperbolic and has a local unstable manifold of dimension 2. □

Corollary 3.6. *The differential system* (3.1) *does not exhibit an Andronov–Hopf bifurcation at the equilibrium point* $p_1 = \left(x_0, y_0, \frac{2k_3}{d_2k_8+6d_2x_0}\right)$.

Whereas the equilibrium point p_1 does not have an Andronov–Hopf bifurcation, we will show that the equilibrium point p_2 can have a pair of purely imaginary eigenvalues and consequently it can exhibit an Andronov–Hopf bifurcation.

Lemma 3.7. *If the parameters k_8, b_2, k_3, c_2, ρ and d_2 satisfy the conditions*

$$\begin{aligned} & k_8 = x_0, \quad b_2 = \frac{c_3x_0y_0(35c_2+\rho)+60k_3}{3c_3\rho x_0}, \quad k_3 = c_2c_3x_0y_0, \quad c_2 = \frac{581875-5877\rho^2}{143640\rho}, \\ & \rho < \sqrt{\frac{581875}{5877}}, \quad d_2 = d_{20}(\rho) := \frac{320060160\rho^3}{(116375-873\rho^2)(581875-5877\rho^2)}, \end{aligned} \tag{3.7}$$

then the equilibrium point p_2 of differential system (3.1) is given by

$$p_2 = \left(\frac{3x_0}{2}, y_0, \frac{c_3\left(581875 - 5877\rho^2\right)^2\left(116375 - 873\rho^2\right)y_0}{84687918336000\rho^4}\right)$$

and the eigenvalues of the linear approximation of system (3.1) at p_2 are

$$\alpha = -\frac{2592\rho^3}{475\left(9\rho^2 + 30625\right)} \quad \text{and} \quad \pm i.$$

Proof. Let M_{p_2} be the Jacobian matrix of the differential system (3.1) evaluated at the equilibrium point p_2, then the characteristic polynomial $P(\lambda) = \det(\lambda I - M_{p_2}) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$, where

$$\begin{aligned}
A_1 &= -\frac{\lambda^2\left((k_8+4x_0)(c_2c_3k_8y_0(k_8+6x_0)+4k_3(k_8+2x_0)) - c_3k_8{}^2\rho(b_2+y_0)(k_8+2x_0)\right)}{c_3(b_2+y_0)(k_8+4x_0)^2(k_8+6x_0)},\\
A_2 &= -\frac{-b_2d_2(k_8+6x_0)(k_8+4x_0)^2(c_2c_3k_8y_0(k_8+6x_0)+4k_3(k_8+2x_0)) - B_1}{c_3y_0(b_2+y_0)(k_8+4x_0)^3(k_8+6x_0)^2}\\
&\quad -\frac{\rho y_0(k_8+2x_0)(k_8+4x_0)\left(c_2c_3y_0(k_8+6x_0)\left(k_8{}^2+4k_8x_0-16x_0{}^2\right)+4k_3(k_8+4x_0)(k_8-2x_0)\right)}{c_3y_0(b_2+y_0)(k_8+4x_0)^3(k_8+6x_0)^2},\\
A_3 &= \frac{b_2d_2k_8{}^2\rho(k_8+2x_0)(c_2c_3k_8y_0(k_8+6x_0)+4k_3(k_8+2x_0))}{c_3y_0(b_2+y_0)(k_8+4x_0)^3(k_8+6x_0)^2},\\
B_1 &= 8b_2\rho x_0(k_8+2x_0)\left(8x_0{}^2 - k_8{}^2\right)(c_2c_3y_0(k_8+6x_0)+2k_3).
\end{aligned}$$

Taking k_8, b_2 and k_3 satisfying (3.7), then A_1, A_2 and A_3 are reduced to

$$A_1 = \frac{12\rho^2}{175(95c_2+4\rho)}, \qquad A_2 = \frac{c_2(665c_2(475d_2+216\rho)+\rho(3325d_2+5877\rho))}{6125(95c_2+4\rho)},$$

$$A_3 = \frac{57c_2d_2\rho(95c_2+\rho)}{6125(95c_2+4\rho)}.$$

Following the proof of Lemma 2.3, the characteristic polynomial $P(\lambda)$ has a pair of purely imaginary roots $\pm i\omega$ and a real root α if and only if $A_2 > 0$ and

$$A_1A_2 - A_3 = 0, \tag{3.8}$$

where $\omega = \sqrt{A_2}$ and $\alpha = -A_1$.

Since $A_1A_2 - A_3 = -\frac{3c_2\rho\left(30008125c_2{}^2d_2+665c_2\rho(475d_2-864\rho)-23508\rho^3\right)}{1071875(95c_2+4\rho)^2}$, solving equation (3.8) for the parameter d_2, we have that

$$d_2 = \frac{36\rho^2(15960c_2+653\rho)}{315875c_2(95c_2+\rho)}.$$

Taking into account this assignment for d_2, the coefficient $A_2 = \frac{9\rho(15960c_2+653\rho)}{581875} > 0$, which is equal to 1, when $c_2 = \frac{581875-5877\rho^2}{143640\rho}$. Moreover, $c_2 > 0$, if

$$\rho < \sqrt{\frac{581875}{5877}}.$$

Therefore, the characteristic polynomial $P(\lambda)$ has a pair of purely imaginary roots $\pm i$ and a real root $\alpha = -\frac{2592\rho^3}{475(9\rho^2+30625)}$. The equilibrium point p_2 of system (3.1) becomes

$$p_2 = \left(\frac{3x_0}{2}, y_0, \frac{c_3 \left(581875 - 5877\rho^2\right)^2 \left(116375 - 873\rho^2\right) y_0}{84687918336000\rho^4} \right).$$

□

Remark 3.8. If the assumptions of Lemma 3.7 are satisfied, then the linear approximation of the differential system (3.1) at p_2 has two eigenvalues purely imaginary when $d_2 = d_{20}(\rho)$. Hence, by continuity on the eigenvalues, the linear approximation of the differential system at p_2 has a pair of complex eigenvalues,

$$\lambda(p_2, d_2, \rho) = \xi(p_2, d_2, \rho) \pm i\omega(p_2, d_2, \rho),$$

when d_2 is in a neighborhood of $d_{20}(\rho)$.

In order to compute the first Lyapunov coefficient ℓ_1, we make the following assignations.

$$c_3 = 1, \qquad y_0 = 1, \quad \text{and} \quad x_0 = 1.$$

Applying the Kuznetsov formula, (see Ref. [7]) and using the Mathematica software, we obtain the first Lyapunov coefficient $\ell_1(p_2, d_{20}(\rho), \rho)$, of the differential system (3.1) at the equilibrium point p_2.

Lemma 3.9. *If we have the assumptions given in Lemma 3.7, then the eigenvalues of the linear approximation of system (3.1) at the equilibrium point p_2 are $\alpha = -\frac{2592\rho^3}{475(9\rho^2+30625)}$ and $\pm i$, and the first Lyapunov coefficient*

$$\ell_1(p_2, d_{20}(\rho), \rho) = \frac{37791360\rho^5 \left(9\rho^2 + 30625\right) s_3(\rho)}{49 \left(5877\rho^2 - 581875\right) s_4(\rho) s_6(\rho) s_5(\rho)},$$

where

$$\begin{aligned} s_3(\rho) = {} & 66791173348816998488577446 4\rho^{14} \\ & + 617401358762851620995638930875\rho^{12} \\ & - 4417446758799213089587643934375 00\rho^{10} \\ & + 2332879346419733295386537989746093 75\rho^8 \\ & + 177275906705124540208423981933593750000\rho^6 \\ & - 841493581845049255957524390220642089843 75\rho^4 \\ & + 1078380414207792101994178462028503417968 7500\rho^2 \\ & - 4331922607349956064094400331377983093261 71875 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} s_4(\rho) &= 211611572265625 + 9\rho^2(13819531250 + 2030625\rho^2 + 186624\rho^4), \\ s_5(\rho) &= 211611572265625 + 9\rho^2(13819531250 + 2030625\rho^2 + 746496\rho^4), \\ s_6(\rho) &= 4585416449713134765625 + 9\rho^2(874195168467578125 00 \\ &\quad + 27\rho^2(2877811231676281250 + 792497391678300\rho^2 + 108326221587\rho^4)). \end{aligned}$$

Corollary 3.10. *If we have the assumptions given in Lemma 3.7, then there exists a unique real number $0 < \rho_0 < \sqrt{\frac{581875}{5877}}$, such that the first Lyapunov coefficient $\ell_1(p_2, d_{20}(\rho_0), \rho_0) = 0$.*

Proof. By Lemma 3.9 and equation (3.9), the first Lyapunov coefficient $\ell_1(p_2, d_{20}(\rho), \rho) = 0$ if and only if $s_3(\rho) = 0$. By Descartes rule of signs, there are 1, 3 or 5 positive real numbers ρ such that $s_3(\rho) = 0$. Indeed, numerically this equation has three positive solutions, but only ρ_0 (≈ 9.76907) is less than $\sqrt{\frac{581875}{5877}}$. □

Lemma 3.11 (Bautin regularity condition)**.** *If the parameters k_8, b_2, k_3, c_2 and ρ satisfy the relations (3.7) of Lemma 3.7, then the map $(d_2, \rho) \mapsto (\xi(p_2, d_2, \rho), \ell_1(p_2, d_2(\rho), \rho))$ is regular at (d_0, ρ_0), where $\xi(p_2, d_2, \rho)$ is given in Remark 3.8 and $d_0 := d_{20}(\rho_0)$.*

Proof. By hypothesis, the linear approximation of the differential system (3.1) at p_2 depends only on the parameters d_2 and ρ, let $M_{p_2}(d_2, \rho)$ be this linear approximation. By Lemma 3.7, the real part of the complex eigenvalues of $M_{p_2}(d_0, \rho_0)$ are

$$\xi(p_2, d_0, \rho_0) = 0. \tag{3.10}$$

Let $\mathbf{p}$ and $\mathbf{q}$ be eigenvectors of $M_{p_2}(d_0, \rho_0)$ for the corresponding eigenvalues $-i$ and i, respectively, such that

$$\overline{\mathbf{q}}^{tr} \cdot \mathbf{q} = 1 \quad \text{and} \quad \overline{\mathbf{p}}^{tr} \cdot \mathbf{q} = 1. \tag{3.11}$$

By (3.10) and (3.11), we can apply Lemma 2.7 to the linear approximation $M_{p_2}(d_2, \rho)$. Taking into account the values of $\mathbf{q}$, $\overline{\mathbf{p}}$ and $\frac{\partial M_{p_0}(d_2,\rho)}{\partial d_2}$, and using the Mathematica software, we obtain the partial derivative of the real part of the eigenvalues $\xi(d_2, \rho) \pm i\omega(d_2, \rho)$ of $M_{p_2}(d_2, \rho)$,

$$\frac{\partial \xi}{\partial d_2}(d_0, \rho_0) = \frac{5\left(581875 - 5877\rho_0^2\right)^2 \left(116375 - 873\rho_0^2\right)}{Q_2}, \tag{3.12}$$

$$Q_2 = 49392\left(9\left(746496\rho_0^4 + 2030625\rho_0^2 + 13819531250\right)\rho_0^2 + 211611572265625\right).$$

Applying Lemma 2.7 one more time, it follows from the values of $\mathbf{q}$, $\overline{\mathbf{p}}$ and $\frac{\partial M_{p_2}(d_2,\rho)}{\partial \rho}$ that

$$\frac{\partial \xi}{\partial \rho}(d_0, \rho_0) = -\frac{97200\rho_0^2\left(1710207\rho_0^4 + 397304250\rho_0^2 - 67715703125\right)}{Q_3}, \tag{3.13}$$

$$Q_3 = \left(873\rho_0^2 - 116375\right)\left(9\left(746496\rho_0^4 + 2030625\rho_0^2 + 13819531250\right)\rho_0^2 + 211611572265625\right).$$

Using the Wolfram Mathematica software, we have that $\mathrm{Re}\left(\frac{\partial \mathbf{C}_1}{\partial d_2}(d_0, \rho_0)\right) \approx -0.22637$ and $\mathrm{Re}\left(\frac{\partial \mathbf{C}_1}{\partial \rho}(d_0, \rho_0)\right) \approx 158.86065$, where, $\mathbf{C}_1(d_2, \rho)$ is the function given in the proof of Lemma 2.8, and by Corollary 3.10, $\rho_0 \approx 9.76907$. Then by (3.12), (3.13) and the analogous of formula (2.13) given in the proof of Lemma 2.8, we have that

$$\det\begin{pmatrix} \frac{\partial \xi}{\partial d_2}(d_0, \rho_0) & \frac{\partial \xi}{\partial \rho}(d_0, \rho_0) \\ \frac{\partial \ell_1}{\partial d_2}(d_0, \rho_0) & \frac{\partial \ell_1}{\partial \rho}(d_0, \rho_0) \end{pmatrix} \approx -0.00291456.$$

Hence, the map $(d_2, \rho) \mapsto (\xi(p_2, d_2, \rho), \ell_1(p_2, d_2, \rho))$ is regular at (d_0, ρ_0). □

Theorem 3.12. *If the parameters k_8, b_2, k_3, c_2 and ρ satisfy the relations (3.7) of Lemma 3.7, then the differential system (3.1) exhibits an Andronov–Hopf bifurcation at*

$$p_2 = \left(\frac{3}{2}, 1, \frac{\left(581875 - 5877\rho^2\right)^2 \left(116375 - 873\rho^2\right)}{84687918336000\rho^4}\right),$$

with respect to the parameter d_2 and its critical bifurcation value is $d_{20}(\rho)$, where $\rho \in \left(0, \sqrt{\frac{581875}{5877}}\right)$ and $\rho \neq \rho_0$. Moreover, if $\rho < \rho_0$ the bifurcation is subcritical and if $\rho > \rho_0$ the bifurcation is supercritical.

Proof. From Lemma 3.7, the linear approximation $M_{p_2}(d_2, \rho)$ of differential system (3.1) at p_2 has a negative real eigenvalue and a pair of purely imaginary eigenvalues if $d_2 = d_{20}(\rho)$. From Lemma 3.11, the derivative of the real part of the complex eigenvalues

$$\frac{\partial \xi}{\partial d_2}(d_{20}(\rho), \rho) = \frac{5\left(581875 - 5877\rho^2\right)^2 \left(116375 - 873\rho^2\right)}{Q_4},$$

$$Q_4 = 49392\left(9\left(746496\rho^4 + 2030625\rho^2 + 13819531250\right)\rho^2 + 211611572265625\right),$$

which is positive if $\rho \in \left(0, \sqrt{\frac{581875}{5877}}\right)$, and hence the transversality condition holds. By Corollary 3.10 the first Lyapunov coefficient is negative if $\rho > \rho_0$, and is positive if $\rho < \rho_0$. Then the hypotheses of Andronov–Hopf bifurcation Theorem hold and we conclude the proof (see Refs. [7–9]). □

Lemma 3.13 (Second Lyapunov coefficient)**.** *If we have the assumptions given in Lemma 3.7 and $\rho = \rho_0$, then the second Lyapunov coefficient of differential system (3.1) at the equilibrium point p_2, $\ell_2(p_2, d_{20}(\rho_0), \rho_0) \neq 0$.*

Proof. In order to compute the second Lyapunov coefficient ℓ_2, we apply the Kuznetsov formula, (see Ref. [4]). Taking into account the assumptions of this Lemma and using the Mathematica software, we obtain that the second Lyapunov coefficient $\ell_2(p_2, d_{20}(\rho), \rho)$, of the differential system (3.1) at the equilibrium point p_2 takes the value $\ell_2(p_2, d_{20}(\rho_0), \rho_0) \approx 8894.15$, if $\rho = \rho_0$. □

Corollary 3.10, Lemma 3.11 and Lemma 3.13 provide the validity of the necessary and sufficient conditions to apply the Bautin bifurcation theorem (see Ref. [4]). Then we have obtained the following.

Theorem 3.14. *If the parameters k_8, b_2, k_3, c_2 and ρ satisfy the relations (3.7) of Lemma 3.7, then the differential system (3.1) exhibits a Bautin bifurcation at p_2, with respect to the parameters d_2 and ρ and its critical bifurcation value is $(d_{20}(\rho_0), \rho_0)$.*

3.3 Three equilibrium points of the differential system

From now on in this subsection, we assume that the parameters a_1, a_2, b_1, c_1, R, k_2, k_4, k_5 and k_6 satisfy the conditions (3.2) and (3.3) of Lemma 3.1, $k_7 > 0$, $k_8 > 0$ and $k_7 \neq k_8$.

Then by Remark 3.2,

$$p_1 = \left(x_0, y_0, \frac{2k_3}{d_2(k_7+k_8+6x_0)}\right),$$
$$p_2 = \left(\frac{k_7}{2}+x_0, y_0, \frac{c_2c_3k_7y_0(k_8+2x_0)(k_7+k_8+6x_0)+2k_3(k_7+2x_0)(k_8+4x_0)}{2d_2x_0(k_7+k_8+4x_0)(k_7+k_8+6x_0)}\right),$$
$$p_3 = \left(\frac{k_8}{2}+x_0, y_0, \frac{c_2c_3k_8y_0(k_7+2x_0)(k_7+k_8+6x_0)+2k_3(k_7+4x_0)(k_8+2x_0)}{2d_2x_0(k_7+k_8+4x_0)(k_7+k_8+6x_0)}\right)$$

are the unique three equilibrium points of differential system (3.1) in Ω.

3.3.1 Local dynamics and bifurcation at p_1

Lemma 3.15. *If the parameters k_7, k_8, b_2, k_3, c_2 and ρ satisfy the conditions*

$$\begin{aligned} &k_7 = 2x_0, \quad k_8 = x_0/2, \quad b_2 = \frac{27k_3}{c_3\rho x_0}+y_0, \quad k_3 = c_2c_3x_0y_0, \quad c_2 = \frac{459^2-10358\rho^2}{140049\rho}, \\ &\rho < \frac{459}{\sqrt{10358}}, \quad d_2 = d_{20}(\rho) := \frac{12349380771\rho^3}{2\left(459^2-5171\rho^2\right)\left(459^2-10358\rho^2\right)}, \end{aligned} \tag{3.14}$$

then the equilibrium point

$$p_1 = \left(x_0, y_0, \frac{8c_3\left(459^2-10358\rho^2\right)^2\left(459^2-5171\rho^2\right)y_0}{29401813269162243\rho^4}\right)$$

and the eigenvalues of the linear approximation at p_1 are

$$\alpha = -\frac{13832\rho^3}{2448\rho^2+32234193} \quad \textit{and} \quad \pm i.$$

Proof. The characteristic polynomial of the linear approximation M_{p_1} of differential system (3.1), at the equilibrium point p_1 is $P(\lambda) = \det(\lambda I - M_{p_1}) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$, where,

$$\begin{aligned} A_1 &= -\frac{2\left(\frac{k_3}{c_3(b_2+y_0)} - \frac{k_7k_8\rho x_0}{(k_7+4x_0)(k_8+4x_0)}\right)}{B_3}, \\ A_2 &= \frac{\rho y_0\left(c_2c_3y_0B_3B_2+2k_3(k_7+4x_0)(k_8+4x_0)(k_7+k_8+2x_0)\right)}{c_3y_0(b_2+y_0)(k_7+4x_0)(k_8+4x_0)(B_3)^2} \\ &\quad + \frac{b_2\rho B_2(c_2c_3y_0B_3+2k_3)+2b_2d_2k_3(k_7+4x_0)(k_8+4x_0)B_3}{c_3y_0(b_2+y_0)(k_7+4x_0)(k_8+4x_0)(B_3)^2}, \\ A_3 &= \frac{4b_2d_2k_3k_7k_8\rho x_0}{c_3y_0(b_2+y_0)(k_7+4x_0)(k_8+4x_0)(B_3)^2}, \\ B_2 &= (k_7+k_8+4x_0)\left(4x_0(k_7+k_8)+k_7k_8+8{x_0}^2\right), \\ B_3 &= k_7+k_8+6x_0. \end{aligned}$$

By hypothesis k_7, k_8, b_2 and k_3 satisfy (3.14), then A_1, A_2 and A_3 reduce to

$$A_1 = \frac{8\rho^2}{459(27c_2 + 2\rho)}, \qquad A_2 = \frac{c_2(81c_2(612d_2 + 1729\rho) + 2\rho(918d_2 + 5179\rho))}{7803(27c_2 + 2\rho)},$$
$$A_3 = \frac{16c_2d_2\rho(27c_2 + \rho)}{7803(27c_2 + 2\rho)}.$$

As in the proof of Lemma 2.3, the characteristic polynomial $P(\lambda)$ has a pair of purely imaginary roots $\pm i\omega$ and a real root α if and only if $A_2 > 0$ and

$$A_1A_2 - A_3 = 0, \tag{3.15}$$

where $\omega = \sqrt{A_2}$ and $\alpha = -A_1$. In this case,

$$A_1A_2 - A_3 = \frac{8c_2\rho\left(-669222{c_2}^2d_2 + 81c_2\rho(1729\rho - 306d_2) + 10358\rho^3\right)}{3581577(27c_2 + 2\rho)^2}.$$

Solving the equation (3.15) for the parameter d_2, we have that

$$d_2 = \frac{\rho^2(140049c_2 + 10358\rho)}{24786c_2(27c_2 + \rho)}.$$

Taking $c_2 = \frac{459^2 - 10358\rho^2}{140049\rho}$, we have $A_2 = 1$. Since $c_2 > 0$, then the parameter ρ must satisfy

$$\rho < \frac{459}{\sqrt{10358}}.$$

Therefore, the characteristic polynomial $P(\lambda)$ has a pair of purely imaginary roots $\pm i$ and a real root $\alpha = -\frac{13832\rho^3}{2448\rho^2 + 32234193}$. If $k_7, k_8, b_2, k_3, c_2, \rho$ and d_2 satisfy the relations given by (3.7), then

$$p_1 = \left(x_0, y_0, \frac{8c_3\left(459^2 - 10358\rho^2\right)^2\left(459^2 - 5171\rho^2\right)y_0}{29401813269162243\rho^4}\right).$$

□

Remark 3.16. If the assumptions of Lemma 3.15 are satisfied, then the linear approximation of the differential system (3.1) at p_1 has two purely imaginary eigenvalues when $d_2 = d_{20}(\rho)$, hence, by continuity on the eigenvalues, the linear approximation of the differential system at p_1 has a pair of complex eigenvalues,

$$\lambda(p_1, d_2, \rho) = \xi(p_1, d_2, \rho) \pm i\omega(p_1, d_2, \rho),$$

when d_2 is in a neighborhood of $d_{20}(\rho)$.

In order to compute the first and second Lyapunov coefficients we make the following assignations

$$c_3 = 1, \qquad y_0 = 1, \quad \text{and} \quad x_0 = 1.$$

Applying the Kuznetsov formula and using the Mathematica software, we obtain the next result.

Lemma 3.17. *If the assumptions given in Lemma 3.15 hold, then the eigenvalues of the linear approximation of the differential system (3.1) at the equilibrium point* p_1 *are* $\alpha = -\frac{13832\rho^3}{153(16\rho^2+210681)}$ *and* $\pm i$*. The first Lyapunov coefficient is*

$$\ell_1(p_1, d_{20}(\rho), \rho) = \frac{312947271\rho^5\left(16\rho^2 + 459^2\right)s_3(\rho)}{4\left(10358\rho^2 - 459^2\right)s_4(\rho)s_5(\rho)s_6(\rho)},$$

where

$$\begin{aligned}
s_3(\rho) = {} & 17074288205865381069386373529 6\rho^{14} \\
& + 4922629722966755284361616596302 08\rho^{12} \\
& - 1081614546367206122000558628504866 88\rho^{10} \\
& - 2116707589786078001750294619917568712 26\rho^{8} \\
& - 9485999858302719829966772948728271221 506\rho^{6} \\
& + 8965980529109774755477528919410737731 2320\rho^{4} \\
& + 1444139284193694522974863803490461620260 1802\rho^{2} \\
& - 1840628614286306738237682177416464020964 30491, \\
s_4(\rho) = {} & 45900865178296384\rho^8 + 55675243889896695 84\rho^6 \\
& + 147414277124069435267049\rho^4 + 2469518584478549676288 81\rho^2 \\
& + 31522559050648267281936, \\
s_5(\rho) = {} & 16\left(2989441\rho^4 + 374544\rho^2 + 9863663058\right)\rho^2 + 1039043198361249, \\
s_6(\rho) = {} & 32\left(5978882\rho^4 + 187272\rho^2 + 4931831529\right)\rho^2 + 1039043198361249.
\end{aligned}$$

Corollary 3.18. *If the assumptions given in Lemma 3.15 hold, then there exists a unique real number* $0 < \rho_0 < \frac{459}{\sqrt{10358}}$ *such that the first Lyapunov coefficient* $\ell_1(p_1, d_{20}(\rho_0), \rho_0) = 0$.

Proof. By Lemma 3.17, the first Lyapunov coefficient $\ell_1(p_1, d_{20}(\rho), \rho) = 0$ if and only if $s_3(\rho) = 0$. According to Descartes rule of signs, there are 1 or 3 positive real numbers ρ such that $s_3(\rho) = 0$. Indeed, numerically this equation has three positive solutions, but only $\rho_0\,(\approx 3.71999)$ is less than $\frac{459}{\sqrt{10358}}$. □

Lemma 3.19 (Bautin regularity condition)**.** *If the parameters* k_3, k_7, k_8, b_2, c_2 *and* ρ *satisfy the relations (3.14) of Lemma 3.15, then the map* $(d_2, \rho) \mapsto (\xi(p_1, d_2, \rho), \ell_1(p_1, d_2(\rho), \rho))$ *is regular at* (d_0, ρ_0)*, where* $\xi(p_1, d_2, \rho)$ *is given in Remark 3.16 and* $d_0 := d_{20}(\rho_0)$.

Proof. By hypothesis, the linear approximation of the differential system (3.1) at p_1 depends only on the parameters d_2 and ρ. Let $M_{p_1}(d_2, \rho)$ be this linear approximation. By Lemma 3.15, the complex numbers $\pm i$ are eigenvalues of $M_{p_1}(d_0, \rho_0)$. Hence, the real part of the complex eigenvalues of $M_{p_1}(d_0, \rho_0)$ is

$$\xi(p_1, d_0, \rho_0) = 0.$$

Let $\mathbf{p}$ and $\mathbf{q}$ be eigenvectors of $M_{p_1}(d_0, \rho_0)$ for the corresponding eigenvalues $-i$ and i, respectively, such that

$$\overline{\mathbf{q}}^{tr} \cdot \mathbf{q} = 1 \quad \text{and} \quad \overline{\mathbf{p}}^{tr} \cdot \mathbf{q} = 1.$$

By Lemma 2.7 and taking into account the values of $\mathbf{q}$, $\overline{\mathbf{p}}$, $\frac{\partial M_{p_1}(d_2,\rho)}{\partial d_2}$ and $\frac{\partial M_{p_1}(d_2,\rho)}{\partial \rho}$, we obtain

$$\frac{\partial \xi}{\partial d_2}(d_0,\rho_0) = \frac{8\left(459^2-10358\rho_0^2\right)^2\left(459^2-5171\rho_0^2\right)}{46683Q_5}, \tag{3.16}$$
$$Q_5 = 32\left(5978882\rho_0^4+187272\rho_0^2+4931831529\right)\rho_0^2+1039043198361249,$$

$$\frac{\partial \xi}{\partial \rho}(d_0,\rho_0) = -\frac{1058148\rho_0^2\left(53561218\rho_0^4+3271665249\rho_0^2-133159451283\right)}{\left(5171\rho_0^2-210681\right)Q_7}, \tag{3.17}$$
$$Q_7 = \left(32\left(5978882\rho_0^4+187272\rho_0^2+4931831529\right)\rho_0^2+1039043198361249\right).$$

Numerically, taking $\rho_0 \approx 9.76907$, we have that $\mathrm{Re}\left(\frac{\partial \mathbf{C}_1}{\partial d_2}(d_0,\rho_0)\right) \approx -0.60234$ and $\mathrm{Re}\left(\frac{\partial \mathbf{C}_1}{\partial \rho}(d_0,\rho_0)\right) \approx 14.78256$, where $\mathbf{C}_1(d_2,\rho)$ is as in the proof of Lemma 2.8. Then by (3.16), (3.17) and the analogous formula of (2.13) given in Lemma 2.8,

$$\det\begin{pmatrix} \frac{\partial \xi}{\partial d_2}(d_0,\rho_0) & \frac{\partial \xi}{\partial \rho}(d_0,\rho_0) \\ \frac{\partial \ell_1}{\partial d_2}(d_0,\rho_0) & \frac{\partial \ell_1}{\partial \rho}(d_0,\rho_0) \end{pmatrix} \approx -0.00291.$$

Hence, the map $(d_2,\rho) \mapsto (\xi(p_1,d_2,\rho),\ell_1(p_1,d_2,\rho))$ is regular at (d_0,ρ_0). □

Theorem 3.20. *If the parameters k_3, k_7, k_8, b_2, c_2 and ρ satisfy the relations (3.14) of Lemma 3.15, then the differential system (3.1) exhibits an Andronov–Hopf bifurcation at*

$$p_1 = \left(1, 1, \frac{8\left(459^2-10358\rho^2\right)^2\left(459^2-5171\rho^2\right)}{29401813269162243\rho^4}\right),$$

with respect to the parameter d_2 and its critical bifurcation value is $d_{20}(\rho)$, where $\rho \in (0, 459/\sqrt{10358})$ and $\rho \neq \rho_0$. Moreover, if $\rho < \rho_0$ the bifurcation is subcritical and if $\rho > \rho_0$ the bifurcation is supercritical.

Proof. From Lemma 3.15, the linearization $M_{p_1}(d_2,\rho)$ of differential system (3.1) at p_1 has a negative real eigenvalue and a pair of purely imaginary eigenvalues if $d_2 = d_{20}(\rho)$. From Lemma 3.19, the derivative of the real part of the complex eigenvalues

$$\frac{\partial \xi}{\partial d_2}(d_{20}(\rho),\rho) = \frac{8\left(459^2-10358\rho^2\right)^2\left(459^2-5171\rho^2\right)}{46683Q_8},$$
$$Q_8 = 32\left(5978882\rho^4+187272\rho^2+4931831529\right)\rho^2+1039043198361249,$$

which is positive if $\rho \in (0, 459/\sqrt{10358})$, and hence the transversality condition holds. Lemma 3.17 and Corollary 3.18 imply that the first Lyapunov coefficient is negative if $\rho > \rho_0$, and is positive if $\rho < \rho_0$, (see Figure 3.1). Then the hypotheses of Andronov–Hopf bifurcation theorem hold and we conclude the proof. □

In order to show the Bautin bifurcation, we compute the second Lyapunov coefficient ℓ_2. Applying the Kuznetsov formula, and using the Mathematica software, we obtain that the second Lyapunov coefficient $\ell_2(p_1, d_{20}(\rho), \rho)$, of the differential system (3.1) at the equilibrium point p_1 takes the value $\ell_2(p_1, d_{20}(\rho_0), \rho_0) \approx -26718.1$.

Lemma 3.21 (Second Lyapunov coefficient)**.** *If we have the assumptions given in Lemma 3.15, then the second Lyapunov coefficient of differential system (3.1) at the equilibrium point p_1, $\ell_2(p_1, d_{20}(\rho_0), \rho_0) \neq 0$.*

From Corollary 3.18, Lemma 3.19 and Lemma 3.21 we have the necessary and sufficient conditions to apply the Bautin bifurcation theorem. Therefore we obtain the following.

Theorem 3.22. *If the parameters k_3, k_7, k_8, b_2, c_2 and ρ satisfy the relations (3.14) of Lemma 3.15, then the differential system (3.1) exhibits a Bautin bifurcation at p_1, with respect to the parameters d_2 and ρ and its critical bifurcation value is $(d_{20}(\rho_0), \rho_0)$.*

From Theorems 3.20 and 3.22 we have shown the existence of limit cycles in Ω to differential system (3.1) near to p_1. Now, we will analyze the local dynamics at equilibrium point p_2.

3.3.2 Local dynamics and bifurcation at p_2

In this subsection we assume that the parameters k_3, k_7, k_8, b_2, c_2 and ρ satisfy the relations (3.14) of Lemma 3.15, and $x_0 = y_0 = c_3 = 1$. In the same way of the previous subsection, we obtain the next results relative to p_2.

Lemma 3.23. *If*

$$d_2 = d_{21}(\rho) := \frac{42\rho\left(39818709 - 152092\rho^2\right)\left(1629662\rho^2 + 12430179\right)}{5447429\left(459^2 - 5171\rho^2\right)\left(459^2 - 10358\rho^2\right)},$$

then the equilibrium point p_2 of the differential system (3.1) is given by

$$p_2 = \left(2, 1, \frac{3869893\left(459^2 - 10358\rho^2\right)^2\left(459^2 - 5171\rho^2\right)}{5882058\rho^2\left(39818709 - 152092\rho^2\right)\left(1629662\rho^2 + 12430179\right)}\right)$$

and the eigenvalues of the linear approximation of system (3.1) at p_2 are

$$\alpha = -\frac{\rho\left(1629662\rho^2 + 12430179\right)}{5967\left(16\rho^2 + 210681\right)} \quad \text{and} \quad \pm\frac{4\sqrt{39818709 - 152092\rho^2}}{221\sqrt{38779}}i.$$

Lemma 3.24. *If we have the assumptions given in Lemma 3.23, the first Lyapunov coefficient at p_2 is*

$$\ell_1(p_2, d_{21}(\rho), \rho) = -\frac{15877775277\sqrt{247}\rho^5\left(16\rho^2 + 210681\right)\left(152092\rho^2 - 39818709\right)\sigma_3(\rho)}{314\sqrt{6251537313 - 23878444\rho^2}\left(10358\rho^2 - 210681\right)\sigma_4(\rho)\sigma_5(\rho)\sigma_6(\rho)},$$

where

$$\begin{aligned}
\sigma_3(\rho) = {} & 26742294352244146788269524839873927270064311865 60\rho^{14} \\
& + 805184870865477880770429208091467190331922797852960\rho^{12} \\
& - 833251051787696247262478453678965672779490503858506 40\rho^{10} \\
& - 39908334741334892754366139655879276980939381695819791 04\rho^{8} \\
& + 189394169702700882757415404950712927285393366774987535 2320\rho^{6} \\
& - 103337284251342816394773872075621179984010459502936989864334\rho^{4} \\
& + 2045265511014986633744532946163166674530762170351157348636599\rho^{2} \\
& - 13861057129694950151794339970517627657596357576143945686701523,
\end{aligned}$$

$$\sigma_4(\rho) = 1029873831486339 64\rho^6 + 152372746581714 5724\rho^4 - 2964501861396032 99565\rho^2 + 824603966861248791 18144,$$
$$\sigma_5(\rho) = 102988745581469548\rho^6 + 1559250610762430844\rho^4 - 69618757972976491437\rho^2 + 206150991715312197 79536,$$

and

$$\sigma_6(\rho) = 893093474288628472 0660\rho^8 - 2937571167497258615 8626756\rho^6 + 7535731011874228165827361581\rho^4 + 112158685875537178 0236256518\rho^2 + 7624308163261898438530822053.$$

Corollary 3.25. *There exists a unique real number $0 < \rho_1 < \frac{459}{\sqrt{10358}}$ such that the first Lyapunov coefficient $\ell_1(p_2, d_{21}(\rho_1), \rho_1) = 0$. Indeed $\rho_1 \approx 4.36757$.*

Lemma 3.26 (Bautin regularity condition)**.** *The map $(d_2, \rho) \mapsto (\xi(p_2, d_2, \rho), \ell_1(p_2, d_2, \rho))$ is regular at (d_1, ρ_1), where $\xi(p_2, d_2, \rho)$ is as in Lemma 3.19 and $d_1 := d_{21}(\rho_1)$.*

Theorem 3.27. *The differential system (3.1) exhibits an Andronov–Hopf bifurcation at p_2 with respect to the parameter d_2 and its critical bifurcation value is $d_{21}(\rho)$, where $\rho \in (0, 459/\sqrt{10358})$ and $\rho \neq \rho_1$. Moreover, if $\rho < \rho_1$ the bifurcation is subcritical and if $\rho > \rho_1$ the bifurcation is supercritical.*

In order to show a Bautin bifurcation, we compute the second Lyapunov coefficient.

Lemma 3.28 (Second Lyapunov coefficient)**.** *If we have the assumptions given in Lemma 3.23, then the second Lyapunov coefficient of differential system (3.1), $\ell_2(p_2, d_{21}(\rho_1), \rho_1)$ is negative. Indeed $\ell_2(p_2, d_{21}(\rho_1), \rho_1) \approx -161.216$.*

We summarize the results in the following theorem.

Theorem 3.29. *The differential system (3.1) exhibits a Bautin bifurcation at p_2, with respect to the parameters d_2 and ρ and its critical bifurcation value is $(d_{21}(\rho_1), \rho_1)$.*

3.3.3 Local dynamics at p_3

Theorem 3.30. *If the parameters k_3, k_7, k_8, b_2, c_2 and ρ satisfy the relations (3.14) of Lemma 3.15, then, the differential system (3.1) does not exhibit a Hopf bifurcation at the equilibrium point*

$$p_3 = \left(\frac{5}{4}, 1, \frac{94\,(210681 - 10358\rho^2)}{30950829 d_2 \rho}\right).$$

Proof. A necessary condition for a differential system to exhibit an Andronov–Hopf bifurcation at an equilibrium point is that the characteristic polynomial of its linear approximation has a pair of purely imaginary roots. According to the proof of Lemma 2.3, the characteristic polynomial $P(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$ has a pair of purely imaginary roots $\pm i\omega$ and a real root α if and only if $A_2 > 0$ and

$$A_1 A_2 - A_3 = 0,$$

where $\omega = \sqrt{A_2}$ and $\alpha = -A_1$. By hypothesis, if M_{p_3} is the linear approximation of differential system (3.1) at p_3 then

$$P(\lambda) = \det(\lambda I - M_{p_3}) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3,$$

where,

$$A_1 = -\frac{\rho(417c_2+10\rho)}{663(27c_2+2\rho)}, \qquad A_2 = \frac{2c_2(81c_2(10387d_2+14070\rho)+\rho(31161d_2+84655\rho))}{146523(27c_2+2\rho)},$$
$$A_3 = -\frac{470c_2d_2\rho(27c_2+\rho)}{146523(27c_2+2\rho)}.$$

Since $c_2 = \frac{459^2-10358\rho^2}{140049\rho} > 0$, when $0 < \rho < \frac{459}{\sqrt{10358}}$, we have that

$$A_1A_2 - A_3 = -\frac{2c_2\rho Q_9}{97144749(27c_2+2\rho)^2} < 0,$$
$$Q_9 = 237259854{c_2}^2d_2 + 475242390{c_2}^2\rho + 8787402c_2d_2\rho + 46697835c_2\rho^2 + 846550\rho^3.$$

Therefore, the differential system (3.1) does not exhibit an Andronov–Hopf bifurcation at the equilibrium point $p_3 = \left(\frac{5}{4}, 1, \frac{94(210681-10358\rho^2)}{30950829d_2\rho}\right)$. □

Theorem 3.31. *If the parameters k_3, k_7, k_8, b_2, c_2 and ρ satisfy the relations (3.14) of Lemma 3.15, then the equilibrium point $p_3 = \left(\frac{5}{4}, 1, \frac{94(210681-10358\rho^2)}{30950829d_2\rho}\right)$ of differential system (3.1) is locally unstable.*

Proof. From Theorem 3.30, the characteristic polynomial $P(\lambda)$ has three sign changes in its coefficients. By the Descartes rule of signs, we have that there exists at least one positive eigenvalue for the linearization at p_3. Then p_3 is unstable. □

3.3.4 Simultaneous periodic orbits at p_1 and p_2

If the parameters k_3, k_7, k_8, b_2, c_2 and ρ satisfy the relations (3.14) of Lemma 3.15, then according to Theorem 3.20 the differential system (3.1) exhibits an Andronov–Hopf bifurcation at p_1, with respect to the parameter d_2, with critical bifurcation value $d_2 = d_{20}(\rho)$. By Theorem 3.27 the differential system (3.1) exhibits an Andronov–Hopf bifurcation at p_2, with respect to the parameter d_2, with critical bifurcation value $d_2 = d_{21}(\rho)$. In order to find a parameter value where the differential system exhibits a simultaneous Andronov–Hopf bifurcation we solve the equation

$$d_{21}(\rho) - d_{20}(\rho) = 0.$$

The unique solution in the interval $\left(0, \frac{459}{\sqrt{10358}}\right)$ is

$$\rho_* := 8262\sqrt{\frac{2478}{38779\sqrt{10580386691137}+126081609691}} \approx 0.81892.$$

The Figure 3.1 (a), shows the graph of critical bifurcation value in terms of ρ for each equilibrium point and its intersection at $(\rho_*, d_{2,0}(\rho_*))$ Therefore the differential system (3.1) exhibits a simultaneous Andronov–Hopf bifurcation at p_1 and p_2, with respect to the parameter d_2, with critical bifurcation value

$$d_{20}(\rho_*) = d_{21}(\rho_*)$$
$$= \frac{1593299484\sqrt{2478\left(38779\sqrt{10580386691137}+126081609691\right)}}{76178503957\sqrt{10580386691137}+248078464264819189} \approx 0.08032.$$

Since $\rho_* < \rho_0 < \rho_1$, by Theorems 3.20 and 3.27, this simultaneous Andronov–Hopf bifurcation is subcritical at p_1 and p_2 (the Figure 3.1 (b) shows the first Lyapunov coefficient corresponding to p_1 or p_2). In this case, the limit cycle bifurcating from p_1 and the limit cycle bifurcating from p_2 are unstable. By Theorems 3.22 and 3.29 the differential systems exhibits a Bautin bifurcation, then there are two limit cycles bifurcating from p_1 or p_2, where one is stable and the other is unstable.

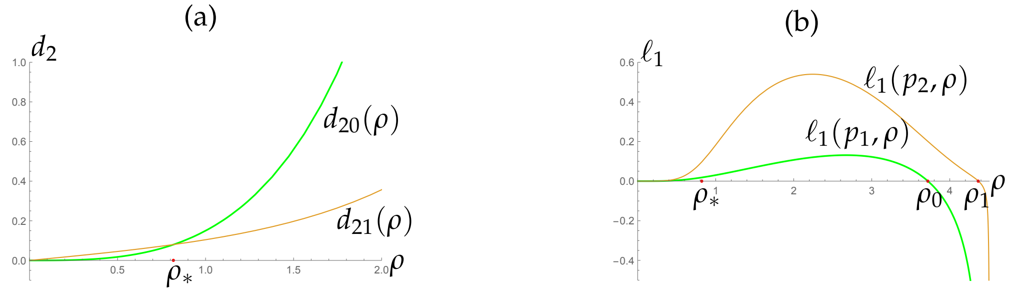

Figure 3.1: (a) Parameter bifurcation value, d_2. (b) First Lyapunov coefficient at p_1 and p_2

Example 3.32. Taking $a_1 = 22.7118$, $a_2 = 984.723$, $b_1 = 5.75$, $b_2 = 1.10108$, $c_1 = 0.0060473$, $c_2 = 0.0164717$, $c_3 = 1$ and $\rho = 4.4 > \rho_1 > \rho_0$, then $d_{20} = 468.666$, $d_{21} = 49.021$ and the parameters involved in the differential system (3.1) satisfy the hypothesis established in the Subsection 3.3. Hence we have three equilibrium points p_1, p_2 and p_3. The first Lyapunov coefficient at p_1 and p_2 are

$$\ell_1(p_1, d_{20}) = -1.24243, \qquad \ell_1(p_2, d_{21}) = -0.0135894.$$

In this case, we have two stable limit cycle each one bifurcating from the equilibrium points p_1 and p_2. In the Figure 3.2 we show two trajectories whose $\omega-$limit are the stable periodic orbits, where $d_2 = d_{20} + 1/100$.

4 Conclusion

When the prey has a linear growth, the differential system has only one equilibrium point in the positive octant Ω and around this point appear an stable periodic orbit generated by an Andronov–Hopf bifurcation or a Bautin bifurcation. On the other hand, if the growth of the prey is logistic, the differential system can have even three equilibrium points in Ω. In particular, when there is only one equilibrium point in Ω, it is not hyperbolic. When there are two equilibria, one is not hyperbolic and the other exhibits an limit cycle generated by an Andronov–Hopf bifurcation or Bautin bifurcation. In the case, when there are three equilibrium points, two of them can present Andronov–Hopf and Bautin bifurcation, in fact they can appear simultaneously. Thus the differential system exhibits bi-stability. The other equilibrium point is always unstable. This analysis shows that the condition to have coexistence of the three populations is better in the logistic growth.

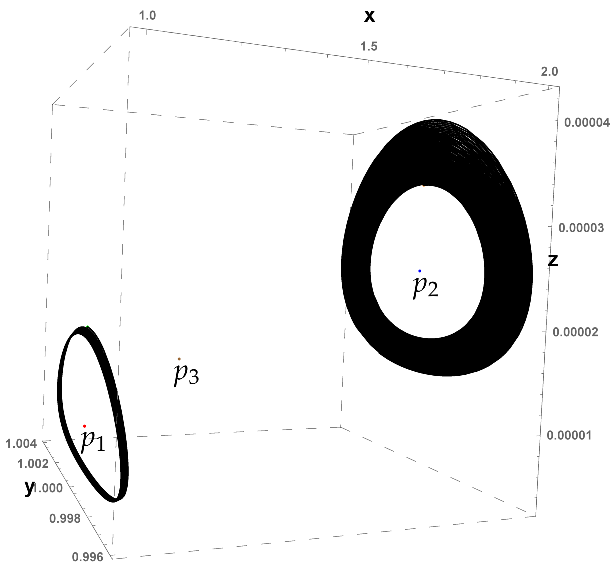

Figure 3.2: Phase space of differential system (3.1) with three equilibria and two limit cycles.

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