The Correspondence Between Varieties of Automata and Semigroups

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For a finite alphabet X, a non-empty class of recognizable languages in the free monoid X^*, closed under Boolean operations, left and right quotients and inverse homomorphic images, is called a *variety of X-languages*, and by a *variety of languages* we mean a mapping $X \mapsto \mathcal{L}(X)$ which to each finite alphabet X associates some variety of X-languages $\mathcal{L}(X)$. On the other hand, automata with the input alphabet X are often considered as algebras of unary type indexed by X, so by an *variety* (resp. *pseudovariety*) *of X-automata* we mean an ordinary variety (resp. pseudovariety) of unary algebras of this type and by a *variety* (resp. *pseudovariety*) *of automata* we mean a mapping $X \mapsto \mathcal{A}(X)$ which to each alphabet X associates some variety (resp. pseudovariety) of X-automata $\mathcal{A}(X)$.

The well-known theorem of S. Eilenberg, proved in 1976, says that there is a bijective, order preserving correspondence $V \mapsto \mathcal{L}_V$ between pseudovarieties of monoids and varieties of languages. A similar correspondence between semigroups and languages in free semigroups was established by J. E. Pin in 1984. On the other hand, it was proved by M. Steinby in 1994 that there is a bijective, order preserving correspondence between pseudovarieties of monogenic automata and varieties of languages. Therefore, the remaining open problem is to study related correspondences between automata and semigroups, This is the main purpose of the present paper.

The main result that we prove here is the following general theorem:

Theorem 1 *There is a bijective, order preserving correspondence $V \mapsto \mathcal{A}_V$ between varieties of semigroups and regular varieties of automata.*

Here by a *regular variety of X-automata* we mean a variety determined by regular identities, that is of identities of the form $pu = pv$, where p is a variable and $u, v \in X^*$.

Similar correspondences between pseudovarieties and generalized varieties of semigroups and regular pseudovarieties and generalized varieties of automata will be also established. For a given cardinal κ we consider semigroups having not more than κ generators, called *κ-generated semigroups*, and classes of κ-generated semigroups closed under homomorphic images and κ-generated subdirect products, called *κ-varieties*, and we establish a correspondence between varieties of X-automata and κ-varieties of semigroups, where κ is the cardinality of X.

We also study non-regular varieties, pseudovarieties and generalized varieties of automata. The greatest non-regular generalized variety of automata is the class of all *directable automata*. Besides the "classical" characteristic semigroups, we introduce and study a new type of characteristic semigroups of directable automata, defined in terms of non-regular identities satisfied on them. We consider the problem of *regularization* of non-regular varieties of automata, studied previously by J. Płonka in 1982–95, E. Graczyńska in 1983, S. R. Kogalovskiĩ in 1991 and others, and we connect it with the concept of *localization* of varieties of automata, introduced by M. Steinby in 1994.

Moreover, we investigate some well-known generalized varieties of automata, consisting of directable, definite, reverse definite, generalized definite and nilpotent automata, and we introduce some their generalizations: generalized directable, trapped, one-trapped, locally directable, locally one-trapped, locally nilpotent and locally definite automata. These types of automata will be completely described in terms of their characteristic (transition) semigroups, and using several well-known decomposition methods, such as direct sum decompositions, subdirect and parallel decompositions and extensions of automata.