Packing of Equal Circles in a Square

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Several optimal packings of diverse shape objects are still open problems in geometry. The packing problem of this work is the optimal packing of n equal and non-overlapping circles in a square, where the radius of the circles are maximal. An equivalent problem is to locate n points in a square, where the minimal pairwise distance of n points is maximal. Let's denote this maximum distance by m. The problem can be described as [1]:

$$\begin{array}{c} (x_i - x_j)^2 + (y_i - y_j)^2 = s_{i,j}, \qquad \forall(1 \le i < j \le n) \\ 0 \le x_i \le 1 \qquad i = 1, \ldots, n \\ 0 \le y_i \le 1 \qquad i = 1, \ldots, n \\ \hline \max\limits_{x_i, x_j} \min\limits_{(1 \le i < j \le n)} s_{i,j} \end{array}$$

where $s_{i,j}$ is the squares distance between the points i and j.

The packing problem have a long history in the mathematical literature, but this instance is only 38 years old. Leo Moser in 1960 was the first who studied this question. The proven optimal solutions given in the literature are only those up to $n = 27$ circles and that of $n = 36$. The approximations for larger configurations were done by numerical procedures. Good packings (the best known without a rigorous proof) over 50 are available for 52, 54-56, 60-62, 72 and the 78 [2, 3]. These good packings were found by repeated patterns method and minimization of the energy function. The most interesting story is for n=10. In the table we can see some people who worked with this case.

n	Year	Authors	m
10	1970	M. Goldberg	0.41666666
	1971	J. Schaer	0.41954209
	1979	K. Schlüter	0.42127954
	1987	R. Milano	0.42014346
	1989	G. Valette	0.42118970
	1989	B. Grünbaum	0.42124996
	1990	M. Grannell	0.42127954
	1990	J. Petris, N. Hungerbühler	0.42127954
	1990	M. Mollard, C. Payan	0.42127954
	1990	de Groot, R. Peikert, D. Würtz	0.42127954

Table 1: The n=10 case.

The method presented here is based on threshold accepting technique maximizing the minimum distance between one center of a circle and that of the others. This procedure is similar to simulated annealing but only not too bad configurations are accepted. The probability to find the optimum grows with the number of trials. To locate better positions of the center of the circles, modification of SASS (Simple Agent Stochastic Search) was applied [4]. This is the first stage of the algorithm. This stage stops when a reasonable number of searches were done. In the second stage more accurate solution is located from the best solution obtained in the previous stage. The accuracy is improved by overlapping a box over each obtained circle center and by the use of an interval branch-and-bound procedure [5, 6].

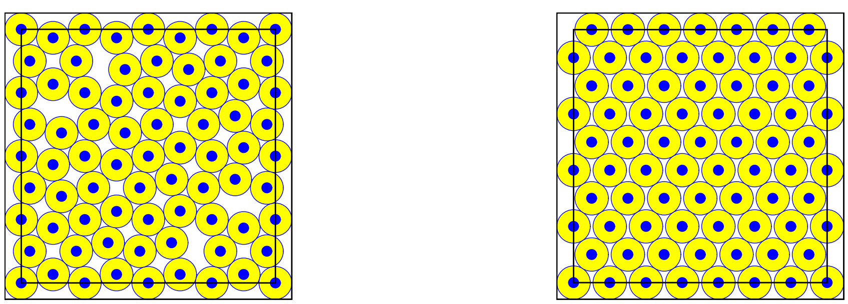
The interval branch-and-bound procedure can not be applied in all the cases because the larger the dimensionality the larger the number of combinatorial subproblems that make the problem intractable, and even reducing the number of combinations by taking into account symmetries will not help much [7, 8].

Beside the numerical results, we determined the exact m values for a lot of optimal packings.

n	m	n	m
2	$\sqrt{2}$	15	$(1+\sqrt{2}-\sqrt{3})/2$
3	$2\sqrt{2-\sqrt{3}}$	16	$1/3$
4	1	18	$\sqrt{13}/12$
5	$\sqrt{2}/2$	20	$(6-\sqrt{2})/16$
6	$\sqrt{13}/6$	23	$\sqrt{2-\sqrt{3}}/2$
7	$2(2-\sqrt{3})$	24	$4+2\sqrt{3}-\sqrt{26+15\sqrt{3}}$
8	$\sqrt{2-\sqrt{3}}$	25	$1/4$
9	$1/2$	27	$\sqrt{89}/40$
12	$\sqrt{34}/15$	36	$1/5$
14	$2(4-\sqrt{3})/13$		

Table 2: The exact values of the optimal packings.

Good packings are found a lot of new cases using a new pattern. For example in the Figure 1 we can see good packings for $n = 75$.



m=0.129409522 m=0.132089817

Figure 3: Packings of 75 circles

The good packings found with the discussed, as well as the best known packings. In Figure 2 we can see the m values as the function of the number of circles. The full circles mean the best results of the literature and the empty ones were found by the present procedure. Notice, that the new solutions fit well to the curve of the earlier values.

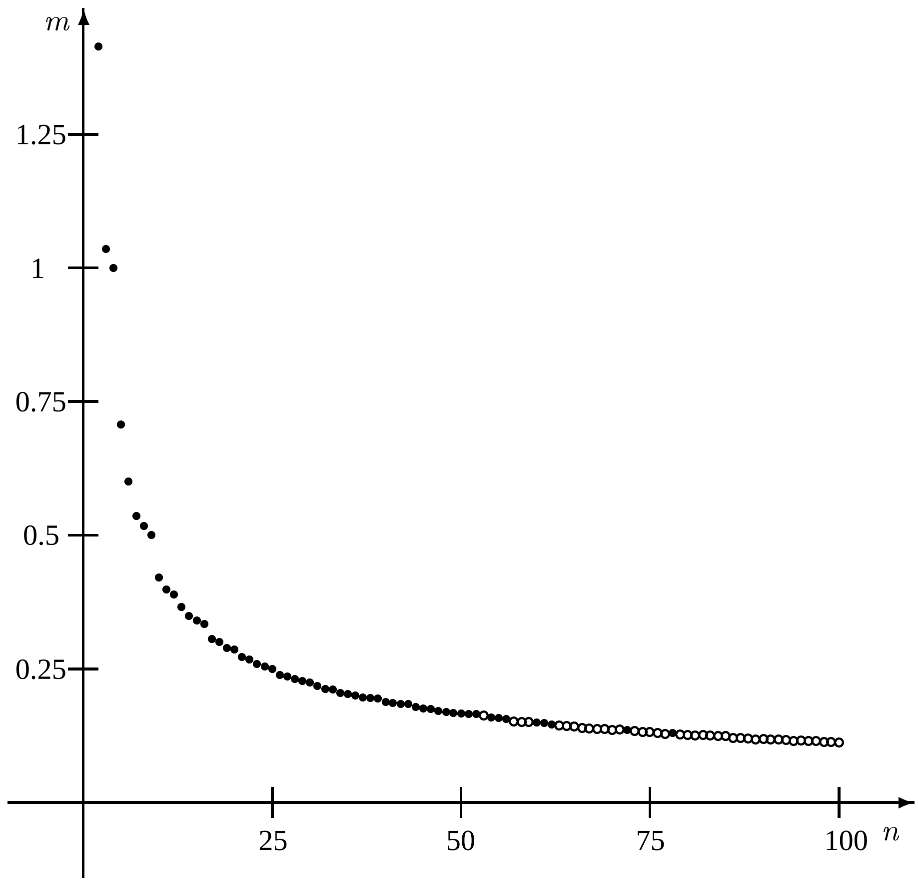


Figure 2: The maximum of the minimal distance as the function of the number of packed circles

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