The Structure of the Univoque Set

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The specification of the univoque numbers is one of the newest fields in the research of generalized number systems. In [1] and [2] Z. Daróczy and I. Kátai have specified the univoque sequences and have presented a method for the computation of the Hausdorff dimension of the univoque set in the cases $1 < \beta \leq 2$, where β is the base of the number system. Now we continue this investigation in the general case, where $\beta > 1$ is the base number of an arbitrary number system, $\Theta = \frac{1}{\beta}$. The first part of this investigation is based on the methods presented in [1] and [2] but later significantly new approaches are needed. In the sequel we work only on the set of the fractions.

For $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots) \in \{0, 1, \dots [\beta]\}^N$ let $\langle \varepsilon, \Theta \rangle = \sum_{n=1}^{\infty} \varepsilon_n \Theta^n$. A sequence ε is said to be univoque (with respect to be univoque (with respect to Θ), if $\langle \varepsilon, \Theta \rangle = \langle \delta, \Theta \rangle$ is only true if $\varepsilon = \delta$, i.e. $\varepsilon_n = \delta_n$, for $n \in N(\delta \in \{0, 1, \dots, [\beta]\}^N)$. In this case the number $\langle \varepsilon, \Theta \rangle$ is said to be univoque, too. The quality $x = (\varepsilon(x), \Theta)$ is called the regular expansion of x, where $\varepsilon(x) = (\varepsilon_1(x), \varepsilon_2(x), \dots)$ and $\varepsilon_i(x)$ is maximal. We denote the set of univoque sequences and the set of regular sequences by $U(\Theta)$ and $R(\Theta)$, respectively. It is true in general, that the set of univoque sequences is symmetrical and self-similar. We can specify the univoque sequences by using the statement that a sequence $\varepsilon \in \{0, 1, \dots, [\beta]\}^N$ is univoque with respect to $\Theta \iff \varepsilon, \underline{[\beta]} - \varepsilon \in R(\Theta)$, where $\underline{[\beta]} - \varepsilon$ is the complementary sequence of ε. Thus, in order to decide whether a sequence and that of the complementary sequence. To establish that an expansion producing a number less than 1 is regular, we use the result of W. Parry [3].

After this we break down the problem into two cases. If the fraction part of the base number is smaller than a suitable bound, then the structure of the univoque sequences is relatively easy to describe. In the other case the situation is much more complicated, but eventually we can present a method for the computation of the Hausdroff dimension of the univoque set in this case, too. We illustrate the theoretical results with interesting examples.

References

[1] Z. Daróczy, I. Kátai, Univoque Sequences, *Publ. Maht. Debrecen,* 42. (1993), 397-407.

[2] Z. Daróczy, I. Kátai, On the Structure of Univoque Numbers, *Publ. Math. Debrecen,* 46. (1995), 385-408.

[3] W. Parry, On the β Expansions of Real Numbers, *Acta Math. Hung.,* 11. (1960), 401-406.